16. *Applications of the theory of pseudo-regular functions to the type-problem of Riemann surfaces.**

By Shizuo KAKUTANI

(Received May 4, 1936)

The object of this paper is to apply the theory of pseudo-regular functions to the type-problem of Riemann surfaces of meromorphic functions.

Pseudo-regular functions (or pseudo-conformal transformations) were first studied by H. Grötzsch⁽¹⁾, who proved that Picard's theorem can be generalized to a class of pseudo-regular transformations ("Abbildung von beschränkter infinitesimaler Verzerrung").

Recently M. Lavrentieff⁽²⁾ proved the same theorem for some wider class of functions.

In § 1 we define pseudo-regular functions and pseudo-conformal transformations and give some properties of these functions.

Then the generalization of Picard's theorem to some class of these functions is established with the aid of Ahlfors'⁽³⁾ method, the result being the same with that obtained by M. Lavrentieff⁽⁴⁾.

This theorem is applied⁽⁵⁾ in § 2 to the problem of the type of simply connected infinitely many sheeted open Riemann surface Σ , and a method to determine the type of Σ is obtained.

From this we obtain some results on the properties of Riemann surfaces of meromorphic functions, which are already studied by A. Speiser and L. Ahlfors⁽⁶⁾.

* The result of this paper was presented to the Meeting of Physico-Mathematical society of Japan on 28, Sept. 1935 (Osaka).

(¹) H. Grötzsch: Über eine Verzerrung bei schlichten nicht konformen Abbildung und über eine damit zusammenhängende Erweiterung des Picardschen Satzes, Leipziger Berichte, 80 (1928).

(²) M. Lavrentieff: Sur une class de représentations continues, C. R. 209 (1935).

(³) L. Ahlfors: Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen, Acta Soc. Scient. Fennicae, Nova Series A. 1 (1930).

(⁴) M. Lavrentieff: loc. cit. (²).

(⁵) In this application we use Picard's theorem only in the form which was given by H. Grötzsch.

(⁶) (a) A. Speiser: Probleme aus dem Gebiet der ganzen transzendenten Funktionen, Comment. Math. Helv., 1 (1929).

(b) L. Ahlfors: Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée, Acta Soc. Scient. Fennicae, Nova Ser. A. 2 (1933).

(c) L. Ahlfors: Quelques propriétés des surfaces de Riemann correspondant aux fonctions méromorphes, Bull. de la soc. math. France, 60 (1932). see also :

In § 3 we deal with Speiser's Problem⁽⁷⁾, although this method can be applied to a much wider class of problems, and a sufficient condition that Σ is hyperbolic is obtained, by which we can show the impossibility of determining the type of Riemann surface Σ only from the order of increment of the function $\mu(\theta)$ (or $n(\theta)$)⁽⁸⁾ of its topological tree; for it is possible to construct two Riemann surfaces Σ_1 and Σ_2 , the one being parabolic and the other hyperbolic, and yet for which the inequality $\mu_1(\theta) \geq \mu_2(\theta)$ holds for all $\theta \geq 0$, $\mu_1(\theta)$ and $\mu_2(\theta)$ denoting the "tree-functions" of topological trees of Σ_1 and Σ_2 respectively.

I wish to express my hearty thanks to Profs. T. Shimizu and K. Yosida for their kind advices in the course of this work. Theorems in § 1 are obtained by K. Yosida.

§ 1.

A uniform continuous function

$$w=f(z)=u(x, y)+iv(x, y), \quad z=x+iy,$$

is called pseudo-regular in a domain D , if it satisfies the following conditions:

1. u_x, u_y, v_x, v_y exist and are continuous in D .

2.
$$J(x, y) \equiv \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$$

except possibly at most the countable set of points $z=x+iy$ where $J(x, y)=0$, which, however, has no point of accumulation inside of D .

3. At the point z_0 where $J(x, y)=0$, a sufficiently small neighbourhood of z_0 is transformed topologically on the piece of Riemann surface which has its only one algebraic singularity of order $n-1$ ⁽⁹⁾ at $w_0=u(x_0, y_0)+iv(x_0, y_0)$.

(d) L. Ahlfors: Über die konforme Abbildung von Überlagerungsflächen, Åttende Skandinaviska Matematikerkongressen, Stockholm 1934, and

(e) L. Ahlfors: Zur Theorie der Überlagerungsflächen, Acta Math., 65 (1935).

(7) (a) A. Speiser, loc. cit., (6) (a) see also:

(b) A. Speiser: Über beschränkte automorphe Funktionen, Comment. Math. Helv., 4 (1933).

(c) Z. Kobayashi: On the type of Riemann surfaces, Science Reports of the Tokyo Bunrika Daigaku, Section A, 2. (1935) No. 42.

(8) as for $\mu(\theta)$ see:

(a) Z. Kobayashi: Theorems on the conformal representation of Riemann surfaces, Science Reports of the Tokyo Bunrika Daigaku, Section A, 2 (1935), No. 39.

as for $n(\theta)$ see:

(b) R. Nevanlinna: Ein Satz über die konforme Abbildung Riemannscher Flächen, Comment. Math. Helv., 5 (1933). see also:

(c) L. Ahlfors: Zur Bestimmung des Typus einer Riemannschen Fläche, Comment. Math. Helv., 3 (1931).

