3. Markoff process with an enumerable infinite number of possible states.

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1. Introduction. Let $P$ be a simple Markoff process with an enumerable infinite number of possible states $\mathcal{R} = (S_1, S_2, \ldots)$. $P$ is represented by an infinite matrix: $P = (p_{ij})$, $i, j = 1, 2, \ldots$, where $p_{ij}$ is the transition probability that the state $S_i$ is transferred to the state $S_j$ after the elapse of a unit-time. Then the probability $p_{ij}^{(n)}$ that the state $S_i$ is transferred to the state $S_j$ after the elapse of $n$ unit-times is given recurrently by:

$$p_{ij}^{(n)} = \sum_{k=1}^{\infty} p_{ik} \cdot p_{kj}^{(n-1)}, \quad n = 2, 3, \ldots; \quad p_{ij}^{(1)} = p_{ij}.$$

We have always

$$0 \leq p_{ij}^{(n)} \leq 1, \quad i, j, n = 1, 2, \ldots,$$

$$\sum_{j=1}^{\infty} p_{ij}^{(n)} = 1, \quad i, n = 1, 2, \ldots,$$

and it will be easily seen that

$$p_{ij}^{(m+n)} = \sum_{k=1}^{\infty} p_{ik}^{(m)} \cdot p_{kj}^{(n)}, \quad i, j, m, n = 1, 2, \ldots.$$

Such a Markoff process was discussed in detail by A. Kolmogoroff(1). He has proved among others that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)} = P_{ij}$$

exists for any $i$ and $j$. In the present paper we shall give two elementary proofs of this result, accompanied with some additional new results.

The first proof is given in §2. This is a direct proof, and the same method is also valid for the analogous problems concerning the iteration of positive linear transformations in Banach spaces. Using a new formula:

$$\sum_{k=1}^{\infty} P_{ik} \cdot p_{kj}^{(n)} = \sum_{k=1}^{\infty} p_{ik}^{(n)} \cdot P_{kj} = \sum_{k=1}^{\infty} P_{ik} \cdot P_{kj} = P_{ij}, \quad i, j, n = 1, 2, \ldots,$$

which we obtained in §2 as a by-product, we shall give in §3 the decomposition of the total state \( \mathcal{R} \) into a dissipative part \( \mathcal{D} \) and ergodic parts \( \mathcal{E} \). It is to be noted that we have first proved the existence of the mean sojourn (1.5) and that then the decomposition of the total state is given, while these were in the reverse order by Kolmogoroff.

The second proof, which is given in §4, is entirely different from the first one. Our method is purely analytic and will probably become a useful tool in the further investigations of such Markoff processes. Moreover, our observations in §5 concerning the construction of a Markoff process with given \( \{p^{(n)}_{ij}\} \) or \( \{k^{(n)}_{ij}\} \) (the meaning of \( k^{(n)}_{ij} \) will be explained in §4) may be noted with some interests.

2. First proof of the existence of the mean sojourn.

Theorem 1. (1.5) exists for any \( i \) and \( j \). This limit \( P_{ij} \) satisfies (1.6) and

\[
\sum_{j=1}^{\infty} P_{ij} \leq 1, \quad i = 1, 2, \ldots.
\]

Proof. Put \( q^{(n)}_{ij} = \frac{1}{n} \sum_{m=1}^{n} p^{(m)}_{ij} \). Then we have from (1.2) and (1.3)

\[
0 \leq q^{(n)}_{ij} \leq 1, \quad i, j, n = 1, 2, \ldots, \tag{2.2}
\]

\[
\sum_{j=1}^{\infty} q^{(n)}_{ij} = 1, \quad i, n = 1, 2, \ldots. \tag{2.3}
\]

Because of (2.2) there exists, by the diagonal method, an increasing sequence of positive integers \( \{n_{\nu}\}, \nu = 1, 2, \ldots \), such that \( \lim_{\nu \to \infty} q^{(n_{\nu})}_{ij} = P_{ij} \) exists for any \( i \) and \( j \). Before coming to the proof of the existence of (1.5), let us first prove that this \( P_{ij} \) satisfies the relations (1.6) and (2.1). (2.1) is a direct consequence of (2.3), and (1.6) may be proved as follows:

From

\[
\sum_{k=1}^{\infty} p_{ik} \cdot q^{(n)}_{kj} = \sum_{k=1}^{\infty} p_{ik} \cdot q^{(n_{\nu})}_{kj} = \frac{1}{n} (p^{(n_{\nu}+1)}_{ij} - p^{(n_{\nu})}_{ij}) \leq \frac{2}{n},
\]

we have

\[
\lim_{\nu \to \infty} \sum_{k=1}^{\infty} p_{ik} \cdot q^{(n_{\nu})}_{kj} = \lim_{\nu \to \infty} q^{(n_{\nu})}_{kj} = P_{ij}.
\]

Since \( \sum_{k=1}^{\infty} p_{ij} = 1 \) is absolutely convergent, we have

\[
\lim_{\nu \to \infty} \sum_{k=1}^{\infty} p_{ik} \cdot (\lim_{\nu \to \infty} q^{(n_{\nu})}_{kj}) = \sum_{k=1}^{\infty} p_{ik} \cdot P_{kj}.
\]

Hence we have

\[
\sum_{k=1}^{\infty} p_{ik} \cdot P_{kj} = P_{ij}, \quad i, j = 1, 2, \ldots,
\]

and consequently

\[
\sum_{k=1}^{\infty} p^{(n)}_{ij} \cdot P_{kj} = P_{ij}, \quad i, j, n = 1, 2, \ldots, \tag{2.4}
\]
Markoff process with an enumerable infinite number of possible states.

(2.5) \[ \sum_{k=1}^{\infty} q_{ik}^{(n)} \cdot p_{kj} = P_{ij}, \quad i, j, n = 1, 2, \ldots. \]

Analogously we have, from \( \sum_{k=1}^{\infty} q_{ik}^{(n)} \cdot p_{kj} - q_{ij}^{(n)} = \frac{1}{n} (p_{ij}^{(n+1)} - p_{ij}) \leq \frac{2}{n}, \)

\[ \lim_{\nu \to \infty} \sum_{k=1}^{\infty} q_{ik}^{(n)} \cdot p_{kj} = \lim_{\nu \to \infty} q_{ij}^{(n)} = P_{ij}. \]

Since \( q_{ik}^{(n)} \) and \( p_{kj} \) are all non-negative, we have, by making \( \nu \) tend to \( \infty \),

(2.6) \[ \sum_{k=1}^{\infty} P_{ik} \cdot p_{kj} \leq P_{ij}, \quad i, j = 1, 2, \ldots. \]

It is to be noted that, from the arguments just given above, the equality in (2.6) does not follow directly; but we must have equal sign here for any \( i \) and \( j \). For, if there exists a couple of integers \( i_0 \) and \( j_0 \) such that \( \sum_{k=1}^{\infty} P_{ik} \cdot p_{kj} < P_{ij} \), then we shall have, by taking a sum with respect to \( j \) in (2.6) \( (i = i_0) \),

\[ \sum_{k=1}^{\infty} P_{ik} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} P_{ik} p_{kj} < \sum_{j=1}^{\infty} P_{ij}, \]

which is clearly a contradiction. Hence we have

\[ \sum_{k=1}^{\infty} P_{ik} \cdot p_{kj} = P_{ij}, \quad i, j = 1, 2, \ldots, \]

and consequently

(2.7) \[ \sum_{k=1}^{\infty} P_{ik} \cdot q_{kj}^{(n)} = P_{ij}, \quad i, j, n = 1, 2, \ldots, \]

(2.8) \[ \sum_{k=1}^{\infty} P_{ik} \cdot q_{kj}^{(n)} = P_{ij}, \quad i, j, n = 1, 2, \ldots. \]

Putting \( n = n_\nu \) in (2.8) and making \( \nu \) tend to \( \infty \), we have, since \( \sum_{k=1}^{\infty} P_{ik} \)

is absolutely convergent,

(2.9) \[ \sum_{k=1}^{\infty} P_{ik} \cdot P_{kj} = P_{ij}, \quad i, j = 1, 2, \ldots. \]

Thus (1.6) is proved by (2.4), (2.7) and (2.9).

