20. On analytic functions regular in the half-plane(I).⁽¹⁾

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1. Introduction. Let $f(z)=f(x+iy)$ be regular analytic in the upper half-plane $y>0$. If, for every positive y , there exists a constant C depending only on p such that

$$(1.1) \quad \int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq C^p,$$

where $p>0$, then we say that $f(z)$ belongs to the class $\mathfrak{S}_p$ and we write as $f(z) \in \mathfrak{S}_p$.

Recently Professors Hille and Tamarkin⁽²⁾ proved the following theorems.

A. (i) A function $f(z) \in \mathfrak{S}_p (p \geq 1)$ for almost all x has a limit function $f(x) \in L_p$ to which it tends along any non-tangential path,

(ii) Any $f(z) \in \mathfrak{S}_p (p \geq 1)$ tends to its limit function $f(x)$ in the mean of order p , that is

$$\lim_{y \rightarrow 0} \int_{-\infty}^{\infty} |f(x+iy) - f(x)|^p dx = 0,$$

(iii) As $y \downarrow 0$,

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \uparrow \int_{-\infty}^{\infty} |f(x)|^p dx.$$

B. A function $f(z) \in \mathfrak{S}_p (p \geq 1)$ can be represented as a product

$$f(z) = b_f(z) h(z),$$

where

$$b_f(z) = \prod_{(v)} \frac{z - z_v}{z - \bar{z}_v} \frac{\bar{z}_v - i}{z_v - i}$$

is the Blaschke function associated with $f(z)$ and $h(z) \in \mathfrak{S}_p$, but does not vanish in the half-plane $y>0$. Hereby $\{z_v\}$ is the sequence of zeros of $f(z)$ in the half-plane $y>0$. And

$$|b_f(z)| \leq 1, \quad \text{for } y>0,$$

$$|b_f(x)| = 1, \quad \text{almost everywhere.}$$

Corresponding theorems concerning functions regular in the unit circle

⁽¹⁾ This is a more detailed report of the paper presented to the Imperial Academy of Japan: On analytic functions regular in the half-plane, Proc. Imperial Acad. of Japan, 12 (1936).

⁽²⁾ Hille and Tamarkin, On the absolute integrability of Fourier transforms, Fund. Math., 25 (1935).

are well known. The theorems of the same type in a strip or in the half-plane were previously obtained by Hardy, Ingham and Pólya⁽³⁾ and Gabriel⁽⁴⁾. Hille and Tamarkin's idea is to prove a lemma due to Gabriel in our case and by applying it, to transform the theorems into those in the case of the unit circle, some other ideas being needed. Their method breaks down in the case of $p < 1$. But by applying a theorem due to Hardy, Ingham and Pólya, we can prove the theorems in the case $p < 1$, using Hille and Tamarkin's theorems *A* and *B*. This is one of the objects of our present paper.

If $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ be regular in the unit circle and

$$\left\{ \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq C(1-r)^{-a},$$

or

$$\left\{ \int_{-\pi}^{\pi} |u(r, \theta)|^p d\theta \right\}^{\frac{1}{p}} \leq C(1-r)^{-a},$$

where $a \geq 0$, $p > 0$, then

$$\left\{ \int_{-\pi}^{\pi} |f(re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq KC(1-r)^{-a - \frac{1}{p} + \frac{1}{q}}, \quad (q > p),$$

$$\max_{-\pi \leq \theta < \pi} |f(re^{i\theta})| \leq KC(1-r)^{-a - \frac{1}{p}},$$

K being independent of r .

These theorems are due to Professors Hardy and Littlewood⁽⁵⁾. We will prove the analogous theorems for the functions regular in the half-plane.

2. We begin with stating a definition and proving some lemmas.

Definition. If $f(z) = f(x + iy)$ is regular in the half-plane $y > 0$ and for any positive number y_0 but fixed,

$$\int_{-\infty}^{\infty} |f(x + iy)|^p dx \leq C^p, \quad (p > 0),$$

(³) Hardy, Ingham and Pólya, Theorems concerning mean values of analytic functions, Proc. Royal Soc. London, (A) **113** (1927).

(⁴) Gabriel, An improved result concerning the zeros of a function regular in a half-plane, Journal London Math. Soc., **4** (1929).