(9) n is an integer greater than 1. It means that n sheets of surface meet in one point z_0 . n is called "multiplicity" of w_0 -point $z=z_0$.

These conditions mean that this function considered as a transformation (or representation) is topologically equivalent to a regular function.

We say that $f(z)$ is pseudo-regular at z_0 if it is pseudo-regular in some neighbourhood of z_0 .

If we put as usual

$$\begin{aligned} E &= u_x^2 + v_x^2, \\ F &= u_x u_y + v_x v_y, \\ G &= u_y^2 + v_y^2 \end{aligned}$$

and
$$p_f(z) \equiv p(z) \equiv p(x, y) = \frac{E+G}{J} \equiv \frac{E+G}{\sqrt{EG-F^2}},$$

then
$$p(x, y) \geq 2^{(10)}.$$

We shall first explain the meaning of $p(x, y)$. At the point where $J(P) \neq 0$, an infinitesimal circle C with centre P is transformed by this function into an infinitesimal ellipse E with centre $f(P)$ (if we neglect infinitesimals of order higher than 1).

Assuming, without loss of generality, $P=0$, $f(P)=0$, a circle $z=re^{i\theta}$ (or $x=r \cos \theta$, $y=r \sin \theta$) is transformed into an ellipse

$$\begin{aligned} u &= r(u_x(0, 0) \cos \theta + u_y(0, 0) \sin \theta), \\ v &= r(v_x(0, 0) \cos \theta + v_y(0, 0) \sin \theta), \end{aligned}$$

whose major and minor axes are of lengths respectively

$$r_1 = \sqrt{\frac{E+G}{2}} + \sqrt{\left(\frac{E+G}{2}\right)^2 - J^2}$$

and

$$r_2 = \sqrt{\frac{E+G}{2}} - \sqrt{\left(\frac{E+G}{2}\right)^2 - J^2}.$$

If we put

$$q_f(z) \equiv q(z) \equiv q(x, y) = \frac{r_1}{r_2} (\geq 1)$$

then

$$p(x, y) = q(x, y) + \frac{1}{q(x, y)}.$$

Thus $p(x, y)$ and $q(x, y)$ denote the magnitude of the ratio of major and minor axes, or the degree of deformation "im kleinen" (Verzerrungsgrad im kleinen), and as $p(x, y)$ becomes small (that is, when it approaches to 2), the transformation tends to the conformal one.

⁽¹⁰⁾ Equality holds when and only when $u_x=v_y$, $u_y=-v_x$, so that, if $p(x, y) \equiv 2$ in D then $f(z)$ is regular inside of D .

Since $r\sqrt{E}$ and $r\sqrt{G}$ are the lengths of the radii of E corresponding to the radii of C on x -axis and y -axis respectively, we have

$$r_1 \geq r\sqrt{E} \geq r_2, \quad r_1 \geq r\sqrt{G} \geq r_2$$

or

$$\frac{r_1^2}{r^2} \geq E \geq \frac{r_2^2}{r^2}, \quad \frac{r_1^2}{r^2} \geq G \geq \frac{r_2^2}{r^2}. \quad (3)$$

On the other hand, as is easily seen by (1) and (2),

$$r^2 \cdot J = r_1 \cdot r_2.$$

This, combined with (3), gives

$$q \cdot J \geq E \geq \frac{J}{q}, \quad q \cdot J \geq G \geq \frac{J}{q}. \quad (4)$$

Now we shall study the properties of pseudo-regular functions.

At first we give three theorems without proof, as these are almost evident from the definitions of $p(z)$ and $q(z)$.

Theorem 1. If $w=f(z)$ is pseudo-regular and $\zeta=g(w)$ is regular, then

$$\zeta = g(f(z)) \equiv h(z)$$

is a pseudo-regular function, which has the same $p(z)$ and $q(z)$ with $f(z)$.

Theorem 2. If $w=f(z)$ is regular and $\zeta=g(w)$ is pseudo-regular, then

$$\zeta = g(f(z)) \equiv h(z)$$

is a pseudo-regular function, whose $p(z)$ and $q(z)$ are the same with $p_g(f(z))$ and $q_g(f(z))$ respectively, $p_g(w)$ and $q_g(w)$ being the functions corresponding to $g(w)$.

Theorem 3. If $w=f(z)$ and its inverse $z=g(w)$ are pseudo-regular, then $q_f(z)$ and $q_g(w)$ (accordingly $p_f(z)$ and $p_g(w)$) are equal in corresponding points.

Next we prove a theorem on pseudo-conformal representation, which is the generalization of Ahlfors' theorem⁽¹⁾.

Theorem 4. Let Ω be a simply connected domain on $z(=x+iy)$ -plane bounded by two segments S_1 and S_2 each on $R(z)=x_1$ and x_2 respectively, and by two curves C_1 and C_2 joining these segments, and let T be a simply connected domain on $w(=u+iv)$ -plane bounded by two segments G_1 and G_2 on $v=\pm\frac{a}{2}$ and two curves L_1 and L_2 joining these segments.

