4. On the Borel’s directions of meromorphic functions of infinite order.

By K. P. Lee

(Received March 4th, 1936.)

In a recent paper, K. L. Hiong(1) has discussed the Borel’s directions of meromorphic functions of infinite order. The object of this paper is to generalize his theory to the possibly general form, by considering some results of A. Rauch.(2)

Every function \( \rho(r) \), satisfying the following conditions:

1° \( \rho(r) \) is a non-decreasing function of \( r \), and \( \lim_{r \to \infty} \rho(r) = \infty \),

\[ T(r, f) \leq r^{\rho(r)} \]

for all sufficiently large values of \( r \),

2° \( T(r, f) = r^{\rho(r)} \)

for at least a sequence of values of \( r \) tending toward infinity,

3° Let

\[ U(r) = r^{\rho(r)} \]

then

\[ U\left(r + \frac{r}{\log U(r)}\right) \leq U(r)\left(1 + \varepsilon(r)\right) \]

for all sufficiently large values of \( r \), and for a suitably chosen infinitesimal \( \varepsilon(r) \), is called an infinite order of the meromorphic function \( f(z) \) of infinite order, with its characteristic function \( T(r, f) \).(3)

We can prove the following

Theorem of Milloux. If \( f(z) \) is a meromorphic function of infinite order \( \rho(r) \), and that \( R \) is a positive number such that

\[ T(R, f) = U(R) = R^{\rho(R)}, \quad (R > R_0); \]

\[ T(R, f) > 12T(r, f), \quad 12 \frac{T(k, f)}{\log k} \log \frac{R}{r} > 1, \]

and \( T(r, f) > c(f) \),

then, in the circular ring \( I(r, R) \), the number \( n(I, a) \), of the zeros of the function \( f(z) - a \), is superior to

---


(2) A. Rauch: Extensions de théorèmes relatifs aux directions de Borel des fonctions méromorphes. J. de Math. 1933.

(3) K. L. Hiong: l. c.
for all values of \( a \) in a certain circle of radius \( \frac{1}{2} \) on the Riemannian sphere, where \( c_1(f) \) is a constant which depends only upon \( f(z) \).

From K. L. Hiong,\(^{(4)}\) we know that for all \( a \), such that \( |a| < e^{T(R,f)} \), \( |a - f(0)| > e^{-T(R,f)} \), the inequality
\[
N(R, a) > \frac{1}{4} T(R,f)
\]
holds good, for all values of \( R \) satisfying the following conditions:

\[(1) \begin{cases} T(R,f) = U(R) = R^{\alpha(r)} \\ T(R,f) > C_1(f) \end{cases},
\]
except the points in at most two circles with their radii less than \( \frac{1}{T(R,f)} \),
where \( C_1(f) \) is a positive constant, which depends only upon the function \( f(z) \).

Considering any such value \( a \), since
\[
N(kr, a) \geq n(r, a) \log k, \quad (k > 1, r < R),
\]
\[
N(R, a) \leq N(r, a) + n(R, a) \log \frac{R}{r},
\]
hence
\[
n(f, a) \geq \frac{N(R, a) - N(r, a)}{\log \frac{R}{r}} - \frac{N(kr, a)}{\log k} \geq \frac{1}{4} \frac{T(R,f)}{\log \frac{R}{r}} - \frac{N(r, a)}{\log \frac{R}{r}} - \frac{N(kr, a)}{\log k}.
\]

Moreover, from Nevanlinna, we know that
\[
m(\rho, a) + N(\rho, a) \leq T(\rho, f) + \log | C_1 | + \log | a | + \log 2
\]
where \( C_1 \) is either a constant depending only upon the function \( f(z) \) or \( f(0) - a \). Hence, when \( R \) is sufficiently large, we can choose a circle \( S \), of radius \( \frac{1}{2} \) on the Riemannian sphere such that for all values \( a \) in \( S \) we have
\[
N(R, a) > \frac{1}{4} T(R,f)
\]
and that
\[
m(r, a) + N(r, a) < T(r,f) + c'(f),
\]
where \( c'(f) \) is a positive constant depending only upon \( f(z) \).

