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Introduction.

In the range of the theory of Bohr almost periodic Fourier series, we have to-day two important problems not yet completely solved. One problem is to find the necessary and sufficient condition that a given trigonometric series $\sum A_n e^{i\lambda_n x}$ represents a Bohr almost periodic Fourier series and the other is the decision of the existence or non-existence of some inner conditions under which the partial sums of a given Bohr almost periodic Fourier series or function may converge in the neighbourhood of a point or in an interval. Now classical Riesz-Fischer theorem due to Besicovitch asserts that the convergence of the series $\sum |A_n|^2$ is the necessary and sufficient condition that a given trigonometric series $\sum A_ne^{i\lambda_n x}$ is $B_2$ almost periodic Fourier series (Besicovitch class with exponent 2). This theorem is a starting point for the research of the first problem. Since then, many efforts have been done with regard to the first problem. Among them, we quote here two results obtained by Paley-Wiener and Bochner-Jessen respectively. Paley-Wiener's theorem$^{(1)}$: Let $\{\lambda_n\}$ be a sequence of real numbers such

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$^{(1)}$ R. E. A. C. Paley-N. Wiener [1]. S. Bochner [7].
that \( g.l.b. |\lambda_m - \lambda_n| > 0 \). Then a trigonometric series \( \sum A_n e^{i\lambda_n x} \) is the Fourier series of a \( S_2 \) almost periodic function (Stepanoff class with exponent 2) only when the series \( \sum |A_n|^2 \) is convergent. Bochner-Jessen's theorem (2): A trigonometric series \( \sum A_n e^{i\lambda_n x} \) with linearly independent Fourier exponents is the Fourier series of a \( B_1 \) almost periodic function (Besicovitch class with exponent 1) only when the series \( \sum |A_n|^2 \) is convergent.

From these two theorems, we may infer that the first problem will be solved at least for a trigonometric series in which the distribution of the Fourier exponents is not quite arbitrary. But the establishment of a theorem analogous to Riesz-Fischer theorem even for the class of uniformly continuous purely periodic functions is yet beyond the scope of our present mathematical weapon.

Now the object of the present paper is not the discussion of the first problem. We treat here much easier existence theorems where the given trigonometric series are very closely connected with Bohr almost periodic Fourier series. In fact, we may invent many concrete examples of the operations with Bohr almost periodic Fourier series. We shall prove here only three cases—the formal differentiation, the formal integration and the existence of Bohr almost periodic Fourier series conjugate to a given Bohr almost periodic Fourier series. These cases are undoubtedly the most interesting ones.

As to the second problem, Bochner is the only mathematician who has studied the problem in greater detail (3). He showed that the tests of convergence of Fourier series of purely periodic functions, based on the behaviour of a function in the neighbourhood of a point or in an interval, have not been extended to the case of Bohr almost periodic functions.

These tests are intimately bound up with the nature of the Fourier series of purely periodic functions and may be generalised only to some particular cases of Bohr almost periodic functions whose Fourier series have analogous properties. Bochner has considered two cases of this type, namely the case in which the difference between any pair of Fourier exponents is larger than a fixed positive number and the case of limit periodic functions.

Generalisations to a more or less general class of Bohr almost periodic functions satisfying only inner conditions, that is to say, some assumptions which affect the nature of the function itself, seem to be very unlikely. This is seen from the existence of a limit periodic function with

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(2) S. Bochner-B. Jessen [1].
(3) S. Bochner [1], [2], [3]. A. S. Besicovitch [1].
limit periodic derivatives of any finite order and yet its "Hauptfolge" diverges at some point. (This example is originally due to H. Bohr).

Bochner has also considered a Bohr almost periodic function

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a(\lambda_n) e^{i\lambda_n x} + a(-\lambda_n) e^{-i\lambda_n x} \right) \quad (\lambda_n > 0) \]

whose Fourier exponents \( \lambda_n \) have an infinite number of non-overlapping intervals \((\mu_n, \mu_n+1), (n = 1, 2 \cdots)\). Here it does not demand any assumption about the density of the exponents outside the intervals. He called such \( f(x) \) as regular function. He then discussed the convergence of a regular sequence

\[ S_{\mu_n}(x) \quad (0 < \mu_1 < \mu_2 < \cdots \to + \infty) \]

of the partial sums

\[ S_{\mu_n}(x) \sim a_0 + \sum_{\lambda_n \leq \mu_n} \left( a(\lambda_n) e^{i\lambda_n x} + a(-\lambda_n) e^{-i\lambda_n x} \right) \]

of \( f(x) \) and proved that the theorem of Riemann-Lebesgue also holds in this case.