Gabriel, An inequality concerning the integrals of positive subharmonic functions along certain curves, *ibid.* **5** (1927).

(⁵) G. H. Hardy and J. E. Littlewood, Some new properties of Fourier constants, Math. Ann., **97** (1927).

G. H. Hardy and J. E. Littlewood, A convergence criteria for Fourier series, Math. Zeits., **28** (1928).

G. H. Hardy and J. E. Littlewood, Some properties of conjugate functions, Crelle's Journal, Hensel Festschrift.

provided $y \geq y_0$, where C may depend on y_0 but independent of y , then we say that $f(z)$ belongs to the class $\mathfrak{G}_p$ and we write $f(z) \in \mathfrak{G}_p$. Moreover let $f(z) = f(re^{i\theta})$ be regular in the unit circle and

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \leq C,$$

C being independent of r . Here we say that $f(z)$ belongs to the class H_p .

Lemma 1. If $p > 0$ and for $y \geq y_0 > 0$,

$$(2.1) \quad \int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq C^p,$$

where C depends only on y_0 , then $f(x)$ is bounded in the closed half-plane $y \leq y_0 + \varepsilon$, ε being an arbitrary but fixed positive number.

This lemma is a corollary of the following theorem due to Hardy, Ingham and Pólya⁽⁶⁾.

If $\int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq C$ for $\alpha \leq y \leq \beta$, then

$$|f(z)|^p \leq \frac{AC}{\delta}$$

for $\alpha + \delta \leq y \leq \beta - \delta$, where A depends only on p .

Lemma 2. If $p > 0$ and $f(z) \in \mathfrak{G}_p$ in any closed half-plane $y \geq \varepsilon > 0$, then $f(z)$ tends to zero uniformly as $z \rightarrow \infty$.

Since $f(z) \in \mathfrak{G}_p$, there exists a constant C for any positive number ε , such that for $y \geq \frac{\varepsilon}{4}$,

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq C^p.$$

Thus by Lemma 1, $f(z)$ is bounded in $y \geq \frac{\varepsilon}{2}$. Suppose that $|f(z)| \leq B$ for $y \geq \frac{\varepsilon}{2}$. Then in that domain, for any $q (\infty \geq q \geq 1, q > p)$, we have

$$\int_{-\infty}^{\infty} |f(x+iy)|^q dx \leq B^{q-p} \int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq B^{q-p} C^p.$$

That is, $f(z) \in \mathfrak{G}_q$ ($q \geq 1$), for $y \geq \frac{\varepsilon}{2}$. Therefore by Hille and Tamarkin's lemma⁽⁷⁾, if $y \geq \varepsilon$, then $f(z)$ tends to zero uniformly as $z \rightarrow \infty$.

Lemma 3. Let $p > 0$. A function in z -plane $f(z) \in \mathfrak{G}_p$ is transformed into a function in w -plane $\varphi(w) \in H_p$ by the transformation

$$(2.2) \quad w = \frac{1+iz}{1-iz}, \quad z = i \frac{1-w}{1+w}.$$

The proof of this lemma is quite similar to that of the corresponding

(⁶) Hardy, Ingham and Pólya, loc. cit., Theorem 11.

(⁷) Hille and Tamarkin, loc. cit., Lemma 2.5

lemma in the case $p \geq 1$ and so we do not repeat it here.

Theorem 1. *If $p > 0$, then*

- (i) *A function $f(z) \in \mathfrak{S}_p$ for almost all x has a limit function $f(x)$ to which it tends along any non-tangential path,*
- (ii) *Any $f(z) \in \mathfrak{S}_p$ tends to its limit function $f(x)$ in the mean of order p ,*
- (iii) *As $y \downarrow 0$,*

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \uparrow \int_{-\infty}^{\infty} |f(x)|^p dx.$$

Let the transform by (2.2) of $f(z) \in \mathfrak{S}_p$ be $\varphi(w)$. Then $\varphi(w) \in H_p$ by Lemma 3. Thus by the well known theorem, there exists a limit function $\varphi(e^{i\theta})$ such that for almost all θ , $\varphi(w)$ tends to $\varphi(e^{i\theta})$ as w tends to $e^{i\theta}$ along any non-tangential path. Let $f(x)$ be the transform into z -plane of $\varphi(e^{i\theta})$. Then $f(z)$ tends to $f(x)$ almost everywhere as z tends to x along any non-tangential path. Moreover, by Fatou's theorem

$$\int_{-\infty}^{\infty} |f(x)|^p dx \leq \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq C.$$

Thus (i) is proved.

