8. Some characterizations of Euclidean space.

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1. Introduction and the main theorems. The object of the present paper is to give some characterizations of Euclidean space among Banach spaces. This will be done from the standpoint of the theory of extension of linear functionals and linear operations.

Let E be a Banach space and G a closed linear subspace of E . By a well-known theorem of Banach⁽¹⁾, every bounded linear functional defined on G can be extended to a bounded linear functional defined on E without the increase of its norm; i.e., for any bounded linear functional $f(x)$ defined on G , there exists a bounded linear functional $F(x)$ defined on E such that $F(x) = f(x)$ for any $x \in G$ and $\|F\|_E = \|f\|_G$, where we denote by $\|F\|_E = \text{l. u. b. } \|F(x)\|$ and $\|f\|_G = \text{l. u. b. } \|f(x)\|$ the norm of linear functionals $F(x)$ and $f(x)$ respectively.

Concerning this theorem two problems arise:

(I) to determine whether it is possible to extend every bounded linear operator $u(x)$, which maps G into another Banach space E^* , to a bounded linear operator $U(x)$, which maps E into E^* and which is of the same norm with $u(x)$: $\|U\|_E = \|u\|_G$, where we denote by $\|U\|_E = \text{l. u. b. } \|U(x)\|$ and $\|u\|_G = \text{l. u. b. } \|u(x)\|$ the norm of linear operators $U(x)$ and $u(x)$ respectively.

(II) to determine whether it is possible to extend all bounded linear functionals $f(x)$ defined on G to bounded linear functionals $F(x)$ defined on E , in such a way that this operation of extension becomes linear and isometric, i.e., (denoting this operation of extension by $f \rightarrow F$) $f \rightarrow F$ and $g \rightarrow G$ imply $f+g \rightarrow F+G$, $af \rightarrow aF$ and $\|F\|_E = \|f\|_G$.

The first problem (I) was discussed by F. J. Murray⁽²⁾ and F. Bohnenblust⁽³⁾, and the second one (II) was investigated by L. Kantor-

⁽¹⁾ S. Banach: *Théorie des opérations linéaires*, Warszawa, 1933, p. 55, Théorème 3.

⁽²⁾ F. J. Murray: On complementary manifolds in space L_p and l_p , *Trans. Amer. Math. Soc.*, **41** (1937), 138-152.

⁽³⁾ F. Bohnenblust: Convex regions and projections in Minkowski spaces *Annals of Math.*, **39** (1938), 301-308.

vitch⁽⁴⁾. In both cases no decisive conclusion was given. In the present paper we shall give a solution to these problems. We shall, indeed, prove the following theorems:

Theorem 1. *Let E be a Banach space. In order that the problem (I) is possible for any closed linear subspace G of E and for any bounded linear operator which maps G into another Banach space E^* (which is also arbitrary), it is necessary and sufficient that E is unitary.*

Theorem 2. *Let E be a Banach space. In order that the problem (II) is possible for any closed linear subspace G of E , it is necessary and sufficient that E is unitary.*

A unitary Banach space is, by definition, a Banach space E , whose norm satisfies the relation: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for any x and $y \in E$. Such a Banach space is called sometimes general Euclidean (allgemein euklidisch). The separability is not assumed. In order that a Banach space E is unitary, it is necessary and sufficient that every finite-dimensional subspace G of E is Euclidean.

The proofs of Theorems 1 and 2 will be given in §§ 2 and 3 respectively. Both proofs are quite elementary, and it is to be noted that we have in essential only to deal with finite dimensional Banach spaces (Minkowski spaces) especially those of three dimensions. In § 4 we shall discuss the quasi-convex Banach space, which was introduced by L. Kantorovitch⁽⁴⁾ in connection with the problem (II), and it will be shown that a quasi-convex Banach space is nothing but a unitary Banach space.

2. Proof of Theorem 1. It is well-known that the problem (I) is in general impossible; and, even if we substitute the condition of the invariance of norm: $\|U\|_E = \|u\|_G$ by the weaker one: $\|U\|_E < \infty$ (i.e., it is only required that $U(x)$ is a bounded linear extension of $u(x)$), our problem remains impossible in this generalized case. To see this, consider the case when $E^* = G$ and when $u(x)$ is the identical transformation: $u(x) = x$. Then the existence of a linear extension $U(x)$ of $u(x)$ is nothing but the existence of a projection $U(x)$ which maps E on G ($\|U\|_E = 1$ in the case of problem (I) and $\|U\|_E < \infty$ in the generalized case); and, since, as is shown by F. J. Murray⁽²⁾, there exists in (L_p) and (l_p) ($p > 1$) a closed linear subspace to which there exists no projection of finite norm, our problem is impossible even in the generalized form.