In the present paper, we shall discuss the problem of similar type, in which the breadth of intervals into which Fourier exponents do not enter, may tend to zero as rapid as we please. On the other hand, the problem of bringing the latter question in its decisive shape is still open.

In conclusion, I express my cordial thanks to Prof. Harald Bohr who has kindly looked through the manuscript of the present paper.

**Part I. Operations with Fourier Series.**

§ 1. A type of theorems on the formal differentiation and integration of Fourier series.

1. In the theory of Bohr almost periodic functions the following theorems are well known.

**Theorem A.** Let \( f(x) \) be a Bohr almost periodic function of a real variable with the Fourier series

\[ f(x) \sim \sum A_n e^{i\lambda_n x} \]

and suppose that all \( \lambda_n \neq 0 \). Then the series

\[ \sum \frac{A_n}{\lambda_n} e^{i\lambda_n x} \]
obtained from the Fourier series of \( f(x) \) by formal integration is also a Bohr almost periodic Fourier series if and only if an indefinite integral \( F(x) \) of \( f(x) \) is almost periodic (which is true if \( F(x) \) is bounded) and we have

\[
F(x) \sim C + \sum \frac{A_n}{i\lambda_n} e^{i\lambda_n x} \quad (C, \text{ a constant}).
\]

**Theorem B.** Let \( f(x) \) be a Bohr almost periodic function of a real variable with the Fourier series

\[
f(x) \sim \sum A_n e^{i\lambda_n x}.
\]

Then the series

\[
\sum i\lambda_n A_n e^{i\lambda_n x}
\]

obtained from the Fourier series of \( f(x) \) by formal differentiation is also a Bohr almost periodic Fourier series if and only if the derivative of \( f(x) \) exists and is almost periodic (which is true if \( f'(x) \) is uniformly continuous), in which case it is the Fourier series of \( f'(x) \).

Now we shall give some generalisations of theorems A and B.

**Theorem C.** Let \( F(x) \) be an indefinite integral of a Bohr almost periodic function \( f(x) \) of a real variable with the Fourier series

\[
f(x) \sim \sum A_n e^{i\lambda_n x}
\]

and suppose that \( F(x) = O(|x|^{1-p}), \quad (0 < p \leq 1) \). Then the series

\[
\sum sgn \lambda_n \frac{A_n}{|\lambda_n|^q} e^{i\lambda_n x} \quad (0 < q < p)
\]

is also a Bohr almost periodic Fourier series.

**Theorem II.** Let \( f(x) \) be a Bohr almost periodic function of a real variable with the Fourier series

\[
f(x) \sim \sum A_n e^{i\lambda_n x}.
\]

If \( f(x) \) satisfies the Lipschitz condition of order \( p (0 < p \leq 1) \), that is to say, there exist a positive constant \( K \) and a positive number \( \delta \) such that

\[
|f(x_2) - f(x_1)| \leq K |x_2 - x_1|^p \quad \text{when} \quad |x_2 - x_1| \geq \delta,
\]

then the series

\[
\sum \frac{A_n}{i\lambda_n} e^{i\lambda_n x}.
\]
Some new properties of Bohr almost periodic Fourier series.

Let \[ \sum \text{sgn} \lambda_n A_n | \lambda_n |^q e^{i \lambda_n x} \quad (0 < q < p) \]
is also a Bohr almost periodic Fourier series.

