6. The affine breadth and its relation to the relative breadth.

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Prof. W. Süss has made an investigation concerning the affine constant breadth.⁽¹⁾ But I will make some considerations of the affine breadth from a different point of view.

In the following lines we use the same notations as in my preceding paper "*On the Relative Breadth.*"⁽²⁾

§ 1. The affine breadth of the oval.

Let us define b :

$$(1) \quad b = 2 \frac{((\xi_2 - \xi_1)\xi_1)}{(\xi_1''\xi_1) + (\xi_2''\xi_2)} = 2 \frac{p_1 + p_2}{\bar{\rho}_1^{-\frac{1}{3}} + \bar{\rho}_2^{-\frac{1}{3}}}.$$

as the affine breadth at the point ξ of an oval $E(\xi)$.

Theorem 1. Let p_i denote the affine distance from ξ_i to ξ_k ($i, k = 1, 2$; $i \neq k$) where ξ_1 and ξ_2 are the points of contact of the parallel tangents, then b is equal to the harmonic mean of p_1 and p_2 , and b is an affine invariant.

Proof. By assumption we have

$$p_1 = \bar{\rho}_1^{\frac{1}{3}} ((\xi_2 - \xi_1)\xi_1), \quad p_2 = \bar{\rho}_2^{\frac{1}{3}} ((\xi_1 - \xi_2)\xi_2). \quad (3)$$

From the definition of b the relation follows

$$\begin{aligned} \frac{2}{b} &= \frac{\bar{\rho}_1^{-\frac{1}{3}} + \bar{\rho}_2^{-\frac{1}{3}}}{((\xi_2 - \xi_1)\xi_1)}, \\ &= \frac{1}{\bar{\rho}_1^{\frac{1}{3}} ((\xi_2 - \xi_1)\xi_1)} + \frac{1}{\bar{\rho}_2^{\frac{1}{3}} ((\xi_1 - \xi_2)\xi_2)}, \\ &= \frac{1}{p_1} + \frac{1}{p_2}. \end{aligned}$$

(1) Über affine Geometrie XL: Eiflächen konstanter Affinbreite; Math. Ann., **96** (1927).

(2) Proceedings of the Physico-Math. Soc. of Japan, **17** (1935).

(3) W. Blaschke, Vorlesungen über Differentialgeometrie II.

Theorem 2. A necessary and sufficient condition that the affine breadth of $\mathfrak{C}$ be constant is that the two vectors $\mathfrak{x}_2 - \mathfrak{x}_1$ and $\mathfrak{x}_1'' - \mathfrak{x}_2''$ are parallel.

Proof. From the definition of b we have

$$(2) \quad \left(\left\{ \mathfrak{x}_1 - \mathfrak{x}_2 + \frac{b}{2} (\mathfrak{x}_1'' - \mathfrak{x}_2'') \right\} \xi_1 \right) = 0.$$

Then it can be put

$$(3) \quad \mathfrak{x}_1 - \mathfrak{x}_2 + \frac{b}{2} (\mathfrak{x}_1'' - \mathfrak{x}_2'') = \sigma \mathfrak{x}_1',$$

where σ is a scalar quantity.

If b is constant, we have by differentiation of (3) with respect to the affine curve length u_1 at $\mathfrak{x}_1$

$$(4) \quad \mathfrak{x}_1' - \mathfrak{x}_2' \frac{du_2}{du_1} + \frac{b}{2} \left(\mathfrak{x}_1''' - \mathfrak{x}_2''' \frac{du_2}{du_1} \right) = \sigma \mathfrak{x}_1'' + \sigma' \mathfrak{x}_1',$$

where u_2 denotes the affine curve length at $\mathfrak{x}_2$.

Hence the relations become

$$(5) \quad \frac{\mathfrak{x}_2'}{|\mathfrak{x}_2'|} = -\frac{\mathfrak{x}_1'}{|\mathfrak{x}_1'|}, \quad \mathfrak{x}_1''' = -k_1 \mathfrak{x}_1', \quad (3)$$

$$\mathfrak{x}_2''' = -k_2 \mathfrak{x}_2' = k_2 \frac{|\mathfrak{x}_2'|}{|\mathfrak{x}_1'|} \mathfrak{x}_1'.$$

From (5) the equation (4) takes the form

$$\left\{ 1 + \frac{|\mathfrak{x}_2'|}{|\mathfrak{x}_1'|} \frac{du_2}{du_1} - \frac{b}{2} \left(k_1 + \frac{|\mathfrak{x}_2'|}{|\mathfrak{x}_1'|} \frac{du_2}{du_1} \right) - \sigma' \right\} \mathfrak{x}_1' = \sigma \mathfrak{x}_1''.$$

By means of $(\mathfrak{x}_1', \mathfrak{x}_1'') = 1$ (3)

$$\sigma = 0.$$

Therefore the relation holds

$$(6) \quad \mathfrak{x}_1 - \mathfrak{x}_2 + \frac{b}{2} (\mathfrak{x}_1'' - \mathfrak{x}_2'') = 0.$$

This equation shows that the vector $\mathfrak{x}_1 - \mathfrak{x}_2$ is parallel to the vector $\mathfrak{x}_1'' - \mathfrak{x}_2''$.