If Ω is represented pseudo-conformally on T by $w=f(z) \equiv \xi(z) + i\eta(z)$ in such a way that S_1, S_2, C_1 and C_2 correspond to L_1, L_2, G_1 and G_2 respectively, then we have

(¹) L. Ahlfors: loc. cit., (³).

$$a \int_{x_1}^{x_2} \frac{dx}{q(x)\theta(x)} \leq \xi_2(x_2) - \xi_1(x_1),$$

where

$$\xi_1(x) = \min_{R(z)=x} \xi(z),$$

$$\xi_2(x) = \max_{R(z)=x} \xi(z)^{(12)},$$

$\theta(x)$ = length of the portion θ_x of a straight line $R(z)=x$ cut by C_1 and C_2

and

$$q(x) = \max_{R(z)=x} q(z).$$

Proof.

The length $\gamma(x)$ of the image γ_x in T of θ_x satisfies

$$\begin{aligned} a^2 \leq \gamma(x)^2 &= \left(\int_{\theta_x} |u_y + iv_y| dy \right)^2 \\ &\leq \theta(x) \int_{\theta_x} (u_y^2 + v_y^2) dy \leq \theta(x) \int_{\theta_x} q(x, y) J(x, y) dy \\ &\leq q(x) \theta(x) \int_{\theta_x} J(x, y) dy. \end{aligned}$$

Dividing the both sides by $q(x) \cdot \theta(x)$ and integrating with respect to x from x_1 to x_2 we have

$$a^2 \int_{x_1}^{x_2} \frac{dx}{q(x)\theta(x)} \leq \int_{x_1}^{x_2} \int_{\theta_x} J(x, y) dy dx,$$

the right hand side being the area of T ,

$$\leq a(\xi_2(x_2) - \xi_1(x_1)).$$

Remark.

Taking the inequality

$$a^2 + \omega(x)^2 \leq \gamma(x)^2,$$

where

$$\omega(x) = \xi_2(x) - \xi_1(x),$$

we get after L. Ahlfors⁽¹³⁾ a relation that:

$$\text{If } \int_{x_1}^{x_2} \frac{dx}{q(x)\theta(x)} > 2$$

then

$$\xi_1(x_2) - \xi_2(x_1) \geq a \int_{x_1}^{x_2} \frac{dx}{q(x)\theta(x)} - 4a. \quad (6)$$

⁽¹²⁾ We assume that $f(z)$ is continuous even on the boundary of Ω .

⁽¹³⁾ L. Ahlfors, loc. cit., (3).

This inequality holds also even if the shape of Ω is more complicated as the case studied by Ahlfors⁽¹⁴⁾.

Before we go to the generalization of Picard's theorem, we must define an isolated essential singularity of a pseudo-regular function.

$z=a$ is called an isolated essential singularity of a pseudo-regular function $f(z)$, if, for a sufficiently small $\delta>0$, $f(z)$ is pseudo-regular for $0<|z-a|<\delta$ and is discontinuous at $z=a$ in such a manner that there exist at least two sequences $\{z_n\}$ and $\{z_n'\}$, each of which tends to $z=a$ as $n\rightarrow\infty$ and for which

$$\lim_{n\rightarrow\infty} f(z_n) = \alpha, \quad \lim_{n\rightarrow\infty} f(z_n') = \beta, \quad \alpha \neq \beta$$

($f(z)$ may be not defined at $z=a$).

Theorem 5. (Generalization of Picard's theorem)⁽¹⁵⁾.

Let $f(z)$ be pseudo-regular for $0<|z|<1$ and let $z=0$ be its isolated essential singularity. Then Picard's theorem holds if

$$\lim_{\varepsilon\rightarrow 0} \int_{\varepsilon}^1 \frac{dr}{rp(r)} \quad \text{or} \quad \lim_{\varepsilon\rightarrow 0} \int_{\varepsilon}^1 \frac{dr}{rq(r)} \quad (16)$$

diverges, where $p(r)$ and $q(r)$ denote the maximum of $p(z)$ and $q(z)$ on $|z|=r$ respectively.

In other words, under this condition $f(z)$ takes every finite value, except possibly at most one, infinitely many times in any neighbourhood of $z=0$ ⁽¹⁷⁾.

Proof. It will be sufficient if we prove the inconsistency of the assumption that $f(z)$ does not take 0 and 1 in $0<|z|\leq\delta$, $\delta>0$ ⁽¹⁸⁾.

The domain $0<|z|\leq\delta$ is transformed by $w=f(z)$ on a doubly connected Riemann surface R_w .

By the theorem of conformal representation, the interior of R_w can be transformed by a regular function

$$\zeta = F(w) \quad (w = \Phi(\zeta))$$

⁽¹⁴⁾ See L. Ahlfors, loc. cit., (3).

⁽¹⁵⁾ This result is the same with M. Lavrentieff's. H. Grötzsch proved it for the case $q(z)<K$, and in §2 and §3 we use this theorem for this case.

⁽¹⁶⁾ These two integrals diverge or converge simultaneously.

⁽¹⁷⁾ This is a generalization of Picard's 2nd theorem for a function which is homomorphic in $0<|z|<1$. In just the same manner pseudo-meromorphic functions can be defined and it will be obvious that Picard's theorem can be extended even for such functions under analogous conditions.

⁽¹⁸⁾ Just as we do in the case of regular functions, we deal with $g(z) = \frac{f(z)-a}{b-a}$ instead of $f(z)$, if $f(z)$ takes a and b , $a\neq b$, only finite times.