2. For the proof of theorem I consider the function defined by

\[ \varphi_A(x) = \int_0^A t^{-1+q} \left\{ f(t+x) - f(-t+x) \right\} dt \]

where \( A \) is any positive number. Then \( \varphi_A(x) \) is a Bohr almost periodic function. Indeed, taking \( \tau \) as a translation number of \( f(x) \) belonging to \( \epsilon \), that is

\[ |f(x+\tau) - f(x)| \leq \epsilon \quad (-\infty < x < +\infty) \]

we have

\[ \varphi_A(x+\tau) - \varphi_A(x) = \int_0^A t^{-1+q} \left[ \left\{ f(t+x+\tau) - f(t+x) \right\} \right. \]

\[ - \left\{ f(-t+x+\tau) - f(-t+x) \right\} \] \( dt \)

from which

\[ | \varphi_A(x+\tau) - \varphi_A(x) | \leq 2 \epsilon q^{-1} A^q. \]

Thus a translation number \( \tau \) of \( f(x) \) belonging to \( \epsilon \) is a translation number of \( \varphi_A(x) \) belonging to \( 2 \epsilon q^{-1} A^q \).

Now consider a Fourier coefficient \( a_A(\ell) \) of \( \varphi_A(x) \). We have

\[ a_A(\ell) = M \left\{ \varphi_A(x) e^{-i \ell x} \right\} \]

\[ = \lim_{T \to \infty} \int_0^A t^{-1+q} dt \left\{ \frac{1}{T} \int_0^T \left[ f(t+x) - f(-t+x) \right] e^{-i \ell x} dx \right\} \]

and

\[ \frac{1}{T} \int_0^T \left[ f(t+x) - f(-t+x) \right] e^{-i \ell x} dx \]

\[ = e^{i \ell t} \cdot \frac{1}{T} \int_t^{t+T} f(x) e^{-i \ell x} dx - e^{-i \ell t} \cdot \frac{1}{T} \int_{t}^{t+T} f(x) e^{-i \ell x} dx. \]

When \( T \) increases infinitely, two expressions
tend uniformly to the same limit \( a(\lambda) \).

Thus for any positive number \( \varepsilon \), one can find an integer \( m \) such that for \( T \geq m \) the following inequalities hold:

\[
\frac{1}{T} \int_{-T}^{T} f(x) e^{-i\lambda x} \, dx = a(\lambda) + \varepsilon'(t, T) \quad \text{with} \quad |\varepsilon'(t, T)| \leq \varepsilon,
\]

\[
\frac{1}{T} \int_{-T}^{T} f(x) e^{-i\lambda x} \, dx = a(\lambda) + \varepsilon''(t, T) \quad \text{with} \quad |\varepsilon''(t, T)| \leq \varepsilon,
\]

Therefore we have

\[
|a(\lambda) - \int_{0}^{A} t^{-1+q} dt \left\{ \frac{1}{T} \int_{0}^{T} \left[ f(t+x) - f(-t+x) \right] e^{-i\lambda x} \, dx \right\} | \leq \varepsilon.
\]

Therefore we have

\[
\int_{0}^{A} t^{-1+q} dt \left\{ \frac{1}{T} \int_{0}^{T} \left[ f(t+x) - f(-t+x) \right] e^{-i\lambda x} \, dx \right\}
\]

\[
= 2i\alpha(\lambda) \int_{0}^{A} t^{-1+q} \sin \lambda t \, dt + \int_{0}^{A} \varepsilon'(t, T) t^{-1+q} e^{-i\lambda t} \, dt - \int_{0}^{A} \varepsilon''(t, T) t^{-1+q} e^{-i\lambda t} \, dt
\]

from which

\[
|a(\lambda) - 2i\alpha(\lambda) \int_{0}^{A} t^{-1+q} \sin \lambda t \, dt| \leq \varepsilon(1 + 2q^{-1}A^q).
\]

But we may take \( \varepsilon \) as small as we please; it immediately follows

\[
a(\lambda) = 2i\alpha(\lambda) \int_{0}^{A} t^{-1+q} \sin \lambda t \, dt,
\]

so that

\[
\varphi_{A}(x) \sim 2i \sum_{n} A_n \int_{0}^{A} t^{-1+q} \sin \lambda_n t \, dt e^{i\lambda_n x}.
\]

When \( A \) increases infinitely, \( \varphi_{A}(x) \) converges uniformly to the limit function \( \varphi(x) \) which is itself a Bohr almost periodic function. This will be proved in the following way.
Some new properties of Bohr almost periodic Fourier series.