\(^{(4)}\) K. L. Hiong: l. c.
If

\[ T(R,f) > 12 T(r,f); \]
\[ T(R,f) > \frac{12 T(kr,f)}{\log k} \log \frac{R}{r}, \]

then

\[ T(R,f) > 6 \left\{ T(r,f) + \frac{T(kr,f) \log R}{\log k} \right\}, \]

hence

\[ \frac{11}{60} \frac{T(R,f)}{\log \frac{R}{r}} > \frac{11}{10} \frac{T(r,f) + \frac{T(kr,f) \log R}{\log k}}{\log \frac{R}{r}}, \]

we get, therefore, for all values \( a \) in the circle \( S \),

\[ n(f, a) > \frac{1}{4} \frac{T(R,f)}{\log \frac{R}{r}} - \frac{T(r,f)}{\log \frac{R}{r}} - \frac{T(kr,f)}{\log k} - \frac{c'(f)}{\log k} - \frac{c'(f)}{\log k}, \]
\[ > \frac{T(R,f)}{15 \log \frac{R}{r}} + \frac{T(R,f)}{60 \log \frac{R}{r}} - \frac{c'(f)}{\log k} - \frac{c'(f)}{\log k}, \]
\[ > \frac{T(R,f)}{15 \log \frac{R}{r}}, \]

supposing that \( T(R,f) > 60 \left( c'(f) + \frac{c'(f)}{\log k} \frac{R}{r} \right). \)

Hence the theorem is proved, if we choose \( T(r,f) > 10 c'(f), C_t(f) \).

If we choose \( \frac{R}{r} > k \), then in the foregoing theorem, it is sufficient to assume

\[ T(r,f) > c_t(f), \quad T(R,f) > 12 \frac{T(kr,f) \log R}{\log k}. \]

Following the same method of A. Rauch\(^{(5)}\), we can prove the following: Theorem I.

Suppose that \( f(z) \) is a meromorphic function of infinite order \( \rho(r) \), and that \( R \) is a positive number such that

\[ T(R,f) = R^{\rho(R)}. \]

and that the following conditions are satisfied:

\(^{(5)}\) A. Rauch, i. c., p. 134.
Then there exists, in the circular ring \( I(r, R) \), a point with suffix \( x(r < |x| < R) \), such that in the circle \( \Omega \) with equation
\[
|z - x| = \alpha |x|
\]
the number \( n(\Omega, \Pi) \) of the zeros of the function \( f(z) - \Pi(z) \) in \( \Omega \), for at least a function \( \Pi(z) \) among the three functions \( P(z), Q(z) \) and \( R(z) \) is greater than
\[
n_1 = \text{const} \alpha^4 \frac{T(R, f)}{(\log R)^2},
\]

The foregoing theorem is satisfied, when \( \frac{1}{\alpha} \) and \( \alpha^4 \frac{T(R, f)}{(\log R)^2} \) are sufficiently large.

II. Let us now consider the condition (M') in the foregoing theorem I. Suppose that \( k = 1 + \beta, \ r = \frac{R}{(1 + \beta)^2} \) then the condition (M') becomes to be
\[
T(R; f) > 24 T\left(\frac{R}{1 + \beta}, f\right)
\]
for sufficiently large value of \( R \).

Since \( \rho(R) \geq \rho\left(\frac{R}{1 + k}\right) \), hence, if we choose \( \beta \) such that
\[
(1 + \beta)^k, (1 + \beta)^{R_1} > 24,
\]
then
\[
T(R, f) = R^{\rho(R)} > \frac{24}{(1 + \beta)^k} R^{\rho\left(\frac{R}{1 + \beta}\right)} = 24 T\left(\frac{R}{1 + \beta}; f\right),
\]
and the condition (M') is then satisfied.
On the Borel's directions of meromorphic functions of infinite order.