Now in order to prove (1.5), let us assume that \( \lim q_{ij}^{(n)} \) does not exist for a certain couple of integers \( i = i_0, j = j_0 \). Then there will be another increasing sequence of positive integers \( \{m_\nu\}, \nu = 1, 2, \ldots, \) such that \( \lim q_{ij}^{(m_\nu)} = Q_{ij} \) exists for any \( i \) and \( j \), and such that \( P_{ij_0} = Q_{ij_0} \). \( Q_{ij} \) clearly satisfies the same conditions as \( P_{ij} \). If we now put \( n = m_\nu \) in (2.5) and (2.8), and let \( \nu \) tend to \( \infty \), then we have

(2.10) \[ \sum_{k=1}^{\infty} Q_{ik} \cdot P_{kj} \leq P_{ij}, \quad i, j = 1, 2, \ldots. \]
Note that in (2.10) the equality does not follow directly, while we have always equal sign in (2.11). (The latter is a consequence of the absolute summability of \( \sum_{k=1}^{\infty} P_{ik} \)).

Since \( Q_{ij} \) has the same properties as \( P_{ij} \), we have, by exchanging \( P_{ij} \) and \( Q_{ij} \) in (2.10) and (2.11),

\[
\sum_{k=1}^{\infty} P_{ik} \cdot Q_{kj} \leq Q_{ij}, \quad i, j = 1, 2, \ldots
\]

\[
\sum_{k=1}^{\infty} Q_{ik} \cdot P_{kj} = Q_{ij}. \quad i, j = 1, 2, \ldots
\]

From (2.10) and (2.13) we have \( Q_{ij} \leq P_{ij} \) and from (2.11) and (2.12) we have \( P_{ij} \leq Q_{ij} \), which will lead us to a contradiction since we have, by construction, \( P_{i0j0} \neq Q_{i0j0} \). Hence the limit (1.5) must exist for any \( i \) and \( j \), and thus Theorem 1 is completely proved.

3. Decomposition of the total state \( \mathcal{R} \) into a dissipative part \( \mathcal{D} \) and ergodic parts \( \mathcal{E}_a \).

Theorem 2. The total state \( \mathcal{R} \) is decomposed into a dissipative part \( \mathcal{D} \) and ergodic parts \( \mathcal{E}_a (a = 1, 2, \ldots, \text{finite or enumerably infinite}) \) in such a way that

\[
\begin{align*}
(3.1) & \quad \text{for any } S_j \in \mathcal{D} \text{ we have } P_{ij} = 0, \quad i = 1, 2, \ldots, \\
(3.2) & \quad \text{for any } S_i \in \mathcal{E}_a \text{ we have } \sum_{S_j \in \mathcal{E}_a} p_{ij}^{(n)} = \sum_{S_j \in \mathcal{E}_a} P_{ij} = 1, \quad n = 1, 2, \ldots, \\
(3.3) & \quad \text{for any } S_i \in \mathcal{E}_a, S_j \in \mathcal{E}_a \text{ we have } p_{ij}^{(n)} = P_{ij} = 0, \quad n = 1, 2, \ldots, \\
(3.4) & \quad \text{for any } S_i, S_j \in \mathcal{E}_a \text{ we have } P_{ij} > 0, \\
(3.5) & \quad \text{for any } S_i, S_j \in \mathcal{E}_a \text{ there exists a positive integer } n \text{ such that } p_{ij}^{(n)} > 0, \\
(3.6) & \quad \text{for any } S_i, S_j \in \mathcal{E}_a \text{ } P_{ij} \text{ is independent of } i.
\end{align*}
\]

Proof. Let \( \mathcal{D} \) be defined as the totality of all the states \( S_j \) such that \( P_{ij} = 0 \) for \( i = 1, 2, \ldots \). In order to prove that \( \mathcal{R} - \mathcal{D} \) is decomposed into ergodic parts \( \mathcal{E}_a \), which are mutually disjoint, we shall first show that

\[
\begin{align*}
(3.7) & \quad \text{for any } S_j, S_k \in \mathcal{R} - \mathcal{D} \text{ } P_{jk} > 0 \text{ implies } P_{kj} > 0.
\end{align*}
\]

