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§ 1. We owe to F. Riesz and J. Schauder the "determinantenfrei" treatment of the Fredholm's theory of integral equations(1). They showed that the so-called Fredholm's alternative is also valid for completely continuous (c.c.) linear functional equations in complex Banach spaces. It is to be remarked that there exists an important class of integral equations to which the Riesz-Schauder's theory cannot be applied directly. Consider the integral equation

\[ f = \lambda K \cdot f \quad \left( f(x) = \lambda \int_{0}^{1} K(x, y)f(y)dy \right), \]

where \( K(x, y) \) is bounded and measurable in the square \( 0 \leq x, y \leq 1 \) and \( f(x) \) belongs to the Banach space \( (L) \). Such linear operator \( K \) is not, in general, c.c. as an operator in \( (L) \)(2). Nevertheless the classical analysis of Fredholm also applies to (1), as was shown by M. Fréchet(3). In a preceding note it is shown that the operator \( K^2 \) is c.c. in \( (L) \)(4). Hence we are lead to introduce the following definition:

A linear operator \( K \) in complex Banach space \( \mathcal{B} \) is called quasi-completely-continuous (q.c.c.) if there exist a positive integer \( m \) and a c.c. linear operator \( V \) in \( \mathcal{B} \) such that

\[ ||K^m - V|| < 1. \]

In particular, \( K \) is called \( m \)-quasi-completely-continuous (\( m \)-q.c.c.) if \( K^m \) is c.c. The 1-q.c.c. is equivalent to the usual c.c.

In the present note I intend to show that the validity of the Riesz-Schauder's theory can be extended to the q.c.c. linear functional equa-

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(4) See the note cited in the footnote 2). Cf. also J. Sirvint: C.R. URSS 18 (1938), 255-257.
tions\(^{(5)}\). Fréchet's results quoted above are of course contained in it. The extension of Fréchet-Kryloff-Bogoliouboff's theorem given in a preceding note will also be generalised\(^{(6)}\).

\(\S\ 2\). In \([A]\) I obtained some results concerning the q.c.c. linear operation \(K\) with \(m = 1\). The Lemma 1, 2 below generalise these results to the case \(m \geq 1\). Throughout this paper we assume that \(K\) satisfies (2) unless otherwise is stated.

**Lemma 1.** The proper values\(^{(7)}\) of \(K\) does not accumulate to the point not exterior of the unit circle of the complex \(\lambda\) plane.

**Proof.** The proof for the case \(m = 1\) \([A],\) Lemma 1) also proves the Lemma for the general case \(m \geq 1\).

**Remark.** The only accumulation point of the proper values of a \(m\)-q.c.c. linear operator \(K\) is the point \(\infty\). For \(\lambda K\) is q.c.c. with any complex number \(\lambda\). The well known theorem of F. Riesz corresponds to the case \(m = 1\).

**Lemma 2.** There exists a positive constant \(\varepsilon\) such that the resolvent \(R_\lambda\) of \(K\) is meromorphic for \(1 - \varepsilon < |\lambda| < 1 + \varepsilon\), the poles of \(R_\lambda\) being the proper values of \(K\) with modulus 1. Let the Laurent expansion of \(R_\lambda\) at the pole \(\lambda_0\) \((|\lambda_0| = 1)\) be \(R_\lambda = \sum_{n=-\infty}^{\infty} (\lambda - \lambda_0)^n K_n\). Then \(K_n(n < 0)\) are all c.c.

**Proof.** This Lemma is proved for the case \(m = 1\) in \([A]\). Hence there exists a positive constant \(\varepsilon\) such that the resolvent \(S_\lambda\) of \(K^m\) is meromorphic for \(1 - \varepsilon < |\lambda| < 1 + \varepsilon\). By the identity

\[
(E - \lambda^m K^m) = (E - \lambda K)(E + \lambda K + \cdots + \lambda^{m-1} K^{m-1})
\]

we see that \(R_\lambda\) defined by

\[
E + \lambda R_\lambda = (E + \lambda K + \cdots + \lambda^{m-1} K^{m-1})(E + \lambda^m S_m)
\]

\[
= (E + \lambda^m S_m)(E + \lambda K + \cdots + \lambda^{m-1} K^{m-1})
\]

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\(^{(5)}\) After the present work is completed, I knew the paper of S. Nikolskij: C.R. URSS 16 (1926), 315-319. He proves that the Riesz-Schauder's results can be extended to \(m\)-q.c.c. linear functional equations. His method is somewhat different from ours.


