6. On the measure of section of the integral curves.

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1. Let a system of differential equations ;

$$\left. \begin{aligned} \frac{dy_1}{dx} &= f_1(x, y_1, y_2, \dots, y_n) \\ &\dots\dots\dots \\ \frac{dy_n}{dx} &= f_n(x, y_1, y_2, \dots, y_n) \end{aligned} \right\} \quad (1)$$

be given, where $f_\nu (\nu = 1, 2, \dots, n)$ are continuous and $|f_\nu| \leq M$ for $0 \leq x \leq a$, $|y_i| \leq b$ ($i = 1, 2, \dots, n$), then by Peano, there exists at least one solution $y_1(x), \dots, y_n(x)$, which passes through the origin $(x, y_1, \dots, y_n) = (0, 0, \dots, 0)$ and is continuous in $0 \leq x \leq a$, where $a = \text{Min.} \left(a, \frac{b}{M} \right)$. But there may exist several such solutions. We denote the set of points in $(x, y_1, y_2, \dots, y_n)$ -space, which is covered by such solutions by $\mathfrak{M}$ and its section with a plane $x = \text{const.}$ ($= x$) by $\mathfrak{M}_x$. Kneser⁽¹⁾ proved that $\mathfrak{M}_x$ is a continuum for $0 \leq x \leq a$.

Let E be any closed set of points on a plane $x = \text{const.}$ We denote its n -dimensional Lebesgue measure by $|E|$. In this paper we investigate the case where $|\mathfrak{M}_x| = 0$. First we will prove the theorem:

Theorem I. *If in (1), $f_\nu(x, y_1, \dots, y_n)$ ($\nu = 1, 2, \dots, n$) have continuous partial derivatives of the first order with respect to $y_1, y_2, \dots, y_n$ for $0 < x \leq a$, $|y_i| \leq b$ ($i = 1, 2, \dots, n$) and there*

$$\sum_{\nu=1}^n \frac{\partial f_\nu}{\partial y_\nu} \leq K \quad (K = \text{const.}) \quad (2)$$

or more generally

$$\sum_{\nu=1}^n \frac{\partial f_\nu}{\partial y_\nu} \leq \frac{n}{x} \quad (3)$$

(1) H. Kneser. Über die Lösungen eines Systems gewöhnlicher Differentialgleichungen, das der Lipschitzschen Bedingung nicht genügt. (Sitzungsberichte der Preussischen Akademie der Wissenschaften, (1923)).

then for $0 \leq x \leq a$

$$|\mathfrak{M}_x| = 0.$$

Proof. Let $\varphi_1(x, \eta_1, \dots, \eta_n), \dots, \varphi_n(x, \eta_1, \dots, \eta_n)$ be a set of solution of (1), which passes through the point $(x_0, \eta_1, \dots, \eta_n)$ ($x_0 > 0$) and put

$$J = \begin{vmatrix} \frac{\partial \varphi_1}{\partial \eta_1} & \dots & \frac{\partial \varphi_n}{\partial \eta_1} \\ \dots & \dots & \dots \\ \frac{\partial \varphi_1}{\partial \eta_n} & \dots & \frac{\partial \varphi_n}{\partial \eta_n} \end{vmatrix}.$$

By Lindelöf's theorem⁽²⁾, $\frac{\partial \varphi_i}{\partial \eta_j} (i, j = 1, 2, \dots, n)$ are continuous and

$$J = e^{\int_{x_0}^x \left(\sum_{\nu=1}^n \frac{\partial f_\nu}{\partial y_\nu} \right) dx}$$

where

$$\frac{\partial f_\nu}{\partial y_\nu} = \left[\frac{\partial f_\nu(x, y_1, \dots, y_n)}{\partial y_\nu} \right]_{y_i = \varphi_i(x, \eta_1, \dots, \eta_n)} \quad (\nu = 1, 2, \dots, n).$$

