13. On the theorems of Valiron and Milloux.

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1. The purpose of this paper is to prove the fundamental theorems of Valiron and Milloux in the theory of meromorphic functions by Ahlfors' method\(^{(1)}\).

Let \(a, b\) be two complex numbers and denote the distance of the corresponding points on the Riemann sphere \(A\) of radius \(\frac{1}{2}\) touching the \(z\)-plane at the origin by \([a, b]\), then

\[
[a, b] = \frac{|a - b|}{\sqrt{(1 + |a|^2)(1 + |b|^2)}}.
\]

Let \(w(z)\) be meromorphic in \(|z| \leq R\) and \(n(r, a)\) be the number of zeros of \(w(z) - a\) in \(|z| \leq r (\leq R)\). We put

\[
m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{w(z)} \right| d\varphi \quad (z = re^{i\varphi}),
\]

\[
N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{r} dt + n(0, a) \log r + C,
\]

where \(C\) is determined by the condition;

\[
\lim_{r \to 0} [m(r, a) + N(r, a)] = 0,
\]

so that, if \(w(0) \neq a\),

\[
N(r, a) = \int_0^r \frac{n(t, a)}{t} dt - \log \left| \frac{1}{w(0)} \right| \leq \int_0^r \frac{n(t, a)}{t} dt.
\]

\[
\frac{dm(r, a)}{dr} - \frac{dm(r, b)}{dr} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} \log \left| \frac{w - b}{w - a} \right| d\varphi = \frac{1}{2\pi r} \int_0^{2\pi} d\arg \left| \frac{w - b}{w - a} \right|
\]

\[
= n(r, b) - n(r, a) / r.
\]

\(^{(1)}\) Ahlfors: Über eine Methode in der Theorie der meromorphen Funktionen. (Societas Scientiarum Fennica. Commentationes Physico-Mathematicae. 8.)
so that by (3)
\[ m(r, a) + N(r, a) = m(r, b) + N(r, b). \]

Hence
\[ T(r) = m(r, a) + N(r, a) \quad (5) \]
is independent of \( a \).

Also we write \( T_{z_0}(r, w(z)), N_{z_0}(r, a, w(z)) \) for \( T(r), N(r, a) \) to denote that the quantity is referred to the center \( z_0 \) and the function \( w(z) \).

2. Let \( d\omega(a) \) be the surface element on the Riemann sphere \( A \), then since \( \int_A \log \frac{1}{[w, a]} d\omega(a) \) is independent of \( w \), we have
\[ T(r) = \frac{1}{\pi} \int_A N(r, a) d\omega(a) + \text{const}, \]
so that \( T(r) \) is an increasing function of \( r \).

Let \( \rho(a) \) be any positive integrable function on \( A \), such that
\[ \int_A \rho(a) d\omega(a) = 1 \quad (6) \]
and
\[ \rho(w) = \int_A \log \frac{1}{[w, a]} \rho(a) d\omega(a), \]
\[ m_r(r) = \frac{1}{2\pi} \int_0^\pi \rho(w(z)) d\phi \quad (z = re^{i\phi}), \]
\[ N_r(r) = \int_A N(r, a) \rho(a) d\omega(a) \]
, then from (5), (6),
\[ T(r) = m_r(r) + N_r(r). \]

Now
\[ N'_r(r) = \frac{1}{r} \int_A n'(r, a) \rho(a) d\omega(a) = \frac{1}{r} \int_0^{2\pi} \int_0^r \frac{|w'(z)|^2}{(1 + |w(z)|^2)^2} \rho(w(z)) t dt d\phi \]
\[ = \frac{1}{r} \int_0^r \frac{\lambda(t)}{t} dt, \quad (z = te^{i\phi}) \quad (11) \]
where
\[ \lambda(r) = \int_0^{2\pi} \frac{|w'(z)|^2}{(1 + |w(z)|^2)^2} \rho(w(z)) \, d\varphi \quad (z = re^{i\varphi}). \] (12)