\(^{(7)}\) \(\lambda\) is called a proper value of \(K\) if there exists \(x \neq 0\) of \(\mathfrak{B}\) such that \(x = \lambda K - x\).

\(^{(8)}\) A linear continuous operator \(R_\lambda\) is called the resolvent of \(K\) if \((E + \lambda R_\lambda)\) constitutes the inverse \((E - \lambda K)^{-1}\) of \(E - \lambda K\): \(E = (E - \lambda K)(E + \lambda R_\lambda) = (E + \lambda R_\lambda)(E - \lambda K)\). Here \(E\) denotes the identity operator in \(\mathfrak{B}\).
is the resolvent of $K$. The rest of the Lemma is easily be proved as it holds for $S_{\lambda}$.

**Remark.** If $K$ is $m$-q.c.c., the resolvent $R_{\lambda}$ of $K$ is meromorphic for $|\lambda| < +\infty$. For $\lambda K$ is q.c.c. with any complex number $\lambda$. This constitutes a generalisation of Nagumo-Nikolskij's theorem (9).

**Lemma 3.** Let $\lambda_0(|\lambda_0| = 1)$ be a proper value of $K$. Then we have the decomposition

$$(3) \quad K = A + B, \quad AB = BA = 0$$

such that the resolvents of $A$, $B$ respectively coincides with the principal part, regular part of $R_{\lambda_0}$ in the vicinity of $\lambda = \lambda_0$. $A$ is completely continuous and $B$ is continuous.

**Proof.** By substituting the Laurent expansion $R_{\lambda} = \sum_{n=-\infty}^{\infty} (\lambda - \lambda_0)^n K_n$ in the resolvent equation $R_{\lambda} - R_{\lambda_0} = (\lambda - \mu) R_{\lambda_0} R_{\lambda_0}$ we obtain the Lemma, as $R_{\lambda}$ is given by $\sum_{n=1}^{\infty} \lambda^{n-1} K^n$ for sufficiently small $\lambda$: $A = \sum_{n=1}^{\infty} (-\lambda_0)^n K_n$, $B = K - A$ (10).

**Remark.** The above Lemma holds at any proper value of $K$ if $K$ is $m$-q.c.c. This corresponds to the classical results in the theory of the integral equations.

**An extension of Fréchet-Kryloff-Bogoliouboff's theorem.** Let a q.c.c. linear operator $K$ in $\mathcal{B}$ satisfy condition

$$(4) \quad \|K^n\| \leq \text{a constant } \alpha \quad (n = 1, 2, \ldots).$$

Then the proper values of $K$ with modulus 1 are isolated proper values of finite multiplicities. Let these proper values be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_k$. There exist c.c. linear operators $T_1, T_2, \ldots, T_k$ and a continuous linear operator $S$ such that

$$(5) \quad \begin{cases} K = \sum_{i=1}^{k} \lambda_i^{-1} T_i + S, \quad T_i^2 = T_i, \quad T_i T_j = 0 (i \neq j), \quad T_i S = ST_i = 0, \\ \|S^n\| \leq \frac{\varepsilon}{(1 + \varepsilon)^n} \quad (n = 1, 2, \ldots) \text{ with positive constants } \varepsilon, \delta. \end{cases}$$

**Proof.** The proof given in [A] for the case $m = 1$ also applies to the general case $m \geq 1$, by virtue of the above Lemmas.

