2. On Casimir's theorem of semi-simple continuous groups.

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Introduction.

Recently, Prof. B. L. van der Waerden⁽¹⁾ gave the algebraic proof for Weyl's theorem⁽²⁾ on semi-simple continuous groups. For this purpose, he made use of the following theorem proved by Prof. H. Casimir⁽³⁾.

Theorem. *If $X_1, X_2, \dots, X_r$ generate a semi-simple continuous group of linear transformations and satisfy the law of compositions such that*

$$(1) \quad [X_i X_k] = \sum_{j=1}^r c_{ikj} X_j, \quad (i, k=1, 2, \dots, r),$$

we can write the bilinear form of Cartan in the form

$$(2) \quad \begin{aligned} \varphi(S, T) &= \sum_{i,k=1}^r \left(\sum_{\rho, \sigma=1}^r c_{i\rho\sigma} c_{k\sigma\rho} \right) \lambda^i \mu^k \\ &= \sum_{i,k=1}^r g_{ik} \lambda^i \mu^k, \end{aligned}$$

where $T = \sum_{k=1}^r \lambda^k X_k$ and $S = \sum_{k=1}^r \mu^k X_k$.

Then, if we form

$$(3) \quad G = \sum_{i,k=1}^r g^{ik} X_i X_k,$$

where (g^{ik}) denote the inverse matrix of (g_{ik}) , we have

$$GX_k = X_k G, \quad (k=1, 2, \dots, r).$$

We will now give an another algebraic proof of the above theorem in the following lines.

§ 1. For the purpose of preparations, we recall the following lemma of Weyl⁽²⁾.

Lemma 1.

$$\varphi([ST], T) = 0.$$

Proof. It is well known that Cartan's quadratic form $\varphi(T, T)$ is invariant under any transformation of the adjoint group. Therefore, if we put

$$[ST] = \Delta^*(S)T,$$

where $\Delta^*(S)$ denotes the linear transformation of the adjoint group corresponding to S such that

$$\Delta^*(S) = \begin{pmatrix} \sum_{\sigma=1}^r \mu^\sigma c_{\sigma 11}, & \cdots & \cdots & \cdots & \sum_{\sigma=1}^r \mu^\sigma c_{\sigma r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{\sigma=1}^r \mu^\sigma c_{\sigma 1r}, & \cdots & \cdots & \cdots & \sum_{\sigma=1}^r \mu^\sigma c_{\sigma rr} \end{pmatrix},$$

we have

$$\begin{aligned} \varphi(T, T) &= \varphi(e^{\kappa \Delta^*(S)} T, e^{\kappa \Delta^*(S)} T) \\ &= \varphi(T, T) + \kappa \left[\varphi(\Delta^*(S)T, T) + \varphi(T, \Delta^*(S)T) \right] + o(\kappa). \end{aligned}$$

Consequently, we get

$$\varphi(\Delta^*(S)T, T) = \varphi([ST], T) = 0,$$

because

$$\varphi(S, T) = \varphi(T, S).$$

§ 2. In place of (1), we will choose the normal form reduced by Cartan⁽²⁾:

$$(a) \quad [H_i H_k] = 0, \quad (i, k = 1, 2, \dots, n);$$

$$(\beta) \quad [H(\lambda) E_\alpha] = \alpha(\lambda) E_\alpha,$$

where
$$H(\lambda) = \sum_{k=1}^n \lambda^k H_k \quad \text{and} \quad \alpha(\lambda) = \sum_{k=1}^n \lambda^k \alpha_k;$$

$$(r) \quad [E_\alpha E_{-\alpha}] = H_\alpha,$$

where
$$H_\alpha = \sum_{k=1}^n \alpha^k H_k;$$

$$(\delta) \quad [E_\alpha E_\beta] = N_{\alpha, \beta} E_{\alpha+\beta},$$

where
$$\begin{aligned} N_{\alpha, \beta} &\neq 0, & \text{if } \alpha + \beta \text{ is a root;} \\ &= 0, & \text{if } \alpha + \beta \text{ is not a root.} \end{aligned}$$

Since Cartan's bilinear form (2) becomes

$$\varphi(S, T) = \sum_{i, k=1}^n q_{ik} \lambda^i \mu^k + \sum_{\rho} N_{\rho} \xi_{\rho} \eta_{-\rho},$$

where
$$T = \sum_{k=1}^n \lambda^k H_k + \sum_{\rho} \xi_{\rho} E_{\rho},$$

and
$$S = \sum_{k=1}^n \mu^k H_k + \sum_{\rho} \eta_{\rho} E_{\rho},$$

the expression (3) can be written as follows :

$$(4) \quad G = \sum_{i,k=1}^n q^{ik} H_i H_k + \sum_{\rho} N_{\rho}^{-1} E_{\rho} E_{-\rho},$$

where (q^{ik}) is the inverse matrix of (q_{ik}) .