Hence
\[ N_\varphi(r) - N_\varphi(0) = \int_0^r \frac{dr}{r} \int_0^r \lambda(t) \, dt, \]
so that
\[ N_\varphi(r) - N_\varphi(0) = T(r) - T(0) - m_\varphi(r) + m_\varphi(0) \leq T(r) + m_\varphi(0) \]
or
\[ B(r) = \int_0^r \frac{dr}{r} \int_0^r \lambda(t) \, dt \leq T(r) + m_\varphi(0), \] (13)
where we put
\[ \int_0^r \lambda(t) \, dt = A(r), \quad \int_0^r \frac{A(r)}{r} \, dr = B(r). \]

Then for \( k > 1, \)
\[ A \left( \frac{1+k}{2} r \right) \geq \int_r^{1+k \sqrt{r}} \lambda(t) \, dt = \lambda(\xi) \int_r^{1+k \sqrt{r}} t \, dt \geq \frac{\lambda(\xi) k(k-1)r^2}{4} \left( r \leq \xi \leq \frac{1+k}{2} r \right), \]
\[ B(kr) \geq \int_r^{kr} \frac{A(t)}{t} \, dt \geq A \left( \frac{1+k}{2} r \right) \int_r^{kr} \frac{dt}{t} \geq \frac{k-1}{4k} A \left( \frac{1+k}{2} r \right). \]

Hence for a suitable \( \xi, \)
\[ \lambda(\xi) \leq \frac{16B(kr)}{r^2(k-1)^2} \leq \frac{16}{r^2(k-1)^2}(T(kr) + m_\varphi(0)) \quad (r \leq \xi \leq kr). \] (14)

3. Since
\[ \frac{1}{b-a} \int_a^b \log f(x) \, dx \leq \log \frac{1}{b-a} \int_a^b f(x) \, dx, \]
we have
\[ \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|w'(z)|^2}{(1 + |w(z)|^2)^2} \, d\varphi + \frac{1}{2\pi} \int_0^{2\pi} \log \rho(w(z)) \, d\varphi \leq \log \frac{\lambda(r)}{2\pi} \leq \log \lambda(r). \] (15)
We put
\[ \mu(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|w'(z)|}{1 + |w(z)|^2} \, dp \quad (z = re^{i\theta}) \]
and we can easily prove that
\[ \mu'(r) = \frac{n_1(r, 0)}{r} - \frac{n_1(r, \infty)}{r} - 2 \frac{dm(r, \infty)}{dr}, \]
where \( n_1(r, 0) \), \( n_1(r, \infty) \) are the numbers of zeros and poles of \( w(z) \) in \( |z| \leq r \) respectively.

Since
\[ n_1(r) = n_1(r, 0) - n_1(r, \infty) + 2n(r, \infty) \geq 0, \]
we have
\[ \mu'(r) = \frac{n_1(r)}{r} - 2T'(r) \geq -2T'(r), \]
or
\[ \mu(r) - \mu(0) \geq -2T(r), \]
and
\[ \mu(r) \geq \log \frac{|w'(0)|}{1 + |w(0)|^2} - 2T(r), \quad (16) \]
where we assume \( \frac{w'(0)}{1 + |w(0)|^2} = 0. \)

Hence from (15)
\[ \frac{1}{2\pi} \int_0^{2\pi} \log \rho(w(z)) \, dp \leq 2 \left( 2T(r) - \log \frac{|w'(0)|}{1 + |w(0)|^2} \right) + \log \lambda(r). \quad (17) \]

Let \( a_1, a_2, a_3 \) be any three different points and we take for \( \rho(w) \):
\[ \log \rho(w) = \frac{3}{2} \sum_{i=1}^3 \frac{1}{[w, a_i]} - C \]
or
\[ \rho(w) = e^{-C} \frac{3}{r-1} \frac{1}{[w, a_i]^{\frac{3}{2}}}, \]
where \( C \) is determined by the condition (6), so that
\[ 0 \leq C = \log \left( \int_{\lambda} \frac{1}{r^{v-1}[a, a_i]^{\frac{3}{2}}} \, d\omega(a) \right). \quad (18) \]