Next we prove that it is sufficient. If the two vectors are parallel, from (3) we get (6). By differentiation of (6) with respect to u_1 we have

$$\mathfrak{x}_1' - \mathfrak{x}_2' \frac{du_2}{du_1} + \frac{b}{2} \left(\mathfrak{x}_1''' - \mathfrak{x}_2''' \frac{du_2}{du_1} \right) + \frac{b}{2} (\mathfrak{x}_1'' - \mathfrak{x}_2'') = 0.$$

By means of (5) we get

$$b' = 0, \quad \text{or} \quad b = \text{constant}.$$

Theorem 3. If b is constant, then the following relation holds

$$b = 2 \frac{\bar{\rho}_1 + \bar{\rho}_2}{k_1 \bar{\rho}_1 + k_2 \bar{\rho}_2}.$$

Proof. By differentiation of (6) with respect to u_1

$$(7) \quad 1 + \frac{|\xi'_2|}{|\xi'_1|} \frac{du_2}{du_1} - \frac{b}{2} \left(k_1 + k_2 \frac{|\xi'_2|}{|\xi'_1|} \frac{du_2}{du_1} \right) = 0.$$

Hence $|\xi'_i| = \bar{\rho}_i^{\frac{1}{3}}$, $\frac{du_i}{d\varphi} = \bar{\rho}_i^{\frac{2}{3}}$ and then $\frac{du_2}{du_1} = \frac{\bar{\rho}_2^{\frac{2}{3}}}{\bar{\rho}_1^{\frac{2}{3}}}$.

Therefore the relation (7) becomes

$$1 + \frac{\bar{\rho}_2}{\bar{\rho}_1} - \frac{b}{2} \left(k_1 + k_2 \frac{\bar{\rho}_2}{\bar{\rho}_1} \right) = 0,$$

then the relation

$$(8) \quad b = 2 \frac{\bar{\rho}_1 + \bar{\rho}_2}{k_1 \bar{\rho}_1 + k_2 \bar{\rho}_2}$$

holds.

Theorem 4. If the affine normals at the points of contact of two tangents are affine double normals of an oval, the oval is an ellipse and

$$b = \frac{2}{k}.$$

Proof. From assumption it can be put

$$\xi''_2 = -\sigma \xi''_1,$$

where σ is a positive scalar quantity.

Then we have

$$(9) \quad \xi''_2 \frac{du_2}{du_1} = -\sigma \xi''_1 - \sigma' \xi'_1.$$

By means of (5) and $(\xi'_1, \xi'_1) = 1$, $\sigma = \text{constant}$ and then $\sigma = 1$.

Therefore $\xi''_1 = -\xi''_2$.

By means of (5) we have

$$1 = (\xi'_1, \xi'_1) = \frac{\bar{\rho}_1^{\frac{1}{3}}}{\bar{\rho}_2^{\frac{1}{3}}} (\xi'_2, \xi'_2) = \frac{\bar{\rho}_1^{\frac{1}{3}}}{\bar{\rho}_2^{\frac{1}{3}}},$$

therefore

$$(10) \quad \bar{\rho}_1 = \bar{\rho}_2.$$

Then we have

$$(11) \quad k_1 = k_2.$$

By the assumption the two vectors $\mathfrak{x}_1 - \mathfrak{x}_2$ and $\mathfrak{x}_1'' - \mathfrak{x}_2''$ are parallel. Therefore from *Theorem 2* the oval must be of affine constant breadth. Then we obtain from *Theorem 3*, by means of (10) and (11),

$$b = \frac{2}{k_1}.$$

Then since k is constant, the oval is an ellipse.

§ 2. The relation between the affine breadth and the r-breadth⁽²⁾ of the oval.