Before we proceed to prove (ii) and (iii), we prove the following theorem.

Theorem 2. *If $p > 0$, then a function $f(z) \in \mathfrak{S}_p$ can be represented as a product*

$$f(z) = b_f(z)h(z),$$

where $b_f(z)$ is the one given in 1. Theorem B, and $h(z) \in \mathfrak{S}_p$, but does not vanish in the half-plane $y > 0$. Hereby $\{z_n\}$ is the sequence of zeros of $f(z)$. And

$$(2.3) \quad |b_f(z)| \leq 1, \text{ for } y > 0,$$

$$(2.4) \quad |b_f(x)| = 1, \text{ almost everywhere.}$$

We prove Theorem 2. It is sufficient to prove the theorem in the case $0 < p < 1$. Let $\varphi(w)$ be the transform by (2.2) of $f(z) \in \mathfrak{S}_p$. Then by Lemma 3, $\varphi(w) \in H_p$. Thus by the known theorem, $\varphi(w)$ can be represented as

$$\varphi(w) = b_\varphi(w)\eta(w),$$

where $b_\varphi(w)$ is the corresponding Blaschke function and $\eta(w)$ belongs to the class H_p and does not vanish in $|w| < 1$. Let $b_f(z) = b_\varphi(w)$ and $h(z) = \eta(w)$. Then properties of $b_f(z)$ are immediate from those of $b(w)$.

Next we will prove that $h(z)$ belongs to the class $\mathfrak{S}_p$. Let $f(z+i\eta) = \psi(z)$, η being a positive number fixed for the moment. Then by the proof of Lemma 2, $\psi(z) \in \mathfrak{S}_1$. And consequently $\psi(z)$ can be represented as

$$(2.5) \quad \psi(z) = b_f(z, \eta)h_\eta(z).$$

As is easily verified from the properties of $b_f(z_0; \eta)$, we have

$$(2.6) \quad \lim_{\eta \rightarrow 0} h_\eta(z) = h(z), \quad \text{for } y > 0.$$

Now since $h_\eta(z)$ does not vanish in $y > 0$, $h_\eta^p(z)$ is regular in $y > 0$. Thus $h_\eta^p(z) \in \mathfrak{S}_{\frac{1}{p}}$. Let the limit function of $h_\eta^p(z)$ when $y \rightarrow 0$, be $\bar{h}_\eta(x)$. Then evidently $|\bar{h}_\eta(x)| = |h_\eta(x)|^p$. Since $h_\eta^p(z) \in \mathfrak{S}_{\frac{1}{p}}$ and its limit function $\bar{h}_\eta(x)$ belongs to the class $L_{\frac{1}{p}}$, $h_\eta^p(z)$ is represented as the Poisson integral of $h_\eta^p(x)$, that is

$$h_\eta^p(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \bar{h}_\eta(t) \frac{y dt}{(t-x)^2 + y^2}.$$

On the other hand, $|h_\eta(x)|^p$ is integrable, for $|h_\eta(x)| = |f(x+i\eta)|$ almost everywhere. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} |h_\eta(z)|^p dx &= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} h_\eta(t) \frac{y dt}{(t-x)^2 + y^2} \right|^p \\ &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |h_\eta(t)|^p \frac{y}{(t-x)^2 + y^2} dt = \int_{-\infty}^{\infty} |h_\eta(x)|^p dx \\ &= \int_{-\infty}^{\infty} |f(x+i\eta)|^p dx \end{aligned}$$

which does not exceed an absolute constant C .

Now by (2.6) and Fatou's theorem, we have

$$\int_{-\infty}^{\infty} |h(z)|^p dx = \int_{-\infty}^{\infty} \lim_{\eta \rightarrow 0} |h_\eta(z)|^p dx \leq \lim_{\eta \rightarrow \infty} \int_{-\infty}^{\infty} |h_\eta(z)|^p dx \leq C.$$

Thus the fact that $h(z) \in \mathfrak{S}_p$ is proved.