Hence
\[ \rho(w) \leq \frac{3}{r-1} \frac{1}{[w, a_i]^{\frac{3}{2}}}. \quad (19) \]
Then from (17),
\[
\frac{3}{2} \sum_{v=1}^{3} m(r, a_v) - C \leq 2 \left( 2T(r) - \log \frac{|w'(0)|}{1 + |w(0)|^2} \right) + \log \lambda(r),
\]
or
\[
T(r) \leq 3 \sum_{v=1}^{3} N(r, a_v) - 4 \log \frac{|w'(0)|}{1 + |w(0)|^2} + 2 \log \lambda(r) + 2C.
\]
By (14)
\[
T(r) \leq T(\xi) \leq 3 \sum_{v=1}^{3} N(\xi, a_v) - 4 \log \frac{|w'(0)|}{1 + |w(0)|^2} + 2 \log \lambda(\xi) + 2C
\]
\[
\leq 3 \sum_{v=1}^{3} N(kr, a_v) - 4 \log \frac{|w'(0)|}{1 + |w(0)|^2} + 4 \log \frac{k}{k-1} + 4 \log \frac{1}{kr}
+ 2 \log T(kr) + 2 \log m_{\nu}(0) + 2C + K,
\]
where $K$ means a numerical constant.

In the next paragraph we prove that
\[
0 \leq C \leq 3 \frac{\sum_{v=1}^{3} \log \frac{1}{|a_{\nu}, a_{\nu}|}}{\nu} + K,
\]
(20)
\[
\log m_{\nu}(0) \leq 3 \sum_{v=1}^{3} \log \frac{1}{|a_{\nu}, a_{\nu}|} + K.
\]
(21)

Writing $kr = R$, we have
\[
T(r) \leq 3 \sum_{v=1}^{3} N(R, a_v) - 4 \log \frac{|w'(0)|}{1 + |w(0)|^2} + 4 \log \frac{R}{R-r} + 4 \log \frac{1}{R}
+ 2 \log T(R) + 12 \sum_{v=1}^{3} \log \frac{1}{|a_{\nu}, a_{\nu}|} + K \quad (r < R),
\]
(22)

where $w(z)$ is meromorphic in $|z| \leq R$ and $\frac{|w'(0)|}{1 + |w(0)|^2} = 0$.

4. We will evaluate $C$ and $\log m_{\nu}(0)$. From (18)
\[
0 \leq C = \log \left( \int_{A} \left[ \prod_{v=1}^{3} \frac{1}{|a_{\nu}, a_{\nu}|} \right]^{2} d\omega(a) \right).
\]

Let $\text{Min.} \{ [a_1, a_2], [a_2, a_3], [a_3, a_1] \} = 2\delta$.

Since for any point $a$:
\[
[a, a_1] + [a, a_2] \geq [a_1, a_2] \geq 2\delta,
\]
we have

\[
\text{Max. } \{[a, a_1], [a, a_2]\} \geq \delta,
\]

so that there are two factors in \( H_{v-1} [a, a_v] \) which are \( \geq \delta \), hence

\[
0 \leq C \leq \log \left( \frac{1}{\delta^2} \int_A \sum_{v=1}^{3} \frac{1}{[a, a_v]^3} d\omega(a) \right)
\]

\[
\leq 3 \log \frac{1}{\delta} + K \leq 3 \sum_{v,v}^{1,3} \log \frac{1}{[a_\mu, a_\nu]} + K.
\] (20)

