2. On Fatou's theorems on Poisson integrals.

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1. Let $f(\theta)$ be integrable in the interval $(-\pi, \pi)$ and

$$u(z) = u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) \frac{1-r^2}{1-2r \cos(\psi-\theta) + r^2} d\psi \quad (0 \leq r < 1). \quad (1)$$

In his well known Thesis Fatou has proved that (i) $\lim_{z \rightarrow e^{i\psi}} u(z) = f(\psi)$ almost everywhere, when z approaches to $e^{i\psi}$ along a Stolz path, (ii) if $u(z)$ be harmonic and bounded in a unit circle, then $\lim u(z)$ exists almost everywhere, when z approaches to the unit circle along a Stolz path. I will here give a simple proof that (i) and (ii) hold, when z approaches to the unit circle along a curve, which touches the unit circle in a certain manner and also prove the analogous theorems for Poisson integrals in space.

2. At first we will prove (i). By Lebesgue

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t |f(\psi+t) - f(\psi)| dt = 0 \quad (2)$$

almost everywhere. Suppose that (2) holds at $\psi = 0$, so that

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t |f(t) - f(0)| dt = 0.$$

Hence if we put

$$F(t) = |f(t) - f(0)|,$$

then

$$\int_{-\delta}^{\delta} F(t) dt = \delta \varepsilon(\delta), \quad \varepsilon(\delta) \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0. \quad (3)$$

Let z approach to $z = 1$ in a domain

$$1-r = \delta \sqrt{\varepsilon(2\delta)}, \quad |\theta| \leq \delta, \quad (0 \leq \delta \leq \delta_0). \quad (4)$$

From (1)

$$\begin{aligned} |u(z) - f(0)| &\leq \frac{1}{2\pi} \int_{0 \leq |\psi| \leq 2\delta} F(\psi) \frac{1-r^2}{1-2r \cos(\psi-\theta) + r^2} d\psi \\ &\quad + \frac{1}{2\pi} \int_{2\delta \leq |\psi| \leq \pi} F(\psi) \frac{1-r^2}{1-2r \cos(\psi-\theta) + r^2} d\psi = I_1 + I_2. \end{aligned}$$

Now for $\frac{1}{2} \leq r < 1$,

$$\begin{aligned} 1 - 2r \cos(\psi - \theta) + r^2 &= (1-r)^2 + 4r \sin^2 \frac{\psi - \theta}{2} \geq (1-r)^2 \quad \text{in } I_1 \\ &\geq \sin^2 \frac{\psi - \theta}{2} \geq \sin^2 \frac{\psi}{4} \geq a\psi^2 \quad (a = \text{const}) \quad \text{in } I_2, \end{aligned}$$

since $\left| \frac{\psi}{4} \right| \leq \left| \frac{\psi - \theta}{2} \right|$ in I_2 in virtue of (4).

From (3) there exists a constant K such that for any ψ ,

$$\left| \int_0^\psi F(t) dt \right| \leq K |\psi|, \quad (5)$$

hence from (3), (4), (5),

$$\begin{aligned} I_1 &\leq \frac{1}{2\pi} \int_{|\psi| \leq 2\delta} F(\psi) \frac{1-r^2}{(1-r)^2} d\psi \\ &\leq \frac{1}{\pi(1-r)} \int_{|\psi| \leq 2\delta} F(\psi) d\psi = \frac{2}{\pi} \sqrt{\varepsilon(2\delta)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \\ I_2 &\leq \frac{1-r}{\pi a} \int_{2\delta \leq |\psi| \leq \pi} \frac{F(\psi)}{\psi^2} d\psi \\ &\leq \frac{1}{\pi a} \sum_{n=1}^{\infty} (1-r) \left(\int_{2^{n\delta}}^{2^{n+1}\delta} \frac{F(\psi)}{\psi^2} d\psi + \int_{-2^{n+1}\delta}^{-2^{n\delta}} \frac{F(\psi)}{\psi^2} d\psi \right) \\ &\leq \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{1-r}{2^{2n}\delta^2} \left(\int_0^{2^{n+1}\delta} F(\psi) d\psi + \int_{-2^{n+1}\delta}^0 F(\psi) d\psi \right) \\ &\leq \frac{2K}{\pi a} \sum_{n=1}^{\infty} \frac{1-r}{2^{2n}\delta^2} 2^{n+1}\delta = \frac{2K\sqrt{\varepsilon(2\delta)}}{\pi a} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \\ &= \frac{4K\sqrt{\varepsilon(2\delta)}}{\pi a} \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