Let us take the negative indicatrix of the affine curvature as the unit oval in the *relative differential geometry*⁽⁴⁾, that is, put

$$e = -\mathfrak{x}''.$$

Then the relation holds

$$b = 2 \frac{((\mathfrak{x}_2 - \mathfrak{x}_1)\xi_1)}{((\mathfrak{x}_1'' - \mathfrak{x}_2'')\xi_1)} = 2 \frac{((\mathfrak{x}_1 - \mathfrak{x}_2)\xi_1)}{((e_1 - e_2)\xi_1)} = B, \quad (2)$$

or

$$b = B.$$

Therefore *Theorem 2* and *Theorem 3* are obtained from the corresponding theorems in my preceding paper.

§ 3. The affine breadth of the ovaloid.

Let $\mathfrak{y}_1$ and $\mathfrak{y}_2$ be respectively the affine normals at the points of contact $\mathfrak{x}_1$ and $\mathfrak{x}_2$ of the parallel tangent planes on the ovaloid $\mathfrak{x}$. Let us put

$$(1) \quad b = 2 \frac{((\mathfrak{x}_2 - \mathfrak{x}_1)\xi_1)}{(\mathfrak{y}_1\xi_1) + (\mathfrak{y}_2\xi_2)},$$

where b is defined as the affine breadth.

Hence we have

$$\mathfrak{y} = G^{ml} \left(\frac{\partial}{\partial u^l} \log |\bar{K}|^{-\frac{1}{2}} \right) \mathfrak{x}_m + |\bar{K}|^{\frac{1}{2}} \xi, \quad (3)$$

⁽⁴⁾ W. Süss, Zur relativen Differentialgeometrie I. Japanese Journ. of Math., **4** (1927).

then since $\bar{K} > 0$, $(\eta_i \xi_i) = \bar{K}^{\frac{1}{2}}$.

Therefore we have

$$(2) \quad b = 2 \frac{p_1 + p_2}{\bar{K}_1^{\frac{1}{2}} + \bar{K}_2^{\frac{1}{2}}}.$$

Theorem 1. Let p_i denote the affine distance from ξ_i to ξ_k ($i, k = 1, 2$; $i \neq k$), then b is equal to the harmonic mean of p_1 and p_2 , and then b is an affine invariant.

Proof. By assumption we have

$$p_1 = \bar{K}_1^{-\frac{1}{2}} ((\xi_2 - \xi_1) \xi_1), \quad p_2 = \bar{K}_2^{-\frac{1}{2}} ((\xi_1 - \xi_2) \xi_2).$$

From the definition of b

$$\begin{aligned} \frac{2}{b} &= \frac{\bar{K}_1^{\frac{1}{2}} + \bar{K}_2^{\frac{1}{2}}}{(\xi_2 - \xi_1) \xi_1} \\ &= \frac{1}{\bar{K}_1^{-\frac{1}{2}} ((\xi_2 - \xi_1) \xi_1)} + \frac{1}{\bar{K}_2^{-\frac{1}{2}} ((\xi_1 - \xi_2) \xi_2)} \\ &= \frac{1}{p_1} + \frac{1}{p_2}. \end{aligned}$$

Theorem 2. A necessary and sufficient condition that the affine breadth of an ovaloid $E(\xi)$ be constant, is that the two vectors $\xi_1 - \xi_2$ and $\eta_1 - \eta_2$ are parallel.

Proof. From (1) we have

$$(3) \quad \left(\left\{ \xi_1 - \xi_2 + \frac{b}{2} (\eta_1 - \eta_2) \right\} \xi_1 \right) = 0.$$

Therefore it can be put

$$(4) \quad \xi_1 - \xi_2 + \frac{b}{2} (\eta_1 - \eta_2) = \alpha \xi_{1u} + \beta \xi_{1v}.$$

By differentiation of (4) with respect to the parameters u and v

$$\xi_{1u} - \xi_{2u} + \frac{b}{2} (\eta_{1u} - \eta_{2u}) = \alpha \xi_{1uu} + \beta \xi_{1uv} + \alpha_u \xi_{1u} + \beta_u \xi_{1v},$$

$$\xi_{1v} - \xi_{2v} + \frac{b}{2} (\eta_{1v} - \eta_{2v}) = \alpha \xi_{1uv} + \beta \xi_{1vv} + \alpha_v \xi_{1u} + \beta_v \xi_{1v}.$$

By means of the relation $\eta_{1i} = B_i^k \xi_{1k}$, ⁽³⁾ in the same manner as in the proof of the corresponding theorem in my preceding paper, ⁽²⁾ we have

$$\alpha = \beta = 0.$$

Conversely it can be easily proved that it is sufficient.