Next by (19)

\[
m_\nu(0) = p(w(0)) = \int_A \log \frac{1}{[w(0), a]} \rho(a) d\omega(a)
\]

\[
\leq \int_A \log \frac{1}{[w(0), a]} \frac{1}{H_{v-1} [a, a_v]^3} d\omega(a)
\]

\[
\leq \frac{1}{\delta^3} \int_A \log \frac{1}{[w(0), a]} \sum_{v=1}^{3} \frac{1}{[a_\mu, a_\nu]^3} d\omega(a).
\]

Now

\[
\log \frac{1}{[w(0), a]} \cdot \frac{1}{[a, a_v]^3} \leq \log \frac{1}{[a, a_v]} \cdot \frac{1}{[a, a_v]^3} \quad \text{or}
\]

\[
\leq \log \frac{1}{[w(0), a]} \cdot \frac{1}{[w(0), a]^3},
\]

according as

\[
[w(0), a] \geq [a, a_v] \quad \text{or} \quad [w(0), a] \leq [a, a_v],
\]

so that

\[
m_\nu(0) \leq \frac{1}{\delta^3} \int_A \left[ \sum_{v=1}^{3} \log \frac{1}{[a, a_v]} \cdot \frac{1}{[a, a_v]^3} + 3 \log \frac{1}{[w(0), a]} \cdot \frac{1}{[w(0), a]^3} \right] d\omega(a).
\]

Since the integral on the right hand side is independent of \( a_v \) and \( w(0) \), we have as in (20),

\[
\log m_\nu(0) \leq 3 \sum_{v,v}^{1,3} \log \frac{1}{[a_\mu, a_\nu]} + K.
\] (21)
5. Since $\log^+ x < \frac{1}{4} x + \log 4$, we have from (22),

$$T(r) \leq \frac{1}{2} T(R') + 4 \log \frac{R}{R' - r} + H \quad (r < R' < R), \quad (23)$$

where

$$H = 3 \sum_{\nu=1}^{3} N(R, \alpha_{\nu}) - 4 \log \frac{|w'(0)|}{1 + |w(0)|}^2 + 4 \log \frac{1}{R} + 12 \sum_{\mu, \nu}^{1,3} \log \frac{1}{[a_{\mu}, a_{\nu}]} + K.$$

We will prove that for $\frac{2}{3} R \leq r \leq R$,

$$T(r) \leq 16 \log \frac{R}{R - r} + 2H. \quad (24)$$

Let us as suppose that (24) does not hold for all values of $r$ and suppose at $r = r_1 \left( \frac{2}{3} R \leq r_1 \leq R \right)$,

$$T(r_1) > 16 \log \frac{R}{R - r_1} + 2H$$

and let

$$r_2 = r_1 + \frac{1}{3}(R - r_1), \quad r_3 = r_2 + \frac{1}{3}(R - r_1), \quad \ldots \quad r_{p+1} = r_p + \frac{1}{3^p}(R - r_1), \quad \ldots,$$

then we will prove that for any $p$,

$$T(r_p) > 4(2p + 4) \log \frac{R}{R - r_1} + 2H. \quad (25)$$

Since (25) holds for $p = 1$, we will prove (25) by induction by assuming that (25) holds for $p$, so that taking $R' = r_{p+1}$, $r = r_p$ in (23) and taking account of (25) and $\frac{R}{R - r_1} \geq 3$,

$$T(r_{p+1}) \geq 2T(r_p) - 8 \log \frac{3^p R}{R - r_1} - 2H > 8(2p + 3) \log \frac{R}{R - r_1} - 8p \log 3 + 2H$$

$$\geq 4(2p + 1 + 4) \log \frac{R}{R - r_1} + 2H,$$

so that (25) holds for $p+1$ and hence (25) holds for all $p$. But since

$$\lim_{p \to \infty} r_p = \frac{r_1 + R}{2} < R$$

and $T(r_p) < T\left( \frac{r_1 + R}{2} \right)$, (25) can not hold for all $p$. 
Hence (24) holds for all \( \frac{2}{3} R \leq r \leq R \). Since \( T(r) \) is an increasing function of \( r \), we have for \( 0 \leq r \leq R \)
\[
T(r) \leq 16 \log \frac{3R}{R-r} + 2H \quad (0 \leq r \leq R).
\]