If $f(z)$ takes a and b only finite times, then for a suitably chosen $\delta>0$, $g(z)$ does not take 0 and 1 in $0<|z|\leq\delta$.

on a domain

$$\bar{r}^* < |\zeta| < \bar{r}, \quad 0 \leq \bar{r}^*, \quad \bar{r} \leq \infty.$$

(Here we may assume that $|\zeta| = \bar{r}$ corresponds to $|z| = \delta$).

Then we have $0 < \bar{r}^*, \quad \bar{r} < \infty$.

For, if $\bar{r}^* = 0$ or $\bar{r} = \infty$, then $\zeta = 0$ or $\zeta = \infty$ must be an isolated essential singularity, a pole or a regular point of $\Phi(\zeta)$.

Since R_w has, however, no point on $w = 0, 1$ and ∞ , $\zeta = 0$ and $\zeta = \infty$ cannot be an isolated essential singularity (by the ordinary Picard's theorem for $\Phi(\zeta)$). On the other hand, if $\zeta = 0$ is a regular point or a pole of $\Phi(\zeta)$, then the corresponding boundary of R_w must be a single point, which is incompatible with the fact that there are at least two sequences $\{z_n\}$ and $\{z_n'\}$, each tending to $z = 0$, such that $f(z_n) \rightarrow \alpha$ and $f(z_n') \rightarrow \beta$, $\alpha \neq \beta$; and in case of $\zeta = \infty$, we find also the assumption incompatible with the fact that $f(z)$ is pseudo-regular even on $|z| = \delta$.

Thus $0 < |z| < \delta$ is transformed pseudo-conformally on

$$0 < \bar{r}^* < |\zeta| < \bar{r} < \infty$$

by

$$\zeta = F(f(z)) \equiv g(z),$$

where $g(z)$ and $f(z)$ have the same $p(z)$ and $q(z)$ by Theorem 1.

If we put

$$s \equiv x + iy = \log z, \quad \sigma \equiv \xi + i\eta = \log \zeta,$$

then a domain

$$T: \log \bar{r}^* < \xi < \log \bar{r}, \quad |\eta| < \pi$$

on σ -plane is transformed pseudo-conformally on a domain Ω on s -plane. (Taking the image C of $\eta = \pm\pi$ in $0 < |z| < \delta$, we cut this domain along C , and transform this by $s = \log z$).

Put $G(s) \equiv \log g(e^s)$.

As $q_G(s)$ is equal to $q_g(e^s)$, the maximum $q(x)$ of $q_G(s)$ on $\theta_x(R(s) = x = \log r)$ is equal to the maximum of $q_g(z)$ on $|z| = r$, that is, equal to $q(r) \equiv q_j(r)$.

Hence we have from Theorem 4 ($\theta(x) \equiv 2\pi$, $a = 2\pi$)

$$\int_{\log r_1}^{\log r_2} \frac{dx}{q(x)} = \int_{r_1}^{r_2} \frac{dr}{rq(r)} \leq \xi_2(r_2) - \xi_1(r_1) \leq \log \bar{r} - \log \bar{r}^*$$

for any

$$0 < r_1 < r_2 < \delta.$$

If we let $r_1 \rightarrow 0$, the integral on the left hand side becomes greater than any pre-assigned number, which is inconsistent because the right hand side is independent of r_1 .

Thus we have arrived at contradiction.

Corollary :

$$\text{and} \quad \begin{aligned} u_x &= \frac{-\alpha x}{\sqrt{x^2+y^2}} + 1, & u_y &= \frac{-\alpha y}{\sqrt{x^2+y^2}}, \\ v_x &= 0, & v_y &= 1. \end{aligned}$$

From this we have

$$\begin{aligned} p(z) &\equiv \frac{E+G}{J} = \frac{(2+\alpha^2)\sqrt{x^2+y^2}-2\alpha x}{\sqrt{x^2+y^2}-\alpha x} \\ &= \frac{(2-\alpha^2)x^2+(2+\alpha^2)y^2+\alpha^3x\sqrt{x^2+y^2}}{(1-\alpha^2)x^2+y^2} \leq \frac{2x^2+3y^2+|x|\sqrt{x^2+y^2}}{(1-\rho^2)(x^2+y^2)} \end{aligned}$$

As $|x| \leq \sqrt{x^2+y^2}$ we have

$$p(z) \leq \frac{4}{1-\rho^2}.$$

Although u_x and v_x each have a discontinuity at $z=0$, we may, by the remark given at the beginning of § 2, consider this function in a class of pseudo-regular functions.⁽¹⁹⁾

Theorem 6.

Let Σ be a simply connected infinitely many sheeted open Riemann surface on z -plane. Then the type of Σ (hyperbolic or parabolic) is not altered if we deform Σ topologically by a pseudo-regular function with bounded $q(z)$.

Here the deformation may be effected to only a small part of Σ .

That is, the type of Σ is not altered if we take a finite or an infinite number of mutually exclusive domains D_i on Σ , and deform Σ in each D_i by a pseudo-regular function $f_i(z)$, which makes the boundary of D_i unaltered and whose $q(z)$ is bounded uniformly in i . (D_i may have common projections on z -plane as it is sufficient that they are mutually exclusive considered on E .)