From § 1, we know that there exists a sequence (Σ) of positive numbers

\[ r_1, r_2, \ldots, r_n, \ldots \]

tending toward infinity, and

\[ T(r_n; f) = r_n^{\beta(n)} = U(r_n), \quad (n = 1, 2, \ldots). \]

Choosing an infinitesimal \( \beta(n) \) such that

\[ \beta(n) \] \[ \beta(n) \]

then for every sufficiently large integer \( n \), the condition \( (M') \) is satisfied. By theorem I, we can easily prove the following

Theorem II. Given a meromorphic function of infinite order \( \rho(r) \), and a family of meromorphic functions satisfying the following conditions:

\[ \mathcal{G}(\alpha(n); f) \]

there exists a sequence of circles, each of which \( \Gamma(n) \) is of equation

\[ |z - x(n)| = \alpha(n)|x(n)| \]

with its centre \( x(n) \) in the circular ring \( \mathcal{C} \left( r_n, \frac{r_n}{1 + \beta(n)} \right) \); where \( \beta(n) \) is an infinitesimal satisfying the condition (2), such that the number \( n \left( \Gamma(n); f - \Pi \right) \) of zeros of the functions \( f(z) - \Pi(z) \) in the circle \( \Gamma(n) \) is superior to

\[ n_1 = \text{const.} \alpha^2(n)T(r_n; f), \]

for every function \( \Pi(z) \) in the family \( \mathcal{G}(\alpha(n), f) \) satisfying the following inequalities

\[ C(\Pi - \Pi_{\alpha,n}) > -\alpha^2(n)T(r_n; f), \]
\[ C(\Pi = \Pi_{\alpha,n}) > -\alpha^2(n)T(r_n; f), \]

where \( \Pi_{\alpha,n}(z), \Pi_{\alpha,n}(z) \) are the two possibly exceptional functions in the family \( \mathcal{G}(\alpha(n); f) \) verifying :

\[ C(\Pi_{\alpha,n}) > -\alpha^2(n)T(r_n; f), \]
\[ C(\Pi_{\alpha,n}) > -\alpha^2(n)T(r_n; f), \]
The theorem is true when \( \frac{1}{\alpha(n)} \) and \( \alpha(n)T(r_n;f) \) are sufficiently large.

III. If in theorem II, we substitute \( \frac{1}{\log T(r_n;f)} \) for \( \alpha(n) \), then for every non exceptional function \( I_2(z) \),

\[
\left\lfloor I(n); f - II \right\rfloor > \text{const.} \frac{T(r_n;f)}{\left[ \log T(r_n;f) \right]^2} \quad (a)
\]

for all sufficiently large integers \( n \).

Denoting the absolute values of the non-vanishing zeros of the function \( f(z) - II(z) \) in \( I(n) \) by

\[
r_q \left[ I(n); f - II \right], \quad (q = 1, 2, \ldots, m),
\]

then

\[
r_q \left[ I(n); f - II \right] < r_n \left[ 1 + \frac{1}{\log T(r_n;f)} \right].
\]

But the variable

\[
\frac{T(r_n;f)^{\delta - g(n)}}{\left[ \log T(r_n;f) \right]^2}
\]

tends toward infinity with \( n \), so that the series

\[
\sum_{n=1}^{\infty} \text{const.} \alpha(n) \frac{T(r_n;f)}{U(r_n^{1-\delta})(1+\alpha(r_n))}
\]

is divergent, for an arbitrarily small positive number \( \delta \).

We shall prove that the series

\[
(3) \sum_{n=1}^{m(z)} \sum_{q=1}^{m(q)} \frac{1}{r_q \left[ I(n); f = II \right]^{\delta + \gamma(n)}} \{ r_q \{ I(n); f = II \} \}^{1 + \beta(n)}
\]

is divergent, for all functions in the family \( S' (\alpha(n);f) \), except at most two.