Hence we have the fundamental theorem of Valiron (1):

**Theorem.** Let \( w(z) \) be meromorphic in \( |z| \leq R \) and \( \frac{|w'(0)|}{1+|w(0)|^2} \neq 0 \), then for any three different values \( a_1, a_2, a_3 \),

\[
T(r) \leq 6 \sum_{\nu=1}^{3} N(R, a_{\nu}) - 8 \log \frac{|w'(0)|}{1+|w(0)|^2} + 16 \log \frac{R}{R-r} + 8 \log \frac{1}{R} + 24 \sum_{\nu, \mu} \log \frac{1}{|a_{\nu}, a_{\mu}|} + K \quad (r < R),
\]

where \( K \) is a numerical constant.

6. We will prove the theorem of Valiron and Milloux by means of (26). The method is identical with that of Valiron in his Memorial (1).

In the proof we use the following theorem of H. Cartan:

Let \( z_1, z_2, \ldots, z_n \) be \( n \) points, then

\[
|z-z_1| \cdot |z-z_2| \cdot \ldots |z-z_n| \geq \left( \frac{H}{e} \right)^n
\]

for every point \( z \) outside at most \( n \) circles, whose sum of radii is \( \leq 2H \). Let \( w(z) \) be meromorphic in \( |z| \leq 1 \) and takes each of the values of \( a_1, a_2, a_3 \) at most \( n \)-times in \( |z| \leq 1 \). The zero points of \( w(z) - a_\nu \) be \( z_1^{(\nu)}, z_2^{(\nu)}, \ldots \), and consider a circle \( C: |z|=r \) (\( 0 < r < 1 \)). We take \( H = \frac{1-r}{60} \) in H. Cartan’s theorem, so that for every \( z \) outside at most \( 3n \) circles, whose sum of radii is \( \leq 6H = \frac{1-r}{10} \),

\[
II \left| z-z_i^{(\nu)} \right| \geq \left( \frac{1-r}{180} \right)^n \quad (\nu = 1, 2, 3).
\]

Hence there exists a point \( x_0 \left( 0 \leq x_0 < \frac{1-r}{5} \right) \) and a circle \( \Gamma: |z-x_0| = R \)

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(1) G. Valiron. Recherches sur le théorème de M. Borel dans la théorie des fonctions méromorphes. (Acta Mathematica, 52 (1929)). Directions de Borel des fonctions méromorphes (Memorial des sciences mathématiques (1938)).
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\[
\left(1 - 2^{1-r\over 5} \leq R \leq 1 + 2^{1-r\over 5}\right) \text{ such that (27) holds at every point on } \Gamma \text{ and on the radius } L \text{ of } \Gamma \text{ perpendicular to the real axis.}
\]

The circle \( C : |z| = r \) is contained in \( \Gamma \), which contains a circle \( |z-x_0| = \rho \) (\( = r + x_0 \)), so that
\[
{R \over \rho} \geq 1 + {1-r\over 5}, \text{ or } \log {R \over \rho} \geq {1-r\over 10}.
\] (28)

Let \( \text{Min. } \{[a_1, a_2], [a_2, a_3], [a_3, a_1]\} = 28. \)

Then there are two cases (‡T) and (‡U).

(‡T). everywhere on \( L \) and on \( \Gamma \).