Lemma 4. If $f(z)$ is bounded and regular in a strip $\alpha \leq y \leq \beta$, and further

$$M_p^p(f, y) = \int_{-\infty}^{\infty} |f(x+iy)|^p dx, \quad (p > 0)$$

exists at $y = \alpha$ and β , then $\log M_p^p(f, y)$ is a convex function of y , that is

$$M_p^p(f, y) \leq \{M_p^p(f, \alpha)\}^{\frac{\beta-y}{\beta-\alpha}} \{M_p^p(f, \beta)\}^{\frac{y-\alpha}{\beta-\alpha}}.$$

This is due to Hardy, Ingham and Pólya⁽⁸⁾. They proved this theorem under the more general conditions. But for our purpose the theorem under these conditions is only needed.

Lemma 5. If $f(z) \in \mathfrak{S}_p$ ($p > 0$), then for $y > 0$,

$$M_p(f, y) = \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p dx \right\}^{\frac{1}{p}}$$

is monotone decreasing function of y .

For since $M_p^p(f, y) \leq C$, for $y \geq \varepsilon$ ($\varepsilon > 0$), this is an immediate consequence of Lemma 4.

(⁸) Hardy, Ingham and Pólya, loc. cit., Theorem 7

Now we will prove Theorem 1, (ii) and (iii). By Theorem 2, we can write

$$f(z) = b_f(z)h(z),$$

where $|b_f(x)| = 1$ almost everywhere and $h(z)$ belongs to the class $\mathfrak{G}_p$ and does not vanish in the half-plane $y > 0$.

If $h^p(z) = \gamma(z)$, then $\gamma(z)$ is regular in $y > 0$ and $\gamma(z)$ belongs to the class $\mathfrak{G}_1$. Let E be any measurable set whose measure may be finite or infinite. Whence we first prove

$$(2.7) \quad \int_E |f(x+iy)|^p dx \rightarrow \int_E |f(x)|^p dx, \quad \text{as } y \rightarrow 0,$$

$f(x)$ being the limit function of $f(z)$.

For,

$$\begin{aligned} & \left| \int_E \{ |f(x+iy)|^p - |f(x)|^p \} dx \right| \\ &= \left| \int_E \{ |b_f(x+iy)|^p |\gamma(x+iy)| - |b_f(x)|^p |\gamma(x)| \} dx \right| \\ &\leq \int_E ||b_f(x+iy)|^p - |b_f(x)|^p| |\gamma(x)| dx \\ &\quad + \int_E |b_f(x+iy)|^p ||\gamma(x+iy)| - |\gamma(x)|| dx. \end{aligned}$$

The first term tends to zero as y tends to zero, for the integrand does not exceed $2|\gamma(x)|$ which is integrable and the well known Lebesgue's theorem is applicable. The second term is not greater than

$$\int_E |\gamma(x+iy) - \gamma(x)| dx,$$

which tends to zero as y tends to zero by Hille and Tamarkin's theorem A in 1, since $\gamma(z) \in \mathfrak{G}_1$. Thus (2.7) is established.

Particularly if we take $E = (-\infty, -A) + (A, \infty)$, then for any given positive number ε , there exist $A = A(\varepsilon)$ and $\delta = \delta(\varepsilon, A)$ such that for $y < \delta$,

$$(2.8) \quad \int_A^\infty + \int_{-\infty}^{-A} |f(x+iy)|^p dx \leq \int_A^\infty + \int_{-\infty}^{-A} |f(x)|^p dx + \varepsilon.$$

Similarly there exists δ' such that if $mE < \delta'$, then

$$(2.9) \quad \int_E |f(x+iy)|^p dx \leq \int_E |f(x)|^p dx + \varepsilon \leq 2\varepsilon$$

for $y < \eta'$ where $\eta' = \eta'(\varepsilon, \delta')$.

Since $f(x+iy)$ tends to $f(x)$ almost everywhere as y tends to zero, by Egoroff's theorem, there exists a set M in $(-A, A)$ such that $f(x+iy)$ tends

uniformly to $f(x)$ in M , and the measure of the set $C \equiv (-A, A) - M$ is less than δ' . Then in the expression

$$\int_{-A}^A |f(x+iy) - f(x)|^p dx = \int_M + \int_C,$$

the first term clearly tends to zero as y tends to zero and the second term is not greater than 2ε , if $y < \eta'$. By combining this with (2.8), we obtain

$$\overline{\lim}_{y \rightarrow 0} \int_{-\infty}^{\infty} |f(x+iy) - f(x)|^p dx < 5\varepsilon.$$

Thus (ii) is proved.