Hence

$$u(z) - f(0) \rightarrow 0 \quad \text{as } \delta \rightarrow 0$$

Theorem I. If $\int_{-\delta}^{\delta} |f(t) - f(0)| dt = \delta \varepsilon(\delta)$, $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, then $u(z) \rightarrow f(0)$ uniformly, when $z = r e^{i\theta}$ approaches to $z = 1$ in a domain bounded by a curve

$$1 - r = \theta \sqrt{\varepsilon(2\theta)}. \quad (6)$$

This curve touches the unit circle, so that any Stolz path is contained in the above domain.

3. We pass on to the Poisson integral in space. Let

$$u(P) = \frac{1}{4\pi} \iint_E f(Q) \frac{1-r^2}{(1-2r \cos \psi_1 + r^2)^{\frac{3}{2}}} d\sigma_Q ,$$

where E is a unit sphere with center at the origin, $d\sigma_Q$ the surface element on E and $OP = r$ and ψ_1 is the angle between OP and OQ and $f(Q)$ is integrable on E . By Lebesgue

$$\frac{1}{\text{area } \sigma} \iint_{\sigma} |f(Q) - f(Q_0)| d\sigma_Q \rightarrow 0 \quad \text{as } \sigma \rightarrow 0 \quad (7)$$

almost everywhere on E , where σ is a circle on E about Q_0 .

Suppose that (7) holds at the north pole P_0 of E and put

$$F(Q) = |f(Q) - f(P_0)|$$

so that

$$\iint_{0 \leq \psi \leq \delta} F(Q) d\sigma_Q = \delta^2 \varepsilon(\delta) , \quad \varepsilon(\delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0 , \quad (8)$$

where $(1, \psi, \varphi')$ being the polar coordinates of Q .

Let P approach to the north pole P_0 in a domain

$$1-r = \delta \sqrt{\varepsilon(2\delta)} , \quad 0 \leq \theta \leq \delta , \quad (0 \leq \delta \leq \delta_0) , \quad (9)$$

where (r, θ, φ) being the polar coordinates of P .

Now

$$\begin{aligned} |u(P) - f(P_0)| &\leq \frac{1}{4\pi} \iint_{0 \leq \psi \leq 2\delta} F(Q) \frac{1-r^2}{(1-2r \cos \psi_1 + r^2)^{\frac{3}{2}}} d\sigma_Q \\ &+ \frac{1}{\psi \pi} \iint_{2\delta \leq \psi \leq \pi} F(Q) \frac{1-r^2}{(1-2r \cos \psi_1 + r^2)^{\frac{3}{2}}} d\sigma_Q = I_1 + I_2 . \end{aligned}$$

Let (r, θ, φ) , $(1, \psi, \varphi')$ be the polar coordinates of P and Q respectively, then

$$\cos \psi_1 = \sin \theta \sin \psi \cos (\varphi - \varphi') + \cos \theta \cos \psi$$

so that $\cos(\psi + \theta) \leq \cos \psi_1 \leq \cos(\psi - \theta)$, hence for $0 \leq \theta \leq \delta$, $2\delta \leq \psi \leq \pi$,

$$\frac{\psi}{2} \leq \psi - \theta \leq \psi_1 \leq \psi + \theta \quad (10)$$

and for $\frac{1}{2} \leq r < 1$,

$$\begin{aligned}
1 - 2r \cos \psi_1 + r^2 &= (1-r)^2 + 4r \sin^2 \frac{\psi_1}{2} \geq (1-r)^2 \quad \text{in } I_1, \\
&\geq \sin^2 \frac{\psi_1}{2} \geq \sin^2 \frac{\psi}{4} \geq a^2 \psi^2 \quad (a = \text{const}) \quad \text{in } I_2.
\end{aligned}$$