(I). \[
{\left| w'(z) \right| \over \left| w(z) \right|^2} \leq \delta \over 10 \text{ everywhere on } L \text{ and on } \Gamma .
\]

Then for every point \( z \) on \( L \) or \( \Gamma \),
\[
[w(z), w(x_0)] \leq \left| \int_{L, \Gamma} {w'(z)} \over 1 + |w(z)|^2 |dz| \right| \leq \delta \over 2 .
\]

As we have seen before, for one of \( a_1, a_2, a_3, a_1 \) say,
\[
[w(x_0), a_1] \geq \delta ,
\]
so that everywhere on \( L \) and \( \Gamma \),
\[
[w(z), a_1] \geq [w(x_0), a_1] - [w(z), w(x_0)] \geq \delta - \delta \over 2 = \delta \over 2 ,
\]
and hence
\[ T_{x_0}(R) \leq \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{[w(z), a_1]} \, d\rho + \int_0^R \frac{N_{x_0}(t, a_1)}{t} \, dt \leq \log \frac{2}{\delta} + \log \frac{1}{|x_0 - z_1^{(1)}|} \]
\[ \leq \log \frac{2}{\delta} + n \log \frac{180}{1-r}, \quad \text{where} \quad z = x_0 + Re^{iz}. \]

Let \( a \) be any value such that \( w(x_0) = a \), then by (28)
\[ n_{x_0}(\rho, a) \frac{1-r}{10} \leq \frac{10}{1-r} \left( n \log \frac{180}{1-r} + \log \frac{2}{\delta} + \log \frac{1}{[w(x_0), a]} \right) \]
\[ \leq \frac{10}{1-r} \left( n \log \frac{180}{1-r} + \sum_{\nu, \nu} \log \frac{1}{[a_{\nu}, a_{\nu}]} + \log \frac{1}{[w(x_0), a]} + K \right). \] (29)

(II). At some point \( z_0 \) on \( L \) or \( \Gamma \),
\[ \frac{|w'(z_0)|}{1 + |w(z_0)|^2} \geq \frac{\delta}{10}. \]

Let
\[ \zeta = \frac{z - z_0}{1 - z_0 z}, \quad w(z) = F(\zeta), \]
where
\[ |z_0| \leq R + x_0 \leq 1 - \frac{1-r}{5}. \]

It can easily be proved that the circle \( |z| = r \) is contained in a circle
\[ \left| \frac{z - z_0}{1 - z_0 z} \right| = \rho_0, \quad \text{where} \quad 1 - \rho_0 \geq \frac{(1-r)^2}{40}, \quad \text{so that} \quad \log \frac{1 + \rho_0}{2\rho_0} \geq \frac{(1-r)^2}{40}. \]

Now
\[ \frac{|F'(0)|}{1 + |F(0)|^2} = \frac{|w'(z_0)|}{1 + |w(z_0)|^2} (1 - |z_0|^2) \geq \frac{\delta}{10} (1 - |z_0|) \geq \frac{\delta(1-r)}{50}. \]

\[ N_0(1, a, F') \leq \sum_k \log \left| \frac{1 - z_0 z_k^{(\nu)}}{z_k^{(\nu)} - z_0} \right| \leq \sum_k \log \frac{2}{|z_k^{(\nu)} - z_0|} \]
\[ \leq n \log 2 + n \log \frac{180}{1-r} = n \log \frac{360}{1-r} \quad (\nu = 1, 2, 3). \]

By (26)

(1) \( n_0(r, a) \) means \( n(r, a) \) with center at \( z = 0 \) Similarly for \( N_0(r, a), T_0(r) \).
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\[ T_0\left(\frac{1+\rho_0}{2}, F\right) \leq 6 \sum_{\nu=1}^{3} N_\nu(1, a_\nu, F) - 8 \log \frac{|F'(0)|}{1 + |F(0)|^2} \]

\[ + K \left( \log \frac{1}{1-1+\rho_0} + \frac{1.2.3}{2} \log \frac{1}{[a_\mu, a_\nu]} + 1 \right) \]

\[ \leq K \left( n \log \frac{360}{1-r} + \log \frac{1}{1-r} + \frac{1.2.3}{2} \log \frac{1}{[a_\mu, a_\nu]} + 1 \right). \]