(iii) is nothing but Lemma 5.

3. Some inequalities concerning the means M_p . Let $f(z) = f(x+iy)$ belong to the class $\mathfrak{G}_p$ in the half-plane $y > 0$ and

$$M_p(f, y) = \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p dx \right\}^{\frac{1}{p}},$$

$$M(f, y) = \max_{-\infty < x < \infty} |f(x+iy)|.$$

The fact that $M(f, y)$ is finite when $f(z)$ belongs to the class $\mathfrak{G}_p$, is an immediate consequence from Lemma 1.

Theorem 3. If $p > 0$, $a \leq 0$ and

$$(3.1) \quad M_p(f, y) \leq Cy^{-a} \quad (C = \text{a constant})$$

and further $q > p$, then

$$(3.2) \quad M_q(f, y) \leq KCy^{-a - \frac{1}{p} + \frac{1}{q}},$$

$$(3.3) \quad M(f, y) \leq KCy^{-a - \frac{1}{p}},$$

where K is independent of y .

The analogous theorem concerning the regular function in the unit circle is due to Professors Hardy and Littlewood. Theorem 3 is also proved by the completely similar method to that of Hardy and Littlewood. This theorem is a special case of Theorem 4 which is also the analogous one to a theorem due to Hardy and Littlewood. We do not give the proof of Theorem 3.

4. A theorem concerning conjugate functions. Let $f(z) = f(x+iy) \in \mathfrak{G}_p$ in the half-plane $y > 0$, p being positive and $f(x+iy) = u(x, y) + iv(x, y)$, u and v being real and imaginary parts respectively. The object of the present section is to establish some inequalities concerning $M_p(f, y)$ from the inequality concerning $M_p(u, y)$. Our argument depends upon the use of the Fourier transform.

Theorem 4. If $p > 0$, $a \geq 0$ and

$$(4.1) \quad M_p(u, y) \leq Cy^{-a}, \quad (C = \text{a constant}),$$

and further $p > q$, then

$$(4.2) \quad M_q(f, y) \leq K C y^{-a - \frac{1}{p} + \frac{1}{q}},$$

$$(4.3) \quad M(f, y) \leq K C y^{-a - \frac{1}{p}},$$

where K is independent of y .

It is sufficient to prove the theorem under the condition $0 < p \leq 1$, and $C=1$. Indeed if $p > 1$, then the theorem is a corollary of M. Riesz's theorem concerning the conjugate function of a function regular in the half-plane. No loss of generality of assuming $C=1$ is trivial.

We require following lemmas which are due to Hille and Tamarkin⁽⁹⁾.

Lemma 6. Let $f(x) \in L_p (1 \leq p < \infty)$. If $f(x)$ has a Fourier transform $F(x)$ in some $L_q (1 \leq q \leq \infty)$, then in order that $f(x)$ be the limit function of a function $f(z) \in \mathfrak{S}_p$ (or what is the same, of a function regular in the half-plane $y > 0$ and representable by its Cauchy or Poisson integral), it is necessary and sufficient that $F(x) = 0$ for $x < 0$.

Lemma 7. If $f(x) \in L_p (1 \leq p \leq \infty)$ has a Fourier transform $F(x)$ in some $L_q (1 \leq q \leq \infty)$, the Poisson integral associated with $f(x)$ can be represented as

$$(4.4) \quad P(z; f) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y dt}{(t-x)^2 + y^2} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ixt} e^{-y|t|} F(t) dt,$$

for $y > 0$.

Thus if $f(x)$ is the limit function of $f(z) \in \mathfrak{S}_p (2 \leq p \leq 1)$, we can write in the form

$$P(z, f) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y dt}{(t-x)^2 + y^2} = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} e^{ixt} e^{-yt} F(t) dt,$$

for $y > 0$.