Since $F(t) = O(|t|^{1-p})$, $(0 < p \leq 1)$, the integrals

$$\int_{-A}^{A} \frac{f(t)}{t^{1-q}} \, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{f(t)}{t^{1-q}} \, dt$$

are convergent for any fixed positive number $A$.

On the other hand, we have

$$\int_{A}^{\infty} \frac{f(t + x) - f(t)}{t^{1-q}} \, dt = (1 - q) \int_{A}^{\infty} \frac{F(t + x) - F(t)}{t^{2-q}} \, dt - \frac{F(A + x) - F(A)}{A^{1-q}}$$

and since

$$F(t + x) - F(t) = \int_{t}^{t+x} f(t) \, dt$$

is a Bohr almost periodic function of a real variable $t$ uniformly in $x$, we obtain

$$\int_{A}^{\infty} \frac{F(t + x) - F(t)}{t^{2-q}} \, dt < + \infty$$

uniformly in $x$ and the integral

$$\int_{A}^{\infty} \frac{f(t + x)}{t^{1-q}} \, dt$$

is convergent uniformly in $x$. Similarly the convergence of the integral

$$\int_{-\infty}^{-A} \frac{f(t)}{t^{1-q}} \, dt$$

shows us the existence of the integral

$$\int_{A}^{\infty} \frac{f(-t + x)}{t^{1-q}} \, dt.$$

Thus we get

$$\phi(x) = \int_{0}^{\infty} t^{-1+q}[f(t + x) - f(-t + x)] \, dt \quad (-\infty < x < + \infty).$$

On the other hand, the Fourier series of the limit function of uniformly convergent sequence of Bohr almost periodic functions is the formal
limit of the Fourier series of the almost periodic functions in question. Therefore we have

\[ \varphi(x) \sim 2i \sum \left\{ A_n \int_0^\infty t^{-1+q} \sin \lambda_n t \, dt \right\} e^{i\lambda_n x} \]

\[ = 2i \sum \left\{ A_n \text{sgn} \lambda_n \frac{\Gamma(q)}{\lambda_n^q} \sin \left( q \frac{\pi}{2} \right) \right\} e^{i\lambda_n x} . \]

Putting

\[ \psi(x) = \varphi(x) \left[ 2i \Gamma(q) \sin \left( q \frac{\pi}{2} \right) \right]^{-1} \]

which is obviously a Bohr almost periodic function, we finally get

\[ \psi(x) \sim \sum \text{sgn} \lambda_n \frac{A_n}{\lambda_n^q} e^{i\lambda_n x} \]

and the theorem is thus proved.

3. We proceed to the proof of theorem II. Take now the function defined by

\[ \varphi_\rho(x) = \int_0^\infty \frac{f(t+x)-f(-t+x)}{t^q} \, dt \]

where \( \rho \) is any positive number. Then \( \varphi_\rho(x) \) is a Bohr almost periodic function. Indeed, taking \( \tau \) as a translation number of \( f(x) \) belonging to \( \varepsilon \), we have

\[ |\varphi_\rho(x+\tau) - \varphi_\rho(x)| \leq 2q^{-1} \rho^{-q} \varepsilon . \]

Thus a translation number \( \tau \) of \( f(x) \) belonging to \( \varepsilon \) is a translation number of \( \varphi_\rho(x) \) belonging to \( 2q^{-1} \rho^{-q} \varepsilon . \)

Let \( a_\rho(\lambda) \) be a Fourier coefficient of \( \varphi_\rho(x) \). Then it is not difficult to see

\[ a_\rho(\lambda) = M \left\{ \varphi_\rho(x)e^{-i\lambda x} \right\} = 2ia(\lambda) \int_0^\infty \frac{\sin \lambda t}{t^q} \, dt \]