Proof. Evident from Theorem 5, Cor. 1.

According to this theorem, if we have any simply connected infinitely many sheeted open Riemann surface whose type is already known, we can make an infinite number of Riemann surfaces of the same type by deforming Σ by a pseudo-regular function with bounded $q(z)$.

For example, taking the Riemann surface Σ of a modular function, which has an infinite number of logarithmic singularities on 0, 1 and ∞ , we have a class of hyperbolic Riemann surfaces by deforming Σ in each neighbourhood of its logarithmic singularity, translating each singularity in an arbitrary direction by a suitably chosen length.

⁽¹⁹⁾ We can also give a function which satisfies the conditions of lemma and whose four first partial derivatives are continuous. But such an example will be complicated.

This problem was first considered by A. Speiser⁽²⁰⁾, and L. Ahlfors⁽²¹⁾ solved this problem for the case that there exist three mutually exclusive domains E_1 , E_2 and E_3 each containing 0, 1 and ∞ respectively, and that the deformation is effected in each E_i ($i=1, 2, 3$).

Although L. Ahlfors' theorem says much more than this theorem, but his method cannot be applied if there is no such domain as E and if, for example, the translation of singularities is effected in such a manner that two singularities whose projections were at first different (for example 0 and 1 respectively) have, after deformation, the same projections on z -plane, that is, if, regarding only to the projections of singularities, the singularities corresponding to 0, 1 or ∞ are confused by the translation.

We shall now deal with Riemann surfaces which have algebraic singularities. L. Ahlfors⁽²²⁾ proved the theorem:

Let Σ be parabolic and let there be five mutually exclusive domains D_i bounded by C_i ($i=1, 2, \dots, 5$) respectively, then at least one of the section ("Insel") of Σ by D_i is one-sheeted (schlicht).

In other words:

Considering Σ on Riemann's sphere and taking five mutually exclusive domains D_i bounded by C_i ($i=1, 2, \dots, 5$), if each section ("Insel") $D_{i,p}$ ⁽²³⁾ ($i=1, 2, \dots, 5$; $p=1, 2, \dots$) of Σ is at least two sheeted, then Σ is hyperbolic.

If, except for a finite number of $D_{i,p}$, there is at least one point $z_{i,p}$ in each $D_{i,p}$, where all the sheets of $D_{i,p}$ have singularities (that is the case if all $D_{i,p}$ be two sheeted), and if the projection of $z_{i,p}$ is in D'_i , which is entirely inside of D_i , then we can prove this theorem by our method.

We have only to take one point z_i in each D'_i and to deform each $D_{i,p}$ by a pseudo-regular function considered in the lemma such that it makes the boundary of $D_{i,p}$ unaltered, and translate $z_{i,p}$ into $z_{i,p}^*$ whose projection is z_i .

By Theorem 6 the Riemann surface Σ' thus obtained is of the same type with Σ , and Σ' is hyperbolic by the well-known theorem of Carathéodory-Bloch.

Added in proof (5 April, 1937)

While this paper was in the press, Nevanlinna's book "Eindeutige analytische Funktion" has appeared. In Chap. XII and XIII of this book, the type-problem of Riemann surfaces is treated and especially in Chap,

(20) A. Speiser, loc. cit. (6) (a).

(21) L. Ahlfors, loc. cit. (6) (b).

(22) L. Ahlfors: loc. cit. (6) (b). Here he proved this for six domains. The case of five domains is proved in (6), (d), see also (6), (c).

(23) $D_{i,p}$ corresponds to D_i .

XIII § 8 pseudo-conformal representations are discussed under the name of “Abbildungen von beschränkter Exzentrizität.” Here it is proved that most of the theorems on the conformal representation concerning the type problem are also valid for “Abbildungen von beschränkter Exzentrizität.” It is easy to see that

$$q(z) = \frac{1}{\sqrt{1-e^2(z)}}, \quad p(z) = \sqrt{1-e^2(z)} + \frac{1}{\sqrt{1-e^2(z)}}.$$

This concept of “Abbildungen von beschränkter Exzentrizität” comes from L. Ahlfors⁽²⁴⁾.

§ 3.

In this chapter we deal with the class of Riemann surfaces Σ whose singularities are all logarithmic and lie only on 1 , $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$.

Such a class of Riemann surfaces was considered by R. Nevanlinna⁽²⁵⁾, A. Speiser⁽²⁶⁾, E. Ullrich⁽²⁷⁾ and Z. Kobayashi⁽²⁸⁾, and many conditions that Σ is parabolic are obtained. Here we shall give a sufficient condition for hyperbolicity.

These surfaces are characterized by its “topological tree” introduced by A. Speiser⁽²⁹⁾, and almost all the conditions for parabolicity are given by the order of increment of “tree-function” $\mu(\theta)$.

If the increment of $\mu(\theta)$ is sufficiently small then Σ is parabolic, and the greater the increment of $\mu(\theta)$ becomes, the more the surface Σ becomes likely to be hyperbolic. But we can show by an example that it is impossible to determine the type of surfaces of this class only by the order of increment of $\mu(\theta)$.