If we assume (3),

$$J \leq e^{\int_{x_0}^x \frac{n}{x} dx} = \left(\frac{x}{x_0} \right)^n \quad (x_0 \leq x \leq a) \quad (4)$$

Since the segments which connect the origin to the points of $\mathfrak{M}_{x_0}$ tend uniformly to a definite direction as $x_0 \rightarrow 0$, we can take x_0 so small and an n -dimensional hypercube $\mathcal{Q}_{x_0}$ on a plane $x = x_0$, that $\mathcal{Q}_{x_0}$ contains $\mathfrak{M}_{x_0}$ and

$$\frac{|\mathcal{Q}_{x_0}|}{x_0^n} < \sigma \quad (5)$$

where σ is any assigned small quantity.

Consider the set of the integral curves of (1), passing through the points of $\mathcal{Q}_{x_0}$ and let its section with a plane $x = \text{const.}$ ($= x$) be denoted

(2) Journal de mathématique pures et appliquées (5) **6** (1900) or E. Kamke, Differentialgleichungen reeller Funktionen, p. 155.

by Ω_x . Evidently Ω_x contains $\mathfrak{M}_x$ and since, $\frac{\partial \varphi_i}{\partial \eta_j}$ ($i, j = 1, 2, \dots, n$) being continuous,

$$|\Omega_x| = \int_{\Omega_{x_0}} |J| d\eta_1 \dots d\eta_n,$$

we get from (4), (5),

$$|\mathfrak{M}_x| \leq |\Omega_x| = \int_{\Omega_{x_0}} |J| d\eta_1 \dots d\eta_n \leq |\Omega_{x_0}| \left(\frac{x}{x_0} \right)^n < \sigma x^n. \quad (6)$$

Since σ can be taken arbitrarily small, we have

$$|\mathfrak{M}_x| = 0 \quad (0 \leq x \leq a). \quad \text{q. e. d.}$$

2. To prove another theorem we need the following lemmas.

Lemma 1. *Let in (1),*

$$|f_\nu(x, Y_1, Y_2, \dots, Y_n) - f_\nu(x, y_1, y_2, \dots, y_n)| \leq N \sum_{i=1}^n |Y_i - y_i| \quad (\nu = 1, 2, \dots, n) \quad (7)$$

for $0 < x_0 \leq x \leq a$, $|y_i| \leq b$, $|Y_i| \leq b$ ($i = 1, 2, \dots, n$), where N is a constant and let $\{y_1(x), \dots, y_n(x)\}$, $\{\bar{y}_1(x), \dots, \bar{y}_n(x)\}$ be two sets of solutions of (1), which pass through the points

$$(x_0, y_1(x_0), \dots, y_n(x_0)), \quad (x_0, \bar{y}_1(x_0), \dots, \bar{y}_n(x_0))$$

$$\text{then} \quad \tau_1 \leq \tau_0 e^{Nn(x_1 - x_0)} \quad (x_0 \leq x_1), \quad (8)$$

where

$$\tau_0^2 = \sum_{i=1}^n (\bar{y}_i(x_0) - y_i(x_0))^2, \quad \tau_1^2 = \sum_{i=1}^n (\bar{y}_i(x_1) - y_i(x_1))^2.$$

Proof. We rewrite (1) in the form;

$$\left. \begin{aligned} dy_1 &= f_1(x, y_1, \dots, y_n) dx \\ &\dots\dots\dots \\ dy_n &= f_n(x, y_1, \dots, y_n) dx \end{aligned} \right\} \quad (9)$$

and put

$$\tau^2(x) = \sum_{i=1}^n (\bar{y}_i(x) - y_i(x))^2,$$

then we have, by the infinitesimal transformation (9),

$$\begin{aligned}
|\tau d\tau| &= \left| \sum_{i=1}^n (\bar{y}_i(x_n) - y_i(x)) (f_i(x, \bar{y}_1, \dots, \bar{y}_n) - f_i(x, y_1, \dots, y_n)) dx \right| \\
&\leq N \sum_{i=1}^n |\bar{y}_i(x) - y_i(x)| \sum_{i=1}^n |\bar{y}_i(x) - y_i(x)| dx \\
&= N \left(\sum_{i=1}^n |\bar{y}_i(x) - y_i(x)| \right)^2 dx \leq Nn \sum_{i=1}^n (\bar{y}_i(x) - y_i(x))^2 dx = Nn\tau^2(x) dx, \\
\left| \frac{d\tau}{\tau} \right| &\leq Nn dx.
\end{aligned}$$