using the same method of proof as in section 2. Thus it immediately follows

\[ \varphi_\rho(x) \sim 2i \sum \left\{ A_n \int_\rho^\infty \frac{\sin \lambda_n t}{t^{1+q}} \, dt \right\} e^{i\lambda_n x} . \]
Since \( f(x) \) satisfies the Lipschitz condition of order \( p(0 < p \leq 1) \), it is easily to verify that when \( \rho \) tends to zero, \( \varphi_{\rho}(x) \) converges uniformly to the limit function \( \varphi(x) \) which is itself a Bohr almost periodic function. Indeed, taking two positive numbers \( \rho_1 \) and \( \rho_2 \), \( (\rho_1, \rho_2 < \min (1, \delta), \rho_1 < \rho_2) \), we have

\[
| \varphi_{\rho_2}(x) - \varphi_{\rho_1}(x) | \leq \int_{\rho_1}^{\rho_2} | f(t + x) - f(\rho_1 + x) - f(-t + x) | \frac{dt}{t^{1+q}}
\]

\[
\leq 2^p K \int_{\rho_1}^{\rho_2} t^{-1+p-q} dt
\]

\[
\leq \frac{2^p K \rho_2^{p-q}}{p-q} (0 < q < p)
\]

from which

\[
\varphi(x) = \lim_{\rho \to 0} \varphi_{\rho}(x) \quad (-\infty < x < +\infty).
\]

Therefore we get

\[
\varphi(x) \sim 2i \sum A_n \left( \frac{\sin \lambda_n t}{t^{1+q}} dt \right) e^{i\lambda_n x}
\]

\[
= 2i \sum \left( A_n \frac{\pi \text{sgn} \lambda_n \lambda_n}{2i(1+q) \cos \left( q \frac{\pi}{2} \right)} \right) e^{i\lambda_n x}.
\]

It is proved that the series

\[
\sum A_n \text{sgn} \lambda_n \lambda_n |^q e^{i\lambda_n x}
\]

is the Fourier series of a Bohr almost periodic function

\[
\Gamma(1+q) \cos \left( q \frac{\pi}{2} \right) \overline{i^{-1} \pi^{-1} \varphi(x)}.
\]

§ 2. Another type of theorems on the formal differentiation and integration of Fourier series.

1. In this paragraph, we use as a lemma the following important theorem due to Bochner.
Bochner's theorem (4).

Let $K(x)$ denote a function of a real variable such that the integral

$$\int_{-\infty}^{\infty} |K(x)| \, dx$$

is convergent, and let $G(\lambda)$ denote its Fourier transform

$$G(\lambda) = \int_{-\infty}^{\infty} K(x) e^{-i\lambda x} \, dx.$$

Then if $f(t)$ is a Bohr almost periodic function of a real variable with the Fourier series

$$f(t) \sim \sum A_n e^{i\lambda_n t}$$

the series

$$\sum G(\lambda_n) A_n e^{i\lambda_n t}$$

is also the Fourier series of a Bohr almost periodic function, namely of the function

$$g(t) = \int_{-\infty}^{\infty} f(-x + t) K(x) \, dx.$$

2. By use of this theorem, we can now obtain the following two theorems which correspond to the most general theorems on the formal differentiation and integration of a Bohr almost periodic Fourier series (5).

**Theorem III (6).** Let $f(t)$ be a Bohr almost periodic function of a real variable with the Fourier series

$$f(t) \sim \sum A_n e^{i\lambda_n t}.$$ 

Then the two series

$$\sum_{\lambda_n < 0} |\lambda_n|^p A_n e^{i\lambda_n s}, \quad \sum_{\lambda_n > 0} |\lambda_n|^p A_n e^{i\lambda_n s} \quad (s = \sigma + it)$$

where $p$ is any positive number, are the Dirichlet series of two functions $f_1(s), f_2(s)$ respectively almost periodic in $[0, +\infty)$ and in $(-\infty, 0]$.

(4) S. Bochner [4], [5]. J. Favard [4].

(5) This method of proof is originally due to B. Jessen. See B. Jessen [1].

(6) A special case $p = 1$ has already been treated by R. Petersen by means of a direct computation. See R. Petersen [1].
Some new properties of Bohr almost periodic Fourier series.

**Theorem IV** (7). Let \( f(t) \) be a Bohr almost periodic function of a real variable with the Fourier series

\[
f(t) \sim \sum A_n e^{i\lambda n t}.
\]

Then the series

\[
\sum \frac{A_n}{(\sigma + i\lambda_n)^p} e^{i\lambda_n t}
\]

where \( \sigma \) and \( p \) are any positive numbers, is a Bohr almost periodic Fourier series.