We will prove Theorem 4. Let $\varphi(z) = f(z + iy_0)$, where y_0 is a positive number fixed for the moment. Then $\varphi(z)$ is regular in the closed half-plane $y \geq 0$ and belongs to the class $\mathfrak{S}_q (1 \leq q \leq \infty)$. This was proved in the proof of Lemma 2. If $F(x)$ is the Fourier transform of $\varphi(x)$, then $F(x) = 0$ for $x < 0$ by Lemma 6. And by Lemmas 6 and 7, we have

$$(4.5) \quad \varphi(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(t) \frac{y dt}{(t-x)^2 + y^2} = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} e^{ixt} e^{-yt} F(t) dt.$$

Now let

$$\varphi(x + iy) = \bar{u}(x, y) + i\bar{v}(x, y),$$

$$F(t) = \alpha(t) + i\beta(t).$$

Equating the real parts of both sides in (4.5), we get

(⁹) Hille and Tamarkin, loc. cit.

$$u(x, y) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \{ \cos xt \alpha(t) - \sin xt \beta(t) \} e^{-yt} dt,$$

therefore we have

$$\begin{aligned} \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{-\infty}^\infty \bar{u}(x, y) \cos xu \, dx &= \frac{2}{\pi} \int_0^\infty \cos xu \, dx \int_0^\infty \cos xt \alpha(t) e^{-yt} dt \\ &= \alpha(u) e^{-yu}. \end{aligned}$$

Similarly we have

$$\frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{-\infty}^\infty \bar{u}(x, y) \sin xu \, dx = -\beta(u) e^{-yu}.$$

Thus

$$(4.6) \quad F(t) e^{-yt} = \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{-\infty}^\infty u(x, y) e^{-txt} dx.$$

Hence

$$\begin{aligned} (4.7) \quad |F(t) e^{-yt}| &\leq \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{-\infty}^\infty |\bar{u}(x, y)| dx \\ &\leq \frac{1}{2} \sqrt{\frac{\pi}{2}} M^{1-p}(f, \eta) \int_{-\infty}^\infty |\bar{u}(x, y)|^p dx, \end{aligned}$$

where $y_0 + y = \eta$. By (4.1)

$$\int_{-\infty}^\infty |\bar{u}(x, y)|^p dx = \int_{-\infty}^\infty |u(x, y + y_0)|^p dx \leq \eta^{-ap}.$$

Putting this into (4.7), we have

$$(4.8) \quad |F(t) e^{-yt}| \leq \frac{1}{2} \sqrt{\frac{\pi}{2}} M^{1-p}(f, \eta) \eta^{-ap}.$$

Let $\bar{\eta} = \bar{y} + y_0 > y + y_0 = \eta$. We have

$$\begin{aligned} f(x + i\bar{y} + iy_0) &= \frac{1}{\pi} \int_{-\infty}^\infty \varphi(t) \frac{\bar{y} dt}{(t-x)^2 + \bar{y}^2} = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{ixt} e^{-\bar{y}t} F(t) dt \\ &= (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{ixt} e^{-yt} F(t) e^{-(\bar{y}-y)t} dt. \end{aligned}$$

Hence by (4.8)

$$\begin{aligned} |f(x + i\bar{\eta})| &\leq \frac{1}{4} M^{1-p}(f, \eta) \eta^{-ap} \int_0^\infty e^{-(\bar{y}-y)t} dt \\ &= \frac{1}{4} \frac{1}{\bar{y}-y} M^{1-p}(f, \eta) \eta^{-ap}. \end{aligned}$$

Therefore

$$(4.9) \quad M(f, \bar{\eta}) \leq \frac{1}{4} (\bar{\eta} - \eta)^{-1} M^{1-p}(f, \eta) \eta^{-ap}.$$

By Hardy and Littlewood's device, we can easily prove, from (4.9), that

$$M(f, \eta) \leq K \eta^{-\alpha - \frac{1}{p}}.$$

This is (4.3), since y_0 is arbitrary. (4.2) is an immediate consequence from (4.3), for

$$\begin{aligned} M_a(f, y) &\leq M^{\frac{q-p}{q}}(f, y) \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p dx \right\}^{\frac{1}{q}} \\ &\leq K y^{-(\alpha + \frac{1}{p})(1 - \frac{p}{q})} y^{-\frac{ap}{q}} = K y^{-\alpha - \frac{1}{p} + \frac{1}{q}}. \end{aligned}$$

Thus Theorem 4 is proved.