For this purpose, assume that \( P_{jk} > 0 \) and \( P_{kj} = 0 \), and denote by \( \mathcal{A}_k \) the totality, inclusive \( S_k \), of all the states \( S_a \) such that \( P_{ka} > 0 \). We
have $S_i \in \mathcal{K}_k$, and $\mathcal{K}_k$ is closed in the sense that $P_{ss} = 0$ for any $S_s \in \mathcal{K}_k$ and $S_s \in \mathcal{K}_k$. For, by (1.6), $P_{kk} > 0$ and $P_{ss} > 0$ imply $P_{kk} > 0$. Hence, again by (1.6), we have for any $i$

$$
\sum_{S_s \in \mathcal{K}_k} P_{is} = \sum_{S_s \in \mathcal{K}_k} \sum_{a=1}^{\infty} P_{ia} \cdot P_{as} = \sum_{S_s \in \mathcal{K}_k} \sum_{a=1}^{\infty} P_{is} \cdot P_{as} \leq \sum_{S_s \in \mathcal{K}_k} P_{is} + P_{ij} \cdot (1 - P_{jk})
$$

which leads to a contradiction, since $P_{jk} > 0$ and since there exists, by assumption, a state $S_i$ with $P_{ij} > 0$. Hence (3.7) is proved. Consequently, for any two states $S_j$, $S_k \in \mathcal{K} - \mathfrak{D}$, we have either $P_{jk} > 0$, $P_{kj} > 0$ simultaneously or $P_{jk} = P_{kj} = 0$. In the first case, we call $S_j$ and $S_k$ to be mutually dependent, and denote this relation by $S_j \sim S_k$. The relation $\sim$ is symmetric and transitive (since, by (1.6), $P_{jk} > 0$ and $P_{kl} > 0$ imply $P_{jl} > 0$), and it will be easily seen that for any $S_j \in \mathcal{K} - \mathfrak{D}$ we have $S_j \sim S_j$. This is clear from the argument above, if there exists a state $S_k \in \mathcal{K} - \mathfrak{D}$ with $P_{jk} > 0$; and if we have $P_{jk} = 0$ for any $S_k$, then we have for any $i$

$$
\sum_{k=1}^{\infty} P_{ik} = \sum_{k=1}^{\infty} \sum_{a=1}^{\infty} P_{ia} \cdot P_{ak} = \sum_{k=1}^{\infty} \sum_{a=1}^{\infty} P_{ia} \cdot P_{ak} = \sum_{k=1}^{\infty} P_{i} - P_{ij} ,
$$

which is a contradiction, since there exists a state $S_i$ with $P_{ij} > 0$. Thus $\mathcal{K} - \mathfrak{D}$ is divided into the classes—the ergodic parts $\mathfrak{C}_a$ ($a = 1, 2, \ldots$, finite or enumerably infinite), which consist of mutually dependent states and each of which satisfies (3.4). Since (3.3) and (3.5) are the direct consequences of (3.2) and (3.4) respectively, we need only prove that these ergodic parts thus obtained have the properties (3.2) and (3.6).

From (1.6) and the definition of $\mathfrak{C}_a$ we have for any $S_i \in \mathfrak{C}_a$

$$
(3.8) \quad \sum_{S_j \in \mathfrak{C}_a} P_{ij} = \sum_{S_j \in \mathfrak{C}_a} \sum_{k=1}^{\infty} P_{ik} \cdot p_{kj}^{(n)} = \sum_{S_j \in \mathfrak{C}_a} \sum_{k=1}^{\infty} P_{ik} \cdot p_{kj}^{(n)} = \sum_{S_k \in \mathfrak{C}_a} \sum_{S_j \in \mathfrak{C}_a} P_{ik} \cdot (\sum_{S_j \in \mathfrak{C}_a} p_{kj}^{(n)}).
$$

Since $\sum_{S_j \in \mathfrak{C}_a} p_{kj}^{(n)} \leq 1$ for any $k$ and since $P_{ik} > 0$ for any $S_i$, $S_k \in \mathfrak{C}_a$, the equality in (3.8) holds only when $\sum_{S_j \in \mathfrak{C}_a} p_{kj}^{(n)} = 1$ for any $k$ with $S_k \in \mathfrak{C}_a$. Thus the first part of (3.2) is proved and the second part of it may be obtained analogously, if we start from the third relation of (1.6): $\sum_{k=1}^{\infty} P_{ik} \cdot P_{kl} = P_{il}$.