Since the circle \(|z| = r\) is contained in the circle \(|\zeta| = \rho_0\), for any value \(a\) such that \(w(z_0) = a\),

\[ \frac{(1-r)^2}{40} n_0(r, a; w) \leq \frac{(1-r)^2}{40} n_0(\rho_0, a; F) \leq n_0(\rho_0, a; F) \log \frac{1+\rho_0}{2\rho_0} \]

\[ \leq \int_{\rho_0}^{1+r_0} \frac{n_0(t, a, F)}{t} dt \leq T_0\left(\frac{1+\rho_0}{2}, F\right) + \log \frac{1}{[w(z_0), a]}. \]

Hence

\[ n(r, a) \leq \frac{K}{(1-r)^2} \left( n \log \frac{360}{1-r} + \log \frac{1}{1-r} + \frac{1.2.3}{2} \log \frac{1}{[a_\mu, a_\nu]} \right. \]

\[ + \left. \log \frac{1}{[w(z_0), a]} + 1 \right). \] (30)

From (29), (30) we have the theorem of Valiron and Milloux (1).

**Theorem.** Let \(w(z)\) be meromorphic in \(|z| \leq R\) and takes each of the values \(a_1, a_2, a_3\) at most \(n\)-times in \(|z| \leq R\), then for any value \(a\)

\[ n(kR, a) \leq \frac{1}{(1-k)^2} \left[ (An+B) \log \frac{2}{1-k} + C \sum_{\nu, \mu} \log \frac{1}{[a_\mu, a_\nu]} + D \log \frac{1}{[a(k), a]} \right]. \] (31)

where \(0 < k < 1\) and \(A, B, C, D\) are numerical constants and \(a(k)\) is a certain point depending on \(k\) and \(w(z)\).

7. Let \(w(z)\) be meromorphic in \(|z| < \infty\) and of finite order,

\[ \lim_{r \to \infty} \frac{\log T(r)}{\log r} = \rho. \]

We suppose without loss of generality \(\frac{|w'(0)|}{1 + |w(0)|^2} \neq 0\).

(1) Valiron, l.c. Milloux, Les cercles de remplissage des fonctions méromorphes ou entières, (Acta Mathematica, 52, (1929).) Fonctions méromorphes dans un cercles (Journal de mathématiques pures et appliquées, 1938.)
There exist $r_1 < r_2 < \cdots < r_n \to \infty$ such that
\[ r_n^{z-\varepsilon} \leq T(r_n) \quad (\varepsilon > 0). \quad (32) \]

For $r_0 > 0$, $k > 1$, we construct $(\nu_0+1)$ ring regions;
\[ C_{\nu}; \frac{r_n}{k^{\nu+1}} \leq |z| \leq \frac{r_n}{k^{\nu}} \quad (\nu = -1, 0, 1, 2 \ldots \nu_0), \]

where $\nu_0$ is determined by the condition
\[ \frac{r_n}{k^{\nu_0+1}} < r_0 \leq \frac{r_n}{k^{\nu_0}}, \quad \nu_0 = \left[ \frac{1}{\log k} \log \frac{r_n}{r_0} \right]. \]

We divide $C_{\nu}$ by $q$ half-lines; $\arg z = \frac{2\pi l}{q}$ ($l = 0, 1, \ldots, q$), so that we have $(\nu_0+1)q$ curvilinear quadrilaterals $Q_{\nu, l}$,
\[ Q_{\nu, l}: \frac{r_n}{k^{\nu+1}} \leq |z| \leq \frac{r_n}{k^{\nu}}, \quad \frac{2\pi l}{q} \leq \arg z \leq \frac{2\pi (l+1)}{q} \]

Let $C_{\nu, l}$ be the circle passing through the four vertices of $Q_{\nu, l}$ and with radius $r_{\nu, l}$ and $C_{\nu, l}$ be the circle concentric with $C_{\nu, l}$ and having a radius $2r_{\nu, l}$. In (30), with $k = \frac{1}{2}$, we have $(\nu_0+1)q$ certain points $a_{\nu, l}(\frac{1}{2})$ on the Riemann sphere $A$. We draw $(\nu_0+1)q$ circles $\Gamma_{\nu, l}$ about each $a_{\nu, l}(\frac{1}{2})$ with a radius $\frac{1}{r_n}$. The sum of the areas of these circles is $0\left(\frac{\log r_n}{r_n^2}\right) \to 0$ as $r_n \to \infty$.