3. Now the theorem III corresponds to the choice

\[
K(x) = \frac{\Gamma(p)}{2\pi} \cdot \frac{1}{(\sigma + ix)^{p+1}}, \quad (\sigma > 0, \ p > 0).
\]

\[K^*(x) = \frac{\Gamma(p)}{2\pi} \cdot \frac{1}{(\sigma - ix)^{p+1}}
\]

Then

\[
\int_{-\infty}^{\infty} |K(x)| \, dx = \int_{-\infty}^{\infty} |K^*(x)| \, dx = \pi^{-1} \Gamma(p) \frac{p}{2} 2^{p-2} \sigma^{-p} < +\infty
\]

and

\[
G(\lambda) = \int_{-\infty}^{\infty} K(x)e^{-i\lambda x} \, dx = \begin{cases} 0 & \text{for } \lambda > 0, \\ |\lambda|^p e^{i\lambda} & \text{for } \lambda < 0, \end{cases}
\]

\[G^*(\lambda) = \int_{-\infty}^{\infty} K^*(x)e^{-i\lambda x} \, dx = \begin{cases} \lambda^p e^{-\sigma \lambda} & \text{for } \lambda > 0, \\ 0 & \text{for } \lambda < 0. \end{cases}
\]

Again the theorem IV corresponds to the choice

\[
K(x) = \begin{cases} \frac{1}{\Gamma(p)} x^{p-1} e^{-\sigma x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (\sigma > 0, \ p > 0)
\]

in which case

(7) A special case \( p = 1 \) has already been treated by S. Takahashi by means of a direct computation. See S. Takahashi [1].
4. When the absolute values of the Fourier exponents \( \lambda_n \) have not the point 0 as a limiting point, so that

\[
|\lambda_n| \geq \lambda > 0
\]

where \( \lambda \) is independent of \( n \), then by Bohr's theorem, the indefinite integrals

\[
\int f_1(s)ds, \quad \int f_2(s)ds
\]

are almost periodic in \([0, +\infty)\) and \((-\infty, 0]\). Then we have as an immediate corollary of theorem III the following theorem.

**Theorem V.** Let \( f(t) \) be a Bohr almost periodic function of a real variable with the Fourier series

\[
f(t) \sim \sum A_n e^{i\lambda_n t}
\]

where \( \{ |\lambda_n| \} \) have not the point 0 as a limiting point. Then the two series

\[
\sum_{\lambda_n < 0} A_n e^{i\lambda_n t}, \quad \sum_{\lambda_n > 0} A_n e^{i\lambda_n t} \quad (s = \sigma + it)
\]

where \( p \) is any non-negative number, are the Dirichlet series of two functions, almost periodic in \([0, +\infty)\) and \((-\infty, 0]\) respectively.

We remark that the theorems III and V are certain generalisations of Bohr's important theorem on the Laurent separation of Dirichlet series of an analytic almost periodic function.

5. Now we want to give another generalisation of Bohr's theorem on the Laurent separation of Dirichlet series of an analytic almost periodic function, whose enunciation is as follows.

**Theorem VI.** Let \( F(t) \) be an indefinite integral of a Bohr almost periodic function \( f(t) \) with the Fourier series

\[
f(t) \sim \sum A_n e^{i\lambda_n t}
\]
and suppose that \( F(t) = O(|t|^{-\alpha}), \ 0 < \alpha \leq 1 \). Then the two series

\[
\sum_{\lambda_n < 0} A_n e^{\lambda_n t}, \quad \sum_{\lambda_n > 0} A_n e^{\lambda_n t} \quad (s = \sigma + it)
\]

are the Dirichlet series of two functions, almost periodic in \([0, +\infty)\) and \((-\infty, 0]\) respectively.