Now, in order to prove (3.6), it is sufficient (by (1.6)) to show that if $\{\xi_i\}$ and $\{\eta_i\}$ are the systems of real numbers (defined for any $i$ with $S_i \in \mathfrak{C}_a$) such that

$$
\sum_{S_i \in \mathfrak{C}_a} |\xi_i| < \infty , \quad \sum_{S_i \in \mathfrak{C}_a} |\eta_i| < \infty ,
$$

...
and
\[ \sum_{S_i \in \mathcal{G}_a} \xi_i P_{ij} = \xi_j, \quad \sum_{S_i \in \mathcal{G}_a} \eta_i P_{ij} = \eta_j, \]
then we have \( \xi_i = \sigma \eta_i \) (\( \sigma \) : constant) for any \( i \) with \( S_i \in \mathcal{G}_a \); or, it is sufficient to show that
\[ \sum_{S_i \in \mathcal{G}_a} |\xi_i| < \infty, \quad \sum_{S_i \in \mathcal{G}_a} \xi_i P_{ij} = \xi_j \]
imply \( \xi_i \geq 0 \) for any \( i \), or \( \xi_i \leq 0 \) for any \( i \).

For this purpose, let \( \mathcal{G}_b \) (\( \mathcal{G}_c \)) be the totality of all the states \( S_i \in \mathcal{G}_a \) such that \( \xi_i > 0 \) (\( \xi_i < 0 \)), and assume that both \( \mathcal{G}_b \) and \( \mathcal{G}_c \) are not empty. Then, since \( P_{ij} > 0 \) for any \( S_i, S_j \in \mathcal{G}_a \) by (3.4), we have
\[ \sum_{S_j \in \mathcal{G}_b} \xi_j = \sum_{S_j \in \mathcal{G}_b} \sum_{S_i \in \mathcal{G}_a} \xi_i P_{ij} < \sum_{S_j \in \mathcal{G}_b} \sum_{S_i \in \mathcal{G}_c} \xi_i P_{ij} = \sum_{S_i \in \mathcal{G}_c} \xi_i \left( \sum_{S_j \in \mathcal{G}_c} P_{ij} \right) \]
which is a contradiction, since \( \sum_{S_j \in \mathcal{G}_c} P_{ij} \leq 1 \) for any \( i \).

Thus Theorem 2 is completely proved.

4. Second proof of the existence of the mean sojourn.

Following Kolmogoroff, let \( k^{(n)}_{ij} \) be the probability that the state \( S_i \) is transferred to the state \( S_j \) after the elapse of \( n \) unit-times, without being previously transferred to the state \( S_j \). We have clearly
\[
\begin{align*}
(4.1) & \quad 0 \leq k^{(n)}_{ij} \leq 1, \quad i, j, n = 1, 2, \ldots, \\
(4.2) & \quad \sum_{n=1}^{\infty} k^{(n)}_{ij} = L_{ij} \leq 1, \quad i, j = 1, 2, \ldots, \\
(4.3) & \quad P_{ij}^{(n)} = k^{(1)}_{ij} \cdot P_{ij}^{(n-1)} + k^{(2)}_{ij} \cdot P_{ij}^{(n-2)} + \cdots + k^{(n-1)}_{ij} \cdot P_{ij}^{(1)} + k^{(n)}_{ij}.
\end{align*}
\]
From (4.3) we have easily
\[ q^{(n)}_{ij} = k^{(1)}_{ij} \cdot n^{-1} \cdot q^{(n-1)}_{ij} + k^{(2)}_{ij} \cdot n^{-2} \cdot q^{(n-2)}_{ij} + \cdots + k^{(n-1)}_{ij} \cdot n^{-1} \cdot q^{(1)}_{ij} + \frac{1}{n} \sum_{m=1}^{n} k^{(m)}_{ij}, \]
and consequently, by Toeplitz's summation, the existence of \( \lim_{n \to \infty} q^{(n)}_{ij} = P_{jj} \)
implies that of \( \lim_{n \to \infty} q^{(n)}_{ij} = L_{ij} \cdot P_{jj} \), so that we need only prove the existence of the limit (1.5) in the case \( i = j \).

For this purpose, consider two power series:
\[
\begin{align*}
(4.4) & \quad P_{jj}(z) = p^{(1)}_{jj} z + p^{(2)}_{jj} z^2 + \cdots + p^{(n)}_{jj} z^n + \cdots, \\
(4.5) & \quad K_{jj}(z) = k^{(1)}_{jj} z + k^{(2)}_{jj} z^2 + \cdots + k^{(n)}_{jj} z^n + \cdots.
\end{align*}
\]
These are convergent in $|z| < 1$ ((4.5) converges uniformly even in $|z| \leq 1$ by (4.2)), and by virtue of (4.3) ($i = j$), we have the relation:

\[(4.6) \quad 1 + P_{ij}(z) = \frac{1}{1 - K_{ij}(z)}, \quad |z| < 1.\]