From (8) there exists a constant K , such that for any ψ_0

$$\iint_{0 \leq \psi \leq \psi_0} F(Q) d\sigma_Q \leq K \psi_0^2. \quad (11)$$

Hence from (8), (9), (11), we have

$$\begin{aligned}
I_1 &\leq \frac{1}{4\pi} \iint_{0 \leq \psi \leq 2\delta} F(Q) \frac{1-r^2}{(1-r)^3} d\sigma_Q \leq \frac{1}{2\pi(1-r)^2} \iint_{0 \leq \psi \leq 2\delta} F(Q) d\sigma_Q \\
&= \frac{4\delta^2 \varepsilon(2\delta)}{2\pi \delta^2 \varepsilon(2\delta)^{\frac{2}{3}}} = \frac{2}{\pi} \sqrt[3]{\varepsilon(2\delta)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
\end{aligned}$$

$$\begin{aligned}
I_2 &\leq \frac{1}{4\pi a^3} \iint_{2\delta \leq \psi \leq \pi} F(Q) \frac{1-r^2}{\psi^3} d\sigma_Q \\
&\leq \frac{1-r}{2\pi a^3} \iint_{2\delta \leq \psi \leq \pi} \frac{F(Q)}{\psi^3} d\sigma_Q \leq \frac{1-r}{2\pi a^3} \sum_{n=1}^{\infty} \iint_{2^n \delta \leq \psi \leq 2^{n+1} \delta} \frac{F(Q)}{\psi^3} d\sigma_Q \\
&\leq \frac{1-r}{2\pi a^3} \sum_{n=1}^{\infty} \frac{1}{2^{3n} \delta^3} \iint_{0 \leq \psi \leq 2^{n+1} \delta} F(Q) d\sigma_Q \leq \frac{K(1-r)}{2\pi a^3} \sum_{n=1}^{\infty} \frac{2^{2n+2} \delta^2}{2^{3n} \delta^3} \\
&= \frac{2K \sqrt[3]{\varepsilon(2\delta)}}{\pi a^3} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{2K \sqrt[3]{\varepsilon(2\delta)}}{\pi a^3} \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
\end{aligned}$$

Hence

$$u(P) - f(P)_0 \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Theorem II. If

$$\iint_{0 \leq \psi \leq \delta} |f(Q) - f(P_0)| d\sigma_Q = \delta^2 \varepsilon(\delta), \quad \varepsilon(\delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0,$$

then $u(P) \rightarrow f(P_0)$ uniformly, when P approaches to the north pole P_0 in a domain bounded by a surface:

$$1 - r = \theta \sqrt[3]{\varepsilon(2\theta)}, \quad (11)$$

where (r, θ, φ) , $(1, \psi, \varphi')$ being the polar coordinates of P and Q respectively.

Since this surface touches the unit sphere, any Stolz path is contained in the above domain.

4. Let $u(P) = u(r, \theta, \varphi)$ be a harmonic function in a unit sphere E and bounded in E , so that $|u(P)| \leq M$ in E .

We express $u(P)$ in a sphere E_1 , whose radius R is less than 1, by a Poisson integral on E_1 ,

$$u(P) = u(r, \theta, \varphi) = \frac{1}{4\pi R} \iint_{E_1} u_R(Q) \frac{R^2 - r^2}{(R^2 - 2Rr \cos \psi + r^2)^{\frac{3}{2}}} d\sigma'_Q, \quad (12)$$

where $u_R(Q)$ is the value of $u(Q)$ on E_1 and $d\sigma'_Q$ the surface element on E_1 .

We rewrite (12) in the form;

$$u(r, \theta, \varphi) = \frac{1}{4\pi} \iint_E u_R(Q) \frac{1 - \left(\frac{r}{R}\right)^2}{\left(1 - \frac{2r}{R} \cos \psi + \left(\frac{r}{R}\right)^2\right)^{\frac{3}{2}}} d\sigma_Q, \quad (13)$$

where E is the unit sphere and $d\sigma_Q$ the surface element on E .