Hence

$$\log \frac{\tau_1}{\tau_0} = \int_{\tau_0}^{\tau_1} \frac{d\tau}{\tau} \leq \int_{x_0}^{x_1} \left| \frac{d\tau}{\tau} \right| \leq Nn(x_1 - x_0),$$

$$\tau_1 \leq \tau_0 e^{Nn(x_1 - x_0)}. \quad \text{q. e. d.}$$

Lemma 2. Let $f(x_1, x_2, \dots, x_n)$ be a continuous function of $x_1, x_2, \dots, x_n$ in $a_i \leq x_i \leq b_i$ ($i = 1, 2, \dots, n$) and

$$|f(x_1 + h, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)| \leq K|h| \quad (K = \text{const.}) \quad (10)$$

then there exists a continuous function $g(x_1, x_2, \dots, x_n)$, which has continuous partial derivatives of the first order with respect to $x_1, x_2, \dots, x_n$, such that for any $\varepsilon > 0$,

$$|f(x_1, x_2, \dots, x_n) - g(x_1, x_2, \dots, x_n)| < \varepsilon \quad (11)$$

uniformly for $a_i + \delta \leq x_i \leq b_i - \delta$ ($\delta > 0$) ($i = 1, 2, \dots, n$) and

$$\left| \frac{\partial g(x_1, x_2, \dots, x_n)}{\partial x_1} \right| \leq K. \quad (12)$$

Proof. For $a_i + \delta \leq x_i \leq b_i - \delta$ ($i = 1, 2, \dots, n$), $0 < r \leq \delta$, we put

$$g(x_1, x_2, \dots, x_n) = \frac{1}{(2r)^n} \int_{-r}^r \dots \int_{-r}^r f(x_1 + \xi_1, x_2 + \xi_2, \dots, x_n + \xi_n) d\xi_1 \dots d\xi_n. \quad (13)$$

It is evident that $g(x_1, x_2, \dots, x_n)$ is continuous and for $r \rightarrow 0$, $g(x_1, x_2, \dots, x_n) \rightarrow f(x_1, x_2, \dots, x_n)$ uniformly in $a_i + \delta \leq x_i \leq b_i - \delta$, so that for a small r , $g(x_1, x_2, \dots, x_n)$ satisfies the condition (11).

Next

$$\begin{aligned}
 g(x_1+h, x_2, \dots, x_n) - g(x_1, x_2, \dots, x_n) &= \\
 &= \frac{1}{(2r)^n} \int_{-r}^r \dots \int_{-r}^r d\xi_2 \dots d\xi_n \left[\int_{-r}^r \left(f(x_1+h+\xi_1, x_2+\xi_2, \dots, x_n+\xi_n) \right. \right. \\
 &\quad \left. \left. - f(x_1+\xi_1, x_2+\xi_2, \dots, x_n+\xi_n) \right) d\xi_1 \right] \\
 &= \frac{1}{(2r)^n} \int_{-r}^r \dots \int_{-r}^r d\xi_2 \dots d\xi_n \int_0^h \left(f(x_1+r+\xi_1, x_2+\xi_2, \dots, x_n+\xi_n) \right. \\
 &\quad \left. - f(x_1-r+\xi_1, x_2+\xi_2, \dots, x_n+\xi_n) \right) d\xi_1,
 \end{aligned}$$

so that

$$\begin{aligned}
 \frac{\partial g}{\partial x_1} &= \frac{1}{(2r)^n} \int_{-r}^r \dots \int_{-r}^r \left(f(x_1+r, x_2+\xi_2, \dots, x_n+\xi_n) \right. \\
 &\quad \left. - f(x_1-r, x_2+\xi_2, \dots, x_n+\xi_n) \right) d\xi_2 \dots d\xi_n. \quad (14)
 \end{aligned}$$

$\frac{\partial g}{\partial x_1}$ is evidently continuous. Similarly $\frac{\partial g}{\partial x_i}$ ($i=1, 2, \dots, n$) are continuous.