Suppose that, there are two functions \( P(z) \) and \( Q(z) \) in the family \( S' (\alpha(n);f) \), such that \( P(z) \equiv Q(z) \), \( P(z) \equiv 0 \), \( Q(z) \equiv 0 \), for which the two series

\[
(4) \frac{T(r_n;f)}{\left[ \log T(r_n;f) \right]^2}
\]

is arbitrarily large.
are convergent. We conclude easily that the two corresponding series

\[(4') \sum_{n=1}^{\infty} \frac{n[\Gamma(n); f = P]}{U(r_n)^{1-\delta(1+\alpha r_n)}} , \]

\[(5') \sum_{n=1}^{\infty} \frac{n[\Gamma(n); f = Q]}{U(r_n)^{1-\delta(1+\alpha r_n)}} , \]

are convergent, where \( \epsilon(r) \) is the same as before.

The sum of terms in the series (2) for which

\[
\frac{n[\Gamma(n); f = P]}{U(r_n)^{\alpha-\delta(1+\alpha r_n)}} \gg \text{const.} \frac{\alpha^2(n)T(r_n; f)}{U(r_n)^{\alpha-\delta(1+\alpha r_n)}}
\]

is convergent, hence if these terms are neglected, we get another divergent series

\[(6) \sum' \text{const} \frac{\alpha^2(n)T(r_n; f)}{U(r_n)^{1-\delta(1+\alpha r_n)}} , \]

where

\[n[\Gamma(n); f = Q] < \text{const} \alpha^2(n)T(r_n, f) . \]

Moreover, the sum of terms in series (6), for which

\[
\frac{n[\Gamma(n); f = Q]}{U(r_n)^{\alpha-\delta(1+\alpha r_n)}} \gg \text{const.} \frac{T(r_n; f)}{U(r_n)^{\alpha-\delta(1+\alpha r_n)}}
\]

is convergent, hence if these terms are neglected, we get another divergent series

\[(7) \sum'' \text{const} \frac{\alpha^2(n)T(r_n; f)}{U(r_n)^{1-\delta(1+\alpha r_n)}} , \]

where,

\[n[\Gamma(n); f = P] < \text{const} \alpha^2(n)T(r_n; f) , \]
Since \( P(z) \equiv Q(z) \), \( P(z) \equiv 0 \), \( Q(z) \equiv 0 \), hence for sufficiently large values of \( n \), we have
\[
C(P - Q) > -a^4(n)T(r_n; f),
\]
\[
C(P) > -a^4(n)T(r_n; f),
\]
\[
C(Q) > -a^4(n)T(r_n; f).
\]

Therefore, these two functions \( P(z), Q(z) \) serve us as the exceptional functions of the family \( \mathcal{E}(a(n); f) \) with respect to the circles \( I(n) \) corresponding to the series (7), we denote such circles by \( I(n') \).

Choose any function \( R(z) \), such that \( R(z) \equiv Q(z), R(z) \equiv P(z), R(z) \equiv 0 \), and
\[
T\left[(1 + a(n))r_n; R\right] < a(n)^4T(r_n; f), \quad (n > n_0)
\]
which is a function in the family \( \mathcal{E}(a(n); f) \), satisfying the conditions:
\[
C(R - P) > -a^4(n)T(r_n; f),
\]
\[
C(R - Q) > -a^4(n)T(r_n; f), \quad n > n_0.
\]

Applying the foregoing theorem, we get
\[
\frac{n\left[I(n'); f = R\right]}{U(r_{n''})(1-\beta)(1+\alpha r_{n''})} > \text{const} \frac{a^2(n')T(r_{n''}; f)}{U(r_{n''})(1-\beta)(1+\alpha r_{n''})}.
\]