Hence there are three points $a_1, a_2, a_3$ on $A$, which are outside $(\nu_0+1)q$ circles $\Gamma_{\nu, l}$ and $[a_{\nu}, a_{\nu}] \geq \frac{1}{r_n}$ and $w(z) - a_{\nu} \neq 0$ for $|z| \leq r_0$ ($\nu = 1, 2, 3$) for a small $r_0$.

From (26), (32), for such $a_{\nu}$,
\[ r_n^{z-\varepsilon} \leq T(r_n) \leq K \left( \sum_{\nu=1}^{3} N(kr_n, a_{\nu}) + \sum_{\nu=1}^{3} \frac{3}{2} \log \frac{1}{[a_{\nu}, a_{\nu}]} + 1 \right). \]

Hence for at least one of $a_{\nu}$, $a$ say,
\[ r_n^{z-\varepsilon} \leq N(kr_n, a) \leq \int_{r_0}^{kr_n} \frac{n(t, a)}{t} dt \leq n(kr_n, a) \log \frac{kr_n}{r_0}, \]
or
\[ r_n^{z-3\varepsilon} \leq n(kr_n, a). \]
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so that in one of $Q_{n, l}$, a fortiori in $C_{n, l}$, $w(z)$ takes the value $\alpha$ at least $r_n^{-4\varepsilon}$-times. By applying (31) to any three values $a_1, a_2, a_3$, such that $[a_\mu, a_\nu] \geq \frac{1}{r_n}$ and $a = \alpha$, we have

$$r_n^{a-4\varepsilon} \leq K\left(N + \sum_{n=1}^{\infty} \log \frac{1}{[a_\mu, a_\nu]} + \log \left[a_{\nu, l}\left(\frac{1}{2}\right), a\right]\right),$$

where $N$ is the largest of the numbers of zeros of $w(z) - a_\nu$ in $C_{n, l}$.

Since

$$[a_\mu, a_\nu] \geq \frac{1}{r_n}, \quad \left[a_{\nu, l}\left(\frac{1}{2}\right), a\right] \geq \frac{1}{r_n},$$

$$r_n^{a-5\varepsilon} \leq N.$$

Hence $w(z)$ takes one of $a_\nu$ at least $r_n^{a-5\varepsilon}$-times in $C_{n, l}$. Let $r$ be the distance of the center of $C_{n, l}$ from the origin and $\kappa$ be its radius, then

$$\kappa = 2\left(\frac{\sin^2 \frac{\pi}{q} + \left(\frac{k-1}{k+1}\right)^2 \cos^2 \frac{\pi}{q}}{q}\right)^{\frac{1}{2}},$$

so that $\kappa \to 0$ for $k \to 1$, $q \to \infty$.

Hence we have the theorem (1).

**Theorem.** If $w(z)$ be a meromorphic function of finite order $\rho$ in $|z| < \infty$, then there exists a sequence of circles $C$; $|z - z_0| = \kappa |z_0|$ ($|z_0| = r$), such that $w(z)$ takes the values $\alpha$ in $C$ at least $r^{a-\varepsilon}$-times, where $\alpha$ is any point outside two circles of radius $\frac{1}{r}$ on the Riemann sphere.

By taking $\varepsilon \to 0$, $\kappa \to 0$, we have the Valiron's theorem on Borel direction.

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(1) Valiron and Milloux, l.c.