Theorem 3. If b be constant, then we have

$$(5) \quad b = 2 \frac{(D_1 + D_2)R_1^1 R_1^2}{D_1 R_1^2 + D_2 R_1^1} = 2 \frac{(D_1'' + D_2'')R_2^1 R_2^2}{D_1'' R_2^2 + D_2'' R_2^1},$$

where $D_i = (\xi_{iuv} \xi_i)$, $D_i'' = (\xi_{iuv} \xi_i)$ and R_1^i and R_2^i are the radii of the affine curvature at ξ_i .

Proof. Since b is constant, by *Theorem 2* we obtain

$$(6) \quad \xi_1 - \xi_2 + \frac{b}{2}(\eta_1 - \eta_2) = 0.$$

Let the lines of affine curvature be taken as parameter-curves, so that

$$(7) \quad \begin{cases} \xi_{iu} = -R_1^i \eta_{iu}, & (3) \\ \xi_{iv} = -R_2^i \eta_{iv}. \end{cases}$$

By means of (6) and (7)

$$\begin{aligned} \left(1 - \frac{b}{2R_1^1}\right) \xi_{1u} &= \left(1 - \frac{b}{2R_1^2}\right) \xi_{2u}, \\ \left(1 - \frac{b}{2R_2^1}\right) \xi_{1v} &= \left(1 - \frac{b}{2R_2^2}\right) \xi_{2v}. \end{aligned}$$

Let us differentiate (8) respectively with respect to u and v , and let us form the scalar product of both sides of the obtained equation with $\xi_1 = -\xi_2$, so that we have

$$\begin{aligned} \left(1 - \frac{b}{2R_1^1}\right) D_1 &= -\left(1 - \frac{b}{2R_1^2}\right) D_2, \\ \left(1 - \frac{b}{2R_2^1}\right) D_1'' &= -\left(1 - \frac{b}{2R_2^2}\right) D_2''. \end{aligned}$$

Therefore the relation (5) holds.

Theorem 4. If the affine normals at the points of contact of the parallel tangent planes on $E(\xi)$ are an affine double normal, the ovaloid is an ellipsoid, and the relation $\frac{b}{2} = R_1 = R_2 = \text{constant}$ follows.

Proof. By assumption and *Theorem 2* we have

$$b = \text{constant},$$

and then it can be put

$$\xi_1 - \xi_2 = -\frac{b}{2}(\eta_1 - \eta_2) \equiv \lambda \eta_1. \quad (\text{i})$$

$$\text{Then} \quad \left(\lambda + \frac{b}{2}\right)\eta_1 = \frac{b}{2}\eta_2. \quad (\text{ii})$$

By differentiation of (ii) we get

$$\lambda_u \eta_1 + \left(\lambda + \frac{b}{2}\right)\eta_{1u} = \frac{b}{2}\eta_{2u},$$

$$\lambda_v \eta_2 + \left(\lambda + \frac{b}{2}\right)\eta_{1v} = \frac{b}{2}\eta_{2v}.$$

Let us form the scalar product of both members of the above equation with $\xi_1 = -\xi_2$, so that the relation $\lambda_u = \lambda_v = 0$ follows, or $\lambda = \text{constant}$. From (ii) $\lambda = -b$, and then

$$\eta_1 + \eta_2 = 0. \quad (\text{iii})$$

By means of (iii) and the relation $\xi_1 = -\xi_2$ we get

$$\bar{K}_1 = \bar{K}_2.$$

Therefore, by *Prof. T. Kubota's* theorem⁽⁵⁾ $E(\chi)$ must be central. And then from (1) we have

$$\xi_1 = -\frac{b}{2}\eta_1.$$

Let us take the lines of affine curvature as parameter-curves, then we obtain (7). By the relation (7) we have

$$R_1^1 = R_2^1 = \frac{b}{2} = \text{constant}.$$

Sinec $H = \frac{1}{2}\left(\frac{1}{R_1^1} + \frac{1}{R_2^1}\right) = \text{const.}$, the ovaloid is an ellipsoid.⁽³⁾

(5) Einige Probleme über konvex-geschlossene Kurven und Flächen; Tôhoku Math. Journ., **17** (1920).

§ 4. The relation between the affine breadth and the r-breadth of the ovaloid.

Let us take the negative indicatrix of the affine curvature as the unit ovaloid in the relative differential geometry, that is, put $e = -\eta$. Then the relation holds

$$b = 2 \frac{(\xi_2 - \xi_1)\xi_1}{(\eta_1\xi_1) + (\eta_2\xi_2)} = 2 \frac{(\xi_1 - \xi_2)\xi_1}{(e_1 - e_2)\xi_1} = B, \quad (2)$$

or

$$b = B.$$

Theorem 2 and *Theorem 3* can be deduced from the corresponding theorems in my preceding paper.⁽²⁾