This goes as follows:

We consider a Riemann surface of this class which is symmetric and whose “tree-function” is of the following form:

$$\begin{aligned} \mu(0) &= 1, \\ \mu(\theta) &= 3 \cdot 2^k, \quad \pi n_k < \theta \leq \pi n_{k+1} \end{aligned}$$

where $n_k (k=0, 1, 2, \dots)$ are integers and

$$0 = n_0 < n_1 < n_2 < \dots < n_k < \dots$$

The topological tree of Σ is obtained if we take the topological tree of

⁽²⁴⁾ L. Ahlfors, loc. cit., ⁽⁶⁾ (e). Here the notation “quasi-konforme Abbildungen” is used. It is to be remarked, that when I obtained the result of the present paper, the paper of Ahlfors ⁽⁶⁾ (e) had not yet appeared.

⁽²⁵⁾ R. Nevanlinna, loc. cit., ⁽⁶⁾, (b).

⁽²⁶⁾ A. Speiser, loc. cit., ⁽⁷⁾, (b).

⁽²⁷⁾ E. Ullrich: Über ein Problem von Herrn Speiser, Comment. Math. Helv., **7** (1934).

⁽²⁸⁾ Z. Kobayashi, loc. cit., ⁽⁷⁾, (c) and ⁽⁸⁾, (a).

⁽²⁹⁾ A. Speiser: Über Riemannsche Flächen, Comment. Math. Helv., **2** (1930).

Riemann surface of a modular function which is composed of segments of length π , and enlarge the segments of the k -th generation⁽³⁰⁾ of it to the length $\pi(n_k - n_{k-1})$.

Theorem 7. *If*

$$\frac{n_k}{n_{k+1} - n_k} \leq K \quad \text{and} \quad \frac{n_{k+1} - n_k}{n_k - n_{k-1}} < K, \quad k \geq 1 \quad (8)$$

for fixed K , then Σ is hyperbolic.

Remark.

If we put

$$n_k = 3 \cdot 2^{k-1}, \quad k \geq 1,$$

then (8) is satisfied for $K=2$, and the tree-function $\mu_1(\theta)$ of the Riemann surface Σ_1 corresponding to this $\{n_k\}$ is

$$\begin{aligned} \mu_1(\theta) &= 2n_k & \pi n_k < \theta \leq \pi n_{k+1}, \quad k \geq 1, \\ &= 3 & 0 < \theta \leq \pi n_1 = 3\pi, \\ \mu_1(0) &= 1. \end{aligned}$$

On the other hand, for the Riemann surface Σ_2 of the inverse function of $w = e^{z^2}$, we have

$$\begin{aligned} \mu_2(\theta) &= 2n + 3, & n\pi < \theta \leq (n+1)\pi, & \quad n = 0, 1, 2, \dots, \\ \mu_2(0) &= 1, \end{aligned}$$

if we take suitably the "origin" of topological tree⁽³¹⁾ of Σ_2 .

Then

$$\mu_2(\theta) \geq \mu_1(\theta)$$

for all $\theta \geq 0$. And Σ_2 is parabolic, while Σ_1 is hyperbolic.

Proof of Theorem 7.

We deform Σ into an easily manageable form as Z. Kobayashi⁽³²⁾ did.

Here the deformation may not always be partially conformal as in Z. Kobayashi's method, but it can also be pseudo-conformal under the condition that $q(z)$ is bounded.

Considering Σ on w -plane, we first cut Σ by a line running from origin to ∞ :

⁽³⁰⁾ We take a knot-point P of topological tree T as origin and call the segments which meet at P "segments of 1st generation". After the segments of $(n-1)$ -th generation have been determined, by "segments of n -th generation" we mean the segments which have common ends with those of $(n-1)$ -th generation and which are not yet defined. See: A. Speiser, loc. cit., (²⁹).

⁽³¹⁾ We take as origin the knot point representing the sheet of Riemann surface which corresponds to the fundamental domain:

$$|I(z)| < \pi, \quad |I(e^z)| < \pi$$

of z -plane.

⁽³²⁾ Z. Kobayashi, loc. cit. (⁸), (a).

$$\operatorname{Arg} w = \frac{2}{3} p\pi, \quad p=0, 1, \text{ or } 2,$$

which goes over the logarithmic singularity of $\sum$, into an enumerable number of schlicht subdomains, each one of which is congruent to one of the following two domains

$$D_1: \quad -\frac{\pi}{3} < \operatorname{Arg} w < \frac{\pi}{3}, \quad 0 < |w| < \infty;$$

$$D_2: \quad -\frac{2}{3}\pi < \operatorname{Arg} w < \frac{2}{3}\pi, \quad 0 < |w| < \infty.$$

Cutting D_1 and D_2 by a bisecting line $\operatorname{Arg} w=0$ into two congruent parts, we transform each one of them thus obtained by a pseudo-regular function $W=W(w)$:

$$|W|=|w|, \quad \operatorname{Arg} W = \pm \frac{3}{2} \arg w$$

or
$$|W|=|w|, \quad \operatorname{Arg} W = \pm \frac{3}{4} \arg w,$$

(+ is taken for the upper part over $\operatorname{Arg} w=0$, and $-$ is taken for the lower part under $\operatorname{Arg} w=0$) into a domain

$$D: \quad 0 < \operatorname{Arg} W < \frac{\pi}{2}, \quad 0 < |W| < \infty.$$

Here we notice that by this transformation $W=i$ corresponds to the logarithmic singularity of $\sum$, and that for this transformation $q(z)$ is bounded uniformly ($|q| \leq \frac{3}{2}$).