Hence, we have

Lemma 2.

$$(5) \quad N_{\alpha}^{-1} H_{\alpha} = \sum_{i,k=1}^n q^{ik} a_k H_i,$$

$$(6) \quad N_{\alpha}^{-1} N_{\alpha, \rho} = N_{-\alpha-\rho}^{-1} N_{-\alpha-\rho, \alpha}.$$

Proof. From lemma 1 we have

$$\varphi([E_{\alpha} T], T) = 0,$$

where

$$T = \sum_{k=1}^n \lambda^k H_k + \sum_{\rho} \xi_{\rho} E_{\rho};$$

therefore we get

$$\xi_{-\alpha} \sum_{i,k=1}^n q_{ik} \lambda^i \alpha^k - \xi_{-\alpha} N_{\alpha} \sum_{k=1}^n \lambda^k \alpha_k + \sum_{\rho} N_{\rho} N_{\alpha, -\alpha-\rho} \xi_{\rho} \xi_{-\alpha-\rho} = 0.$$

Consequently we have

$$\sum_{i,k=1}^n q_{ik} \lambda^i \alpha^k = N_{\alpha} \sum_{k=1}^n \lambda^k \alpha_k,$$

$$N_{\rho} N_{\alpha, -\alpha-\rho} + N_{-\alpha-\rho} N_{\alpha, \rho} = 0.$$

§ 3. Finally, we will proceed to prove Casimir's theorem :

Theorem.

$$(A) \quad GH_{\lambda} = H_{\lambda} G, \quad (\lambda = 1, 2, \dots, n).$$

$$(B) \quad GE_{\alpha} = E_{\alpha} G, \quad \text{if } \alpha \text{ is any root.}$$

Proof.

(A) First we will show that $GH_{\lambda} = H_{\lambda} G$.

From (α), we have

$$H_i H_k H_{\lambda} = H_{\lambda} H_i H_k,$$

and from (β), we get

$$E_{\rho} E_{-\rho} H_{\lambda} = E_{\rho} H_{\lambda} E_{-\rho} + \rho_{\lambda} E_{\rho} E_{-\rho},$$

$$H_{\lambda} E_{\rho} E_{-\rho} = E_{\rho} H_{\lambda} E_{-\rho} + \rho_{\lambda} E_{\rho} E_{-\rho}.$$

(B) Secondly we show that $GE_{\alpha} = E_{\alpha} G$.

(i) From (β), we get

$$H_i H_k E_{\alpha} = H_i E_{\alpha} H_k + \alpha_k H_i E_{\alpha},$$

$$E_{\alpha} H_i H_k = H_i E_{\alpha} H_k - \alpha_i E_{\alpha} H_k.$$

(ii) From (γ), we get

$$\begin{aligned} E_{-\alpha}E_{\alpha}E_{\alpha} &= E_{\alpha}E_{-\alpha}E_{\alpha} - H_{\alpha}E_{\alpha}, \\ E_{\alpha}E_{\alpha}E_{-\alpha} &= E_{\alpha}E_{-\alpha}E_{\alpha} + E_{\alpha}H_{\alpha}. \end{aligned}$$

(iii) From (δ), we get

$$\begin{aligned} E_{\rho}E_{-\rho}E_{\alpha} &= E_{\rho}E_{\alpha}E_{-\rho} + N_{-\rho, \alpha}E_{\rho}E_{\alpha-\rho}, \\ E_{\alpha}E_{\rho}E_{-\rho} &= E_{\rho}E_{\alpha}E_{-\rho} + N_{\alpha, \rho}E_{\alpha+\rho}E_{-\rho}. \end{aligned}$$

In consequence of the above results (i), (ii), (iii), it is necessary for $GE_{\alpha}=E_{\alpha}G$ that we must have

$$\begin{aligned} \sum_{i, k=1}^n q^{ik} \alpha_k H_i E_{\alpha} - N_{-\alpha}^{-1} H_{\alpha} E_{\alpha} + \sum_{\rho} N_{\rho}^{-1} N_{-\rho, \alpha} E_{\rho} E_{\alpha-\rho} \\ = - \sum_{i, k=1}^n q^{ik} \alpha_i E_{\alpha} H_k + N_{\alpha}^{-1} E_{\alpha} H_{\alpha} + \sum_{\rho} N_{\rho}^{-1} N_{\alpha, \rho} E_{\alpha+\rho} E_{-\rho}. \end{aligned}$$

But, this is evident by lemma 2, because $q^{ik}=q^{ki}$ and $N_{\alpha}^{-1}=N_{-\alpha}^{-1}$.

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(3) H. Casimir: Proc. Kon. Acad. Amsterdam **34** (1931), 844.

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