If (10) holds, we see easily from (14), that $\left| \frac{\partial g}{\partial x_1} \right| \leq K$. q. e. d.

3. Now we are able to prove the following theorem.

Theorem II. *If in (1),*

$$|f_\nu(x, Y_1, \dots, Y_n) - f_\nu(x, y_1, \dots, y_n)| \leq K |Y_\nu - y_\nu| + M(x_0) \sum_{i \neq \nu}^1, \dots, n |Y_i - y_i| \quad (15)$$

or more generally

$$|f_\nu(x, Y_1, \dots, Y_n) - f_\nu(x, y_1, \dots, y_n)| \leq \frac{1}{x} |Y_\nu - y_\nu| + M(x_0) \sum_{i \neq \nu}^1, \dots, n |Y_i - y_i| \quad (16)$$

for $0 < x_0 \leq x \leq a$, $|y_i| \leq b$, $|Y_i| \leq b$ ($\nu, i = 1, 2, \dots, n$), where K is a constant and $M(x_0)$ depends on x_0 only, which may tend to infinity in any manner for $x_0 \rightarrow 0$. Then for $0 \leq x \leq a$

$$|\mathfrak{M}_x| = 0.$$

Proof. By Lemma 2, we can get $g_\nu(x, y_1, \dots, y_n)$ ($\nu = 1, 2, \dots, n$) which is continuous and has continuous partial derivatives of the first order with respect to $y_1, \dots, y_n$ and for

$$0 < x_0 \leq x \leq \alpha - \delta, \quad |y_i| \leq b - \delta \quad (\delta > 0) \quad (i = 1, 2, \dots, n),$$

$$|f_\nu(x, y_1, \dots, y_n) - g_\nu(x, y_1, \dots, y_n)| < \varepsilon, \quad (17)$$

$$\left| \frac{\partial g_\nu}{\partial y_\nu} \right| \leq \frac{1}{x} \quad (\nu = 1, 2, \dots, n). \quad (18)$$

We consider a system of differential equations;

$$\left. \begin{aligned} \frac{dY_1}{dx} &= g_1(x, Y_1, Y_2, \dots, Y_n) \\ &\dots\dots\dots \\ \frac{dY_n}{dx} &= g_n(x, Y_1, Y_2, \dots, Y_n) \end{aligned} \right\} \quad (19)$$

Let $\{y_1(x), \dots, y_n(x)\}$, $\{Y_1(x), \dots, Y_n(x)\}$ be the sets of solutions of (1) and (19) respectively, which pass through the same point on the plane $x = x_0$. Since f_ν ($\nu = 1, 2, \dots, n$) satisfy Lipschitz condition for $x_0 \leq x \leq \alpha$, we have from (17),

$$|Y_i(x) - y_i(x)| < \delta \varepsilon \quad (i = 1, 2, \dots, n) \quad (20)$$

uniformly for $x_0 \leq x \leq \alpha - \delta$, where $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

As in the proof of the theorem I, we take $x_0 > 0$ and an n -dimensional hypercube $\mathcal{Q}_{x_0}$ on the plane $x = x_0$, such that

$$\frac{|\mathcal{Q}_{x_0}|}{x_0^n} < \sigma. \quad (21)$$

Consider the sets of integral curves of (1) and (19), passing through the points of $\mathcal{Q}_{x_0}$ and let their sections with a plane $x = \text{const. } (=x)$, ($x_0 \leq x \leq \alpha - \delta$), be denoted by $\mathcal{Q}_x$ and $\bar{\mathcal{Q}}_x$ respectively.