Again we transform D by

$$\zeta = \log \left(\frac{i-W}{i+W} \right)$$

into a domain

$$S: \quad -\infty < \xi < 0, \quad 0 < \eta < \pi$$

of $\zeta(=\xi+i\eta)$ -plane in such a manner that

$$\operatorname{Arg} W=0 \quad \text{corresponds to} \quad C_1: \xi=0, \quad 0 < \eta < \pi,$$

$$\operatorname{Arg} W = \frac{\pi}{2}, \quad 0 < |W| < 1 \quad \text{,,} \quad \text{,,} \quad C_2: \eta=0, \quad -\infty < \xi < 0,$$

$$\operatorname{Arg} W = \frac{\pi}{2}, \quad 1 < |W| < \infty \quad \text{,,} \quad \text{,,} \quad C_3: \eta=\pi, \quad -\infty < \xi < 0.$$

To $W=i$ corresponds $-\infty$ of ζ -plane.

We take an enumerable number of sheets of S and connect them along

their boundary lines in the same way as D_1 and D_2 are connected in Σ , so that we have in the end a surface Σ' , which is essentially the same with the "cylindrical surface" of Z. Kobayashi, although its form is much simpler than that of Kobayashi's.

Σ' can be constructed in the following way:

Taking a topological tree T of Σ , which is composed of segments of length π , affix to each segments s of T two sheets of S along the

Fig. 2.

boundary line C_1 of S (Fig. 1). Then identify the boundary line C_2 (or C_3) of two consecutive sheets of S , which are facing each other (to face means that they have the same end point on T and that this point is on the same side of T) (Fig. 2.)

Thus we have an open surface Σ' which has no boundary in the finite part.

Then we fold Σ' in such a way that all the segments of T which are at the same distance from the "origin" of T come on the same place and consider it fixed on a $\zeta(=\xi+i\eta)$ -plane such that Σ' lies in a quadrant $\xi>0$, $\eta>0$ (Fig. 3).

As is easily seen, Σ' has $2 \cdot 3 \cdot 2^k \equiv 2 \cdot \mu(\eta)$ sheets for $\pi n_k < \eta < \pi n_{k+1}$ (to each segment of T correspond two sheets of S).

We cut Σ' by a straight line

$$L_k: \quad \xi + \eta = \pi n_k, \quad k=1, 2, \dots,$$

and transform the part $T_k(k=1, 2, \dots)$ of Σ' which is between L_k and L_{k+1}

into a rectangle $\mathcal{Q}_k$ on $S(=\sigma+i\tau)$ plane which is between $\tau=0$ and $\tau=2\pi$ (the length of one side of $\mathcal{Q}_k$ is 2π that of the other is l_k which will be determined in the following).

This transformation is defined as follows:

Fig. 1.

Fig. 3.

T_k ($k \geq 1$) consists of two classes of domains of T'_k and T''_k ($3 \cdot 2^k$ sheets each).

$$\begin{aligned} AB &= CD = \pi(n_{k+1} - n_k) \\ OB &= OC = \pi(n_k - n_j), \quad j < k \end{aligned}$$

Fig. 4.

$$\begin{aligned} EF &= FG \\ &= \pi(n_{k+1} - n_k) \end{aligned}$$

Fig. 5.

T'_k and T''_k are connected in T along the segments AB and EF , and two consecutive T'_k or T''_k are connected along CD or FG respectively.

We transform T'_k and T''_k by pseudo-regular functions into domains $\mathcal{Q}'_k$ and $\mathcal{Q}''_k$ (or into the domains $\overline{\mathcal{Q}}'_k$ and $\overline{\mathcal{Q}}''_k$ which are symmetric to them) in such a way that A, B, C, D of T'_k and E, F, G of T''_k correspond to A', B', C', D' of $\mathcal{Q}'_k$ and E', F', G' of $\mathcal{Q}''_k$ respectively, and that this transformation is linear in each side of T'_k and T''_k .

$$B'C' = \alpha_k$$

$$C'D' = \frac{\alpha_k}{2}$$

$$A'D' = \frac{\alpha_k}{2}$$

Fig. 6.

$$F'G' = E'G' = \frac{\alpha_k}{2}$$

Fig. 7.

Then we connect $\mathcal{Q}'_k, \mathcal{Q}''_k, \overline{\mathcal{Q}}'_k$ and $\overline{\mathcal{Q}}''_k$ in the same way as the corres-

ponding T_k' and T_k'' are connected in T_k , and thus we construct a rectangle Ω_k out of Ω_k' , Ω_k'' , $\overline{\Omega}_k'$ and $\overline{\Omega}_k''$ (Fig. 8).

If we put $\alpha_k = \frac{2\pi}{3 \cdot 2^k}$ then the length of one side of Ω_k is 2π and that of the other is $l_k = \frac{\alpha_k}{2} = \frac{\pi}{3 \cdot 2^k}$.