The further arguments are divided into two cases:

**1st case**: $L_{ij} = 1$. Put $M_{ij} = \sum_{n=1}^{\infty} n \cdot k_{ij}^{(n)}$. $M_{ij}$ might become infinite; in this case put $\frac{1}{M_{ij}} = 0$. Then we have from (4.6)

\[(4.7) \quad \lim_{z \to 1-0} (1-z)(1+P_{ij}(z)) = \lim_{z \to 1-0} \frac{1 - z}{1 - K_{ij}(z)} = \lim_{z \to 1-0} \frac{1}{\sum_{n=1}^{\infty} k_{ij}^{(n)}(1 + z + \cdots + z^{n-1})} = \frac{1}{M_{ij}}\]

if we let $z$ tend to 1 along the real axis. Since $\{p_{ij}^{(n)}\}$ is uniformly bounded, we deduce from (4.7), by a well-known argument, that (2)

\[(4.8) \quad \lim_{n \to \infty} q_{ij}^{(n)} = \lim_{n \to \infty} \frac{1}{n} (p_{ij}^{(1)} + p_{ij}^{(2)} + \cdots + p_{ij}^{(n)}) = \frac{1}{M_{ij}}\]

Thus we have proved the existence of (1.5) and obtained the additional relation:

\[(4.9) \quad P_{ij} = \frac{1}{M_{ij}}.\]

**2nd case**: $L_{ij} < 1$. This case is easier to discuss. Making $z$ tend to 1 along the real axis in (4.6), we have

\[(4.10) \quad 1 + \sum_{n=1}^{\infty} p_{ij}^{(n)} = \frac{1}{1 - L_{ij}} < \infty .\]

Hence, a fortiori, $p_{ij}^{(n)}$ and $q_{ij}^{(n)}$ tend to 0 as $n \to \infty$.

5. **Construction of a Markoff process with given** $\{k_{ij}^{(n)}\}$.

If we consider the sequence of real numbers $\{p_{ij}^{(n)}\}$, $n = 1, 2, \ldots$, where $p_{ij}^{(n)}$ is defined by (1.1), $\{p_{ij}^{(n)}\}$ clearly satisfies the relations:

\[(5.1) \quad 0 \leq p_{ij}^{(n)} \leq 1, \quad n = 1, 2, \ldots,\]

\[(5.2) \quad p_{ij}^{(m+n)} \geq p_{ij}^{(m)} \cdot p_{ij}^{(n)}, \quad m, n = 1, 2, \ldots ,\]

(2) See, for example, N. Wiener: The Fourier integral and certain of its applications, Cambridge, 1933, p. 105-106.
(5.2) follows from (1.4)). These are the necessary conditions that \(\{p^{(n)}_{ij}\}\) should satisfy, and it will be easily seen that these are not sufficient; that is, it is not always possible to construct a simple Markoff process \(P = (p_{ij})\), \(i, j = 1, 2, \ldots\), for which \(\{p^{(n)}_{ij}\}\) is the given one. On the contrary, however, if we consider the sequence of real numbers \(\{k^{(n)}_{ii}\}\), which satisfies the conditions (4.1) and (4.2) \((i = j = 1)\), then it is always possible to construct a simple Markoff process \(P = (p_{ij})\), \(i, j = 1, 2, \ldots\), for which \(\{k^{(n)}_{ii}\}\) (which is defined by the way given at the beginning of \S\ 4) is exactly the given one. This may be done as follows:

\[P = \begin{bmatrix}
  k^{(1)}_{11} & k^{(2)}_{11} & k^{(3)}_{11} & 0 & k^{(4)}_{11} & 0 & 0 & 0 & \cdots & k^{(n)}_{11} & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \end{bmatrix}_{n-1}
\]

1st case: \(L_{11} = 1\). Define the matrix \(P = (p_{ij})\), \(i, j = 1, 2, \ldots\), as follows:
2nd case: \( L_{II} < 1 \). Put \( k_{II} = 1 - L_{II} \) and define the matrix \( P \), whose rows and columns are both of type \( \omega + \omega \), as follows:

\[
P = \begin{pmatrix}
    k_{II} & 0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots \\
    0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

where \( K \) is a matrix whose rows and columns are of type \( \omega \) and which is defined in just the same manner as in the 1st case.

It is to be remarked that our problem is in general impossible if the number of possible states is finite.

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