But the series
\[
\sum_{n''} \text{const} \frac{a^2(n')T(r_{n''}; f)}{U(r_{n''})(1-\beta)(1+\alpha r_{n''})} = \sum_{n''} \text{const} \frac{a^2(n)T(r_n; f)}{U(r_n)(1-\beta)(1+\alpha r_n)}
\]
is divergent, hence the series
\[
\sum_{n''} \frac{n\left[I(n''); f = R\right]}{U(r_{n''})(1-\beta)(1+\alpha r_{n''})}
\]
is divergent, so also is the series...
Therefore we conclude that the series (3) is divergent for every function \( \Pi(z) \), satisfying the same conditions as the foregoing function \( R(z) \).

Finally, we have the following

Theorem IV. Given a meromorphic function \( f(z) \) of infinite order \( \rho(r) \), and a family \( \mathcal{G}_1(a(n); f) \) of meromorphic functions each of which \( \Pi(z) \) satisfies the following conditions

\[
\mathcal{G}_1(a(n); f) = \begin{cases} 
T \left[ (1 + a(n)) r_n; \Pi \right] < a'(n)T(r_n; f), & (r_n > r_0) \\
II(z) \neq 0, &  \\
T(r_n; f) = r_n^{\rho(r_n)}, & \lim_{n \to \infty} r_n = \infty, \\
a(n) = \frac{1}{\log T(r_n; f)}, & 
\end{cases}
\]

there exists a sequence of circles \( P(n) \) with equations

\[
|z - x(n)| = a(n) |x(n)|, \quad \lim_{n \to \infty} |x(n)| = \infty,
\]

such that the series

\[
\sum_{n=1}^{\infty} \sum_{q=1}^{m(n)} \frac{1}{r_n^{\left[ \Gamma(n); f = II \right]^q} \cdot \left[ r_n^{\left[ \Gamma(n); f = II \right]} \right]^{(1 - \delta)}}
\]

is divergent, for all functions in the family \( \mathcal{G}_1(a(n); f) \), with two possibly exceptional functions.

Following the method of Collier, which was also used by A. Rauch\(^{(7)}\), and applying the former theorem, we get the following

Theorem V. Given a meromorphic function \( f(z) \) of infinite order \( \rho(r) \), and a family \( \mathcal{G}_1(a(n); f) \) of meromorphic functions, each of which \( \Pi(z) \) satisfies the following conditions:

\[
\mathcal{G}_1(a(n); f) = \begin{cases} 
T \left[ (1 + a(n)) r_n; II \right] > a'(n)T(r_n; f), & (r_n > r_0) \\
II(z) \neq 0, &  \\
T(r_n; f) = r_n^{\rho(r_n)}, & \lim_{n \to \infty} r_n = \infty, \\
a(n) = \frac{1}{\log T(r_n; f)}, & 
\end{cases}
\]

We can find out at least a semi straight line \( (D) \), such that for every arbitrarily small angle \( \hat{A} \) with its bisector the line \( (D) \), and its vertex the origin, the series

\((7)\) A. Rauch. l. c. p. 149.
On the Borel's directions of meromorphic functions of infinite order.

\[ \sum_{n=1}^{\infty} \frac{1}{r_n(\hat{A}; f = II)^2(r_n(\hat{A}; f = II) - 1)^{1-\delta}} \]

is divergent for every function \( II(z) \) in the family, with two possibly exceptional functions, where \( r_n(\hat{A}; f = II) \) represent the absolute values of the non-vanishing zeros of the function \( f(z) - II(z) \) in the angle \( \hat{A} \).

We define that the line \((D)\) in the foregoing theorem is a Borel's direction with respect to the family \( \mathcal{F}_1(a(n); f) \), of the meromorphic functions infinite order \( \rho(r) \).

Finally the author is indebted to Prof. Valiron for his kind directions in a letter.

Tokyo, Dec. 20, 1935,
Mathematical Institute,
Tokyo Imperial University,
Tokyo.