The transformation from T_k' or T_k'' to Ω_k' , Ω_k'' , $\overline{\Omega}_k'$ or $\overline{\Omega}_k''$ is pseudo-regular and we shall calculate the maximal value of q for each of them. For the sake of simplicity we put

$$a = \frac{\pi}{\sqrt{2}} (n_k - n_l), \quad b = \frac{\pi}{\sqrt{2}} (n_{k+1} - n_k)$$

$$\alpha = \alpha_k$$

and consider T_k' , T_k'' and Ω_k' , Ω_k'' suitably placed in (x, y) - and (u, v) -plane respectively.

I) From T_k' to Ω_k' (Fig. 9):

$$u = \frac{\alpha}{2b} x,$$

$$v = \frac{\alpha}{2} \left(1 + \frac{y}{a+x} \right) \left(1 - \frac{x}{2b} \right).$$

Fig. 9.

In this case

$$p = q + \frac{1}{q}$$

Fig. 8.

$$\begin{aligned}
 &= \frac{\frac{1}{b^2} + \left\{ \frac{y}{(a+x)^2} \left(1 - \frac{x}{2b}\right) + \frac{1}{2b} \left(1 + \frac{y}{a+x}\right) \right\}^2 + \left\{ \frac{1}{a+x} \left(1 - \frac{x}{2b}\right) \right\}^2}{\frac{1}{b(a+x)} \left(1 - \frac{x}{2b}\right)} \\
 &= \left\{ \frac{a+x}{b} + \frac{a+x}{4b} \left(1 + \frac{y}{a+x}\right)^2 \right\} \left(1 - \frac{x}{2b}\right)^{-1} \\
 &\quad + \left\{ \frac{by^2}{(a+x)^3} + \frac{b}{a+x} \right\} \left(1 - \frac{x}{2b}\right) + \frac{y}{a+x} \left(1 + \frac{y}{a+x}\right)
 \end{aligned}$$

and since $0 \leq x \leq b$ and $|y| \leq a+x$ we have

$$\leq 4 \frac{a+b}{b} + 2 \frac{b}{a} + 2 = 6 + 4 \frac{a}{b} + 2 \frac{b}{a}.$$

2) From T_k'' to Ω_k'' (Fig. 10):

$$u = \frac{\alpha}{2b} x, \quad v = \frac{\alpha(x+y)}{4b}.$$

In this case

$$p = q + \frac{1}{q} = 3.$$

$$EH = FH = GH = b$$

(x, y) - plane

$$E'G' = F'G' = \frac{\alpha}{2}$$

(u, v) - plane

Fig. 10.

We have always from 1) and 2)

$$p = q + \frac{1}{q} \leq 6 + 4 \frac{a}{b} + 2 \frac{b}{a}$$

and, since by assumption

$$\begin{aligned}
 \frac{a}{b} &= \frac{n_k - n_j}{n_{k+1} - n_k} \leq \frac{n_k}{n_{k+1} - n_k} \leq K, \\
 \frac{b}{a} &= \frac{n_{k+1} - n_k}{n_k - n_j} \leq \frac{n_{k+1} - n_k}{n_k - n_{k-1}} \leq K
 \end{aligned}$$

for all k , we have

$$p = q + \frac{1}{q} \leq 6(1 + K).$$

Performing these transformations for each $k(k \geq 1)$, we connect the domains $\mathcal{Q}_k$ thus obtained as is shown in Fig. 11.

If we determine the transformation of T_0 suitably so as to fill the part $\mathcal{Q}_0(0 < \tau < 2\pi, -\infty < \sigma < \sigma_1, \text{ where } \sigma = \sigma_1 \text{ corresponds to } L_1 \text{ in the transformation of } T_1 \text{ on } s\text{-plane. See Fig. 11.)}$, then we finally have the pseudo-conformal transfor-

gFig. 11.

mation of all Σ' into a domain $\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 + \dots$ of $s(=\sigma + i\tau)$ -plane which is between $\tau=0$ and $\tau=2\pi$. (Since the transformations from T_k' or T_k'' to $\mathcal{Q}_k'$ or $\mathcal{Q}_k''$ are all linear on the boundary, the transformation φ from Σ' into $\mathcal{Q}$ is continuous. Of course, we must choose the transformation of T_0 to $\mathcal{Q}_0$ so as to make φ continuous on L_1 .)

$\mathcal{Q}$ is bounded on the right side, because

$$\sum_{k=1}^{\infty} l_k = \sum_{k=1}^{\infty} \frac{\alpha_k}{2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{2\pi}{3 \cdot 2^k} = \frac{\pi}{3}.$$

Thus we have a pseudo-conformal transformation φ of Σ' in a domain $\mathcal{Q}$ bounded on the right. As $p = q + \frac{1}{q}$ is uniformly bounded we have from Theorem 4, Cor. 1. that Σ' and accordingly Σ is hyperbolic.

Added by the proof (23. March, 1937)

During the proof many publications concerning the Type-problem have appeared, under which the papers directly related with mine are as follows:

- (1) C. Blanc, Le type des surfaces de Riemann simplement connexes, C. R. **202** (1936), p. 623.
- (2) P. J. Myrberg, Über die Bestimmung des Typus einer Riemannschen Fläche, Ann. Acad. Scient. Fennicae, Ser. A. XIV, No. 3.

In these papers the results of § 3 of my paper are obtained, but the way to get them are different.

Mathematical Institute,
Osaka Imperial University.