The boundary of $\mathcal{Q}_{x_0}$ can be covered by a finite number of n -dimensional hyperspheres $K_1, \dots, K_k$ of radii $r_1, \dots, r_k$ such that $\sum_{i=1}^k r_i^n$ is arbitrarily small. Let K'_i be the transform of K_i on the plane $x = \text{const. } (=x)$, then by lemma 1, K'_i can be enclosed into an n -dimensional hypersphere of radius $r_i e^{Nn(x-x_0)}$, where $N = \text{Max.} \left(\frac{1}{x_0}, M(x_0) \right)$.

Hence the boundary of Ω_x can be enclosed in a domain whose n -dimensional measure can be made arbitrarily small. Moreover by (20), the distance of the points of Ω_x and $\bar{\Omega}_x$, which correspond to the same point of Ω_{x_0} tends to zero as $\varepsilon \rightarrow 0$. So we have

$$|\bar{\Omega}_x| \rightarrow |\Omega_x| \quad \text{as } \varepsilon \rightarrow 0. \quad (22)$$

Similarly as (6),

$$\begin{aligned} |\bar{\Omega}_x| &= \int_{\Omega_{x_0}} |J| d\eta_1 \dots d\eta_n = \int_{\Omega_{x_0}} e^{\int_{x_0}^x \left(\sum_{\nu=1}^n \frac{\partial g_\nu}{\partial y_\nu} \right) dx} d\eta_1 \dots d\eta_n \\ &\leq \int_{\Omega_{x_0}} e^{n \int_{x_0}^x \frac{dx}{x}} d\eta_1 \dots d\eta_n = |\Omega_{x_0}| \left(\frac{x}{x_0} \right)^n \leq \sigma x^n. \end{aligned}$$

Here we make $\varepsilon \rightarrow 0$ and from (22) we have for $x_0 \leq x \leq \alpha - \delta$

$$|\Omega_x| \leq \sigma x^n. \quad (23)$$

Since, similarly as we have deduced (22), $|\Omega_{\alpha-\delta}| \rightarrow |\Omega_\alpha|$ for $\delta \rightarrow 0$, (23) holds good for $x_0 \leq x \leq \alpha$, so that

$$|\mathfrak{N}_x| \leq |\Omega_x| \leq \sigma x^n \quad (x_0 \leq x \leq \alpha). \quad (24)$$

Since σ is arbitrary, we have

$$|\mathfrak{N}_x| = 0 \quad (0 \leq x \leq \alpha) \quad \text{q. e. d.}$$

Similarly we can prove the following theorem which is a generalisation of Theorem I.

Theorem III. *If in (1),*

$$|f_\nu(x, Y_1, \dots, Y_n) - f_\nu(x, y_1, \dots, y_n)| \leq M(x_0) \sum_{i=1}^n |Y_i - y_i|$$

for $0 < x_0 \leq x \leq \alpha$, $|y_i| \leq b$, $|Y_i| \leq b$ ($\nu, i = 1, 2, \dots, n$), where $M(x_0)$ depends on x_0 only, and for positive h

$$\begin{aligned} \sum_{\nu=1}^n (f_\nu(x, y_1, \dots, y_{\nu-1}, y_\nu + h, y_{\nu+1}, \dots, y_n) \\ - f_\nu(x, y_1, \dots, y_{\nu-1}, y_\nu, y_{\nu+1}, \dots, y_n)) \leq Kh \quad (K = \text{const.}) \end{aligned} \quad (25)$$

or more generally

$$\sum_{\nu=1}^n (f_{\nu}(x, y_1, \dots, y_{\nu-1}, y_{\nu} + h, y_{\nu+1}, \dots, y_n) - f_{\nu}(x, y_1, \dots, y_{\nu-1}, y_{\nu}, y_{\nu+1}, \dots, y_n)) \leq \frac{n}{x} h \quad (26)$$

then for $0 \leq x \leq a$

$$|\mathfrak{M}_x| = 0 .$$

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