⁽⁴⁾ L. Kantorovitch: Sur les prolongement des fonctionnelles linéaires, C.R. Acad. Sci. URSS., 1 (1935), 207-210.

Conversely, if there exists a projection P of E on G of norm 1, then our problem (I) is possible. Indeed, we have only to put $U(x) = u(P(x))$ for any $x \in E$.

Thus our Theorem 1 is transformed into the following

Theorem 3. *In order that there exists, for any closed linear subspace G of E , a projection P of E on G of norm 1, it is necessary and sufficient that E is unitary.*

Since in any unitary Banach space E , there exists for any closed linear subspace G of E a projection of the required property, we have only to prove that the condition of the theorem 3 is necessary. For this purpose we have only to consider the three-dimensional part of E , and we need only to prove

Theorem 4. *Let E be a three-dimensional Minkowski space. If for any two-dimensional subspace G of E there exists a projection of E on G of norm 1, then E is Euclidean.*

Or, transforming our problem in geometrical form,

Theorem 5. *Let K be a closed convex region in the three-dimensional linear space, which is symmetric with respect to its centre 0. If there exists, for any two-dimensional plane π through 0, a direction $d(\pi)$ such that the straight line of direction $d(\pi)$, which goes through any one of the points of intersection of π with the surface of K , touches K (= does not go through the inner part of K), then K is an ellipsoid.*

This Theorem is already known when K is assumed to be smooth, that is, when there exists only one touching plane at each boundary point of K ⁽⁵⁾. The general case may be treated by making slight modifications, if we only observe and make use of the fact that the set of boundary points of K , at which the touching plane is not uniquely determined, is of spherical measure zero⁽⁶⁾. Since this may be easily carried out, we omit the proof.

3. Proof of Theorem 2. The sufficiency of the condition is again almost obvious. Taking a projection P of E on G of norm 1, we have only to put $F(x) = f(P(x))$ for any f and $x \in E$. The operation $f \rightarrow F$ thus obtained is the required one.

To prove the necessity of the condition let us consider again the case when E is a three-dimensional Minkowski space and when G is a two-dimensional linear subspace of E . We need only prove the following

⁽⁵⁾ See, for example, W. Blaschke: *Kreis und Kugel*, p. 160.

⁽⁶⁾ T. Bonnesen und W. Fenchel: *Theorie der konvexen Körper*, p. 13.

Theorem 6. *If the problem (II) is possible for a three-dimensional Minkowski space E and a two-dimensional linear subspace G of E , then there exists a projection of E on G of norm 1.*

Proof. Without the loss of generality, we may assume that G is a (ξ, η) -plane in a (ξ, η, ζ) -space E . Each bounded linear functional $f(x)$ defined on G is of the form: $f(x) = a\xi + b\eta$, $x = (\xi, \eta, 0)$, where a and b are constants; and its extension $F(x)$ will be given by $F(x) = a\xi + b\eta + c\zeta$, $x = (\xi, \eta, \zeta)$, where c is a constant which depends only on a and b : $c = \varphi(a, b)$. Since the operation $f \rightarrow F$ is linear, we have $\varphi(a+a', b+b') = \varphi(a, b) + \varphi(a', b')$ and $\varphi(aa', ab) = a\varphi(a, b)$, where a is a real number. Hence we have $\varphi(a, b) = \varphi(a, 0) + \varphi(0, b) = a \cdot \varphi(1, 0) + b \cdot \varphi(0, 1) = aa_0 + bb_0$, where $a_0 = \varphi(1, 0)$, $b_0 = \varphi(0, 1)$, and consequently $F(x) = a\xi + b\eta + (aa_0 + bb_0)\zeta = a(\xi + a_0\zeta) + b(\eta + b_0\zeta)$. We shall now prove that the linear operation $x = (\xi, \eta, \zeta) \rightarrow P(x) = (\xi + a_0\zeta, \eta + b_0\zeta, 0)$ is a projection operator of norm 1 which maps E on G . Since it is clear that P is a projection, we have only to prove that it is of norm 1.