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(10) For the detail see M. Nagumo: loc. cit.
§ 3. Let $T$ be a continuous linear operator in $\mathcal{B}$. The linear functionals $f(x)$ on $\mathcal{B}$ constitutes a Banach space $\mathcal{B}'$ with the norm $||f|| = \text{L.u.b.} |f(x)|$. $\mathcal{B}'$ is called the conjugate space of $\mathcal{B}$. $T$ defines a continuous linear operator $T$ in $\mathcal{B}$, the conjugate operator of $T$, by the equation

$$g = \bar{T} \cdot f, \quad g(x) = f(T \cdot x).$$

Let $K$ be a q.c.c. linear operator in $\mathcal{B}$. Consider the linear homogeneous functional equation

$$(6) \quad x - K \cdot x = 0 \quad (x \in \mathcal{B})$$

and its conjugate equation

$$(7) \quad f - \bar{K} \cdot f = 0 \quad (f \in \mathcal{B})$$

Then

i). The equations (6), (7) admit the same numbers of linearly independent solutions.

Next consider the inhomogeneous equation

$$(8) \quad x - K \cdot x = y \quad (x, y \in \mathcal{B})$$

and its conjugate equation

$$(9) \quad f - \bar{K} \cdot f = g \quad (f, g \in \mathcal{B}).$$

Then

ii). For a given $y$, (8) admits solution $x$ if and only if $y$ is orthogonal to all the solutions $f$ of (7): $f(y) = 0$. For a given $g$, (9) admits solution $f$ if and only if $g$ is orthogonal to all the solutions $x$ of (6): $g(x) = 0$.

Proof. We have, by the Lemma 3, $K = A + B$, $AB = BA = 0$, where $A$ is c.c. and $(E - B)$ admits continuous inverse $(E - B)^{-1}$. Hence we obtain


i). From $x - K \cdot x = (E - B)(E - A)x$, $f(x) - f(K \cdot x) = f(x - K \cdot x) = f((E - A)(E - B)x)$ we see that (6) and (7) are respectively equivalent to

$$x - A \cdot x = 0, \quad f - \bar{A} \cdot f = 0,$$
as \((E-B)^{-1}\) exists. Since \(A\) is c.c., these last two equations admit the same number of linearly independent solutions (Schauder’s result).

Q. E. D.

ii). The necessities of the propositions are evident. We will show the sufficiencies.

Let \(y\) be orthogonal to all the solutions \(f\) of (7). Put \((E-B)^{-1} \cdot y = z\), then \(f \left( (E-B)z \right) = 0\). Hence, as \((E-B)^{-1}\) exists, \(z\) is orthogonal to all the solutions \(f'\) of \(f' - A \cdot f' = 0\). \(A\) being c.c., there exists a solution \(x\) of \(x - A \cdot x = z\) (Schauder’s result). Therefore
\[
x - K \cdot x = (E-B)(x-A \cdot x) = (E-B)z = y.
\]

Secondly let \(g\) be orthogonal to all the solutions \(x\) of (6). \(x - K \cdot x = 0\) is equivalent to \(x - A \cdot x = 0\) as shown above. Thus \(g'(g'(x) = g \left( (E-B)^{-1} \cdot x \right)\) is orthogonal to all the solutions \(x\) of \(x - A \cdot x = 0\). \(A\) being c.c., there exists a solution \(f\) of \(f - A \cdot f = g'\) (Schauder’s result). Therefore \(f(y - A \cdot y) = g \left( (E-B)^{-1} \cdot y \right)\) for all \(y \in \mathfrak{B}\).

Hence
\[
f(x - K \cdot x) = f \left( (E-A)(E-B)x \right) = g(x) \text{ for all } x \in \mathfrak{B}, \text{ viz. } f - K \cdot f = g.
\]

Q. E. D.

Remark. Let \(K\) be a \(m\)-q.c.c. linear operator. Then \(\lambda K\) is q.c.c. for any complex number \(\lambda\). Therefore, in this case, i) and ii) hold for the equations
\[
x - \lambda K \cdot x = 0, \quad f - \lambda \overline{K} \cdot f = 0,
\]
and
\[
x - \lambda K \cdot x = y, \quad f - \lambda \overline{K} \cdot f = g
\]
respectively, \(\lambda\) being any complex numbers. This is Nikolskij’s results. When \(K\) is c.c. (1-c.c.), we obtain thus the Riesz-Schauder’s result.