6. For the proof, consider the function defined by

\[
\varphi_A(t) = \frac{1}{i\pi} \int_{0}^{A} \frac{x[f(x+t) - f(-x+t)]}{\sigma^2 + x^2} \, dx
\]

where \( A \) and \( \sigma \) are any positive numbers. Then \( \varphi_A(t) \) is a Bohr almost periodic function. Indeed, taking \( \tau \) as a translation number of \( f(t) \) belonging to \( \varepsilon \), we have

\[
|\varphi_A(t+\tau) - \varphi_A(t)| \leq \frac{\varepsilon}{\pi} \log \left(1 + \frac{A^2}{\sigma^2}\right) \quad (-\infty < t < +\infty).
\]

Let \( a_A(\lambda) \) be a Fourier coefficient of \( \varphi_A(t) \). We have

\[
a_A(\lambda) = \frac{2a(\lambda)}{\pi} \int_{0}^{A} \frac{x \sin \lambda x}{\sigma^2 + x^2} \, dx
\]

using the same method of proof as in paragraph 1, section 2. Thus it immediately follows

\[
\varphi_A(t) \sim 2\pi^{-1} \sum A_n \left[ \frac{x \sin \lambda_n x}{\sigma^2 + x^2} \right] e^{i\lambda_n t}.
\]

Since \( F(x) = O(|x|^{-\alpha}), \ 0 < \alpha \leq 1 \), it is easily to verify that when \( A \) tends to infinity, \( \varphi_A(t) \) converges uniformly to the limit function \( \varphi(t) \) which is itself a Bohr almost periodic function. Thus we get

\[
\varphi(t) \sim 2\pi^{-1} \sum A_n \left[ \int_{0}^{\infty} \frac{x \sin \lambda_n x}{\sigma^2 + x^2} \, dx \right] e^{i\lambda_n t}
\]

\[
= \sum A_n \text{sgn}\lambda_n e^{-|\lambda_n|} e^{i\lambda_n t}.
\]

On the other hand, a Bohr almost periodic function

\[
\psi(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma f(t+x)}{\sigma^2 + x^2} \, dx
\]
has the Fourier series

\[ \psi(t) = \sum A_n e^{-\sigma |\lambda_n| e^{i\lambda_n t}}. \]

Therefore we obtain

\[ \frac{\psi(t) + \varphi(t)}{2} = \sum_{\lambda_n > 0} A_n e^{-|\lambda_n| e^{i\lambda_n t}} = \sum_{\lambda_n > 0} A_n e^{\lambda_n (-\sigma + i t)}, \]

\[ \frac{\psi(t) - \varphi(t)}{2} = \sum_{\lambda_n < 0} A_n e^{-|\lambda_n| e^{i\lambda_n t}} = \sum_{\lambda_n < 0} A_n e^{\lambda_n (\sigma - i t)}. \]

Thus the theorem VI is proved.

§ 3. Conjugate Fourier series.

1. Let \( f(x) = \sum A_n e^{i\lambda_n x} \) be a Bohr almost periodic function of a real variable \(-\infty < x < +\infty\). We shall call the trigonometric series

\[ \sum i A_n \text{sgn} \lambda_n e^{i\lambda_n x} \]

the conjugate series of \( \sum A_n e^{i\lambda_n x} \). In his theory of harmonic almost periodic functions, Favard has proved the following important theorem. Favard's theorem \((8)\). Let \( f(x) \) be a Bohr almost periodic function of a real variable with the Fourier series \( \sum A_n e^{i\lambda_n x} \). Then its conjugate series is a Bohr almost periodic Fourier series if an indefinite integral \( F(x) \) of \( f(x) \) is almost periodic (which is true if \( F(x) \) is bounded) and moreover, when \( f(x) \) satisfies the Lipschitz condition of order \( \alpha \) \((0 < \alpha \leq 1)\), that is to say, there exist a positive constant \( K \) and a positive number \( \delta \) such that

\[ |f(x_2) - f(x_1)| \leq K |x_2 - x_1|^\alpha \text{ when } |x_2 - x_1| \leq \delta. \]

Now we want to prove a theorem of similar type on the existence of conjugate Bohr almost periodic Fourier series, whose enunciation is as follows:

**Theorem VII.** Let \( f(x) \) be a Bohr almost periodic function of a real variable with the Fourier series \( \sum A_n e^{i\lambda_n x} \) whose Fourier exponents have the point 0 as a limiting point. Further let all Fourier exponents in the neighbourhood of the point 0 have the same sign. Then its con-

\((8)\) J. Favard [1], [2], [3].
jugate series is a Bohr almost periodic Fourier series when \( f(x) \) satisfies the Lipschitz condition of order \( a(0 < a \leq 1) \)\(^{(9)}\).