This may be done as follows:

For any $x_0 = (\xi_0, \eta_0, \zeta_0)$, take a linear functional $f(x) = a\xi + b\eta$ defined on G such that $\|f\|_G = 1$ and $f(P(x_0)) = \|P(x_0)\|$. Let the extension of this $f(x)$ be $F(x)$. $F(x)$ clearly satisfies the relation: $F(x) = f(P(x))$ for any $x \in E$. For these $f(x)$ and $F(x)$ we have $\|P(x_0)\| = f(P(x_0)) = F(x_0) \leq \|F\|_E \cdot \|x_0\|$, and since $\|F\|_E = \|f\|_G = 1$ we have $\|P(x_0)\| \leq \|x_0\|$, which is what we have wanted to prove.

4. Quasi-convex Banach spaces. In order to discuss the second problem (II), L. Kantorovitch⁽⁴⁾ has introduced the notion of quasi-convex Banach spaces. By his definition, a Banach space is called to be *quasi-convex* if there exists, for any finite number of points $x_1, x_2, \dots, x_n$ and $x_0 \in E$, a point $\bar{x}_0 \in E$ of the form $\bar{x}_0 = c_1x_1 + c_2x_2 + \dots + c_nx_n$ ($c_1, c_2, \dots, c_n$: real constants) such that $\|x_i - \bar{x}_0\| \leq \|x_i - x_0\|$ for $i = 1, 2, \dots, n$. He has proved that in order that the problem (II) is possible for any closed linear subspace G of E , it is sufficient that E is quasi-convex. His idea is surely interesting and his proof is very elegant; but, on looking precisely into the properties of the quasi-convex Banach spaces, it has turned out that a quasi-convex Banach space is nothing but a unitary Banach space. This is also clear from the result of Theorem 2. We shall give a direct proof in the following Theorem 7. Thus the result of L. Kantorovitch says only that in unitary Banach spaces the problem (II) is possible, and this is clearly a triviality.

Theorem 7. *In order that a Banach space E is quasi-convex, it is necessary and sufficient that E is unitary.*

Proof. Since it is clear that the condition of the theorem is sufficient, we have only to prove the necessity of the condition. For this purpose it is once more sufficient to consider the case when E is a three-dimensional Minkowski space. Let E be a three-dimensional quasi-convex Minkowski space and let G be a two-dimensional linear subspace of E . We shall prove that there exists a projection of norm 1 which maps E on G . Let $x_1, x_2, \dots, x_n$ be an arbitrary system of points of G and let x_0 be a point of $E-G$. By assumption, there exists a point $\bar{x}_0 \in G$ such that $\|x_i - \bar{x}_0\| \leq \|x_i - x_0\|$ for $i = 1, 2, \dots, n$. In other words, denoting, for any $x \in G$, by $K(x)$ the set of all the points $\bar{x} \in G$ such that $\|x - \bar{x}\| \leq \|x - x_0\|$ (x_0 being fixed), we have $\prod_{i=1}^n K(x_i) \neq \emptyset$ for any finite number of points $x_1, x_2, \dots, x_n \in G$. Consequently, since each $K(x)$ is bounded and closed, all $K(x)$, $x \in G$, have a common point⁽⁷⁾. (That this point is uniquely determined does not follow at once). Let one of these common points be $\bar{x}_0$. Since any point $x \in E$ is uniquely expressible in the form: $\bar{x} = x_1 + \alpha x_0$, where α is a real number and x_1 is a point of G , the mapping $x \equiv x_1 + \alpha x_0 \rightarrow T(x) \equiv x_1 + \alpha \bar{x}_0$ is a projection of E on G . That this T is of norm 1, may be seen from the following relation:

$$\|T(x)\| = \|x_1 + \alpha \bar{x}_0\| = \left\| \alpha \left\| \frac{x_1}{\alpha} - \bar{x}_0 \right\| \right\| \leq \left\| \alpha \left\| \frac{x_1}{\alpha} - x_0 \right\| \right\| = \|x_1 + \alpha x_0\| = \|x\|.$$

Thus we have proved that there exists a projection of norm 1 which maps E on G . Since G is an arbitrary two-dimensional linear subspace of E , we see from Theorem 4 that E is Euclidean, and hereby the proof of Theorem 7 is completed.

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(7) If we require, in the definition of quasi-convexity, that $x_1, x_2, \dots, x_n$ are mutually linearly independent, then we have $\prod_{i=1}^n K(x_i) \neq \emptyset$ only for $n \leq 3$. But using the result of Helly (Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, Jahresber. d.D.M.V., 32 (1923), 175-176), we can deduce from this that $\prod_{x \in G} K(x) \neq \emptyset$.