Expanding the right hand side of (13) in spherical harmonics series, we have

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n Y_n^{(R)}(\theta, \varphi), \quad (14)$$

where $\left(\frac{r}{R}\right)^n Y_n^{(R)}(\theta, \varphi)$ is a spherical harmonic of order n , depending on R .

But since the left hand side is independent of R , $\frac{1}{R^n} Y_n^{(R)}(\theta, \varphi)$ is independent of R , which we may write $\frac{1}{R^n} Y_n^{(R)}(\theta, \varphi) = Y_n(\theta, \varphi)$, hence

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} r^n Y_n(\theta, \varphi), \quad (15)$$

where $r^n Y_n(\theta, \varphi)$ is a spherical harmonic of order n .

Since $|u(P)| \leq M$ and $Y_n(\theta, \varphi)$ are orthogonal on E ,

$$4\pi M^2 \geq \iint_E u^2(r, \theta, \varphi) d\sigma = \sum_{n=0}^{\infty} r^{2n} \iint_E Y_n^2(\theta, \varphi) d\sigma,$$

$d\sigma$ being the surface element on E . Making $r \rightarrow 1$, we have

$$\sum_{n=0}^{\infty} \iint_E Y_n^2(\theta, \varphi) d\sigma \leq 4\pi M^2. \quad (16)$$

And for $r_1 < r_2 < 1$

$$\iint_E (u(r_1, \theta, \varphi) - u(r_2, \theta, \varphi))^2 d\sigma = \sum_{n=0}^{\infty} (r_1^n - r_2^n)^2 \iint_E Y_n^2(\theta, \varphi) d\sigma. \quad (17)$$

From (16), we take n_0 , so large that

$$\sum_{n=n_0}^{\infty} \iint_E Y_n^2(\theta, \varphi) d\sigma < \frac{\varepsilon}{2}$$

and then δ so small that for $1-\delta \leq r_1 < r_2 < 1$,

$$\sum_{n=n_0}^{n_0-1} (r_1^n - r_2^n)^2 \iint_E Y_n^2(\theta, \varphi) d\sigma < \frac{\varepsilon}{2}$$

so that

$$\iint_E (u(r_1, \theta, \varphi) - u(r_2, \theta, \varphi))^2 d\sigma < \varepsilon.$$

Hence

$$\iint_E (u(r_1, \theta, \varphi) - u(r_2, \theta, \varphi))^2 d\sigma \rightarrow 0 \quad \text{as } r_1, r_2 \rightarrow 1. \quad (18)$$

From (18), there exists a sequence $r_1 < r_2 < \dots < r_n < \dots \rightarrow 1$, such that $\lim_{n \rightarrow \infty} u(r_n, \theta, \varphi)$ exists almost everywhere on E . Let the limiting function be $u(\theta, \varphi)$, then for $r < r_n < 1$

$$u(r, \theta, \varphi) = \frac{1}{4\pi r_n} \iint_{E_n} u(r_n, \theta, \varphi) \frac{r_n^2 - r^2}{(r_n^2 - 2r_n r \cos \psi + r^2)^{\frac{3}{2}}} d\sigma_n,$$

where E_n is the sphere of radius r_n and $d\sigma_n$ the surface element on E_n .

Making $n \rightarrow \infty$, $r_n \rightarrow 1$,

$$u(P) = u(r, \theta, \varphi) = \frac{1}{4\pi} \iint_E u(\theta, \varphi) \frac{1 - r^2}{(1 - 2r \cos \psi + r^2)^{\frac{3}{2}}} d\sigma.$$

By Theorem II, $\lim u(P)$ exists almost everywhere on E , when P approaches the unit sphere in a domain bounded by a surface of the form (11).

Theorem III. If $u(P)$ be harmonic and bounded in a unit sphere E , then $\lim u(P)$ exists almost everywhere on E , when P approaches E in a domain bounded by a surface of the form (11).