2. The function defined by

\[
\varphi_\rho(x) = \int_0^{\infty} \frac{\sin \sigma t [f(t + x) - f(-t + x)]}{t^2} \, dt
\]

where \( \sigma \) and \( \rho \) are any positive numbers is a Bohr almost periodic function. Indeed, taking \( \tau \) as a translation number of \( f(x) \) belonging to \( \varepsilon \), i.e.

\[
|f(x + \tau) - f(x)| \leq \varepsilon \quad (-\infty < x < +\infty)
\]

we have

\[
|\varphi_\rho(x + \tau) - \varphi_\rho(x)| \leq 2 \varepsilon \int_0^{\infty} \left| \frac{\sin \sigma t}{t^2} \right| \, dt \leq 2\rho^{-1} \varepsilon.
\]

Thus a translation number \( \tau \) of \( f(x) \) belonging to \( \varepsilon \) is a translation number of \( \varphi_\rho(x) \) belonging to \( 2\rho^{-1} \varepsilon \).

Let \( a_\rho(\lambda) \) be a Fourier coefficient of \( \varphi_\rho(x) \). Then it is easily seen

\[
a_\rho(x) = M \{ \varphi_\rho(x)e^{-ix\lambda} \} = 2i a_\rho(\lambda) \int_\rho^{\infty} \sin \sigma x \sin \lambda x \, dx
\]

using the same method of proof as in paragraph 1, section 2. Thus it immediately follows

\[
\varphi_\rho(x) \sim 2i \sum \left\{ A_n \int_\rho^{\infty} \sin \sigma x \sin \lambda_n x \, dx \right\} e^{i\lambda_n x}.
\]

Since \( f(x) \) satisfies the Lipschitz condition of order \( a(0 < a \leq 1) \), it is not difficult to verify that when \( \rho \) tends to zero, \( \varphi_\rho(x) \) converges uniformly to the limit function \( \varphi(x) \) which is itself a Bohr almost periodic function. Indeed, taking two positive numbers \( \rho_1 \) and \( \rho_2 (\rho_1, \rho_2 < \min(1, \delta), \rho_1 < \rho_2) \), we have

\[
|\varphi_{\rho_2}(x) - \varphi_{\rho_1}(x)| \leq \int_{\rho_1}^{\rho_2} 2\sigma \cdot 2^a K x^{a-1} \, dx \leq 2^a 1^a \sigma K x^{a-1} \rho_2^{a-1}.
\]

\(^{(9)}\) In case where all Fourier exponents \( \{\gamma_n\} \) have not the point 0 as a limiting point, then an indefinite integral of \( f(x) \) is always almost periodic and theorem VII is nothing but Favard's theorem.
from which

$$\varphi(x) = \lim_{\rho \to 0} \varphi_{\rho}(x) \quad (-\infty < x < +\infty).$$

Therefore we get

$$\varphi(x) \sim 2i \sum \left\{ A_n \int_0^\infty \frac{\sin \sigma x \sin \lambda_n x}{x^2} \, dx \right\} e^{i\lambda_n x}.$$  

Since

$$\int_0^\infty \frac{\sin \sigma x \sin \lambda_n x}{x^2} \, dx = \frac{\pi}{2} \text{sgn} \lambda_n \min (\sigma, |\lambda_n|)$$

the trigonometric series

$$\sum_{|\lambda_n| \leq \sigma} A_n \text{sgn} \lambda_n \lambda_n |e^{i\lambda_n x} + \sum_{|\lambda_n| > \sigma} A_n \text{sgn} \lambda_n e^{i\lambda_n x}$$

is the Fourier series of a Bohr almost periodic function $i^{-1} \pi^{-1} \varphi(x)$.

We shall distinguish here two cases.

Case 1. All Fourier exponents lying in the closed interval $(-\sigma, \sigma)$ are positive numbers. Then we have

$$\Phi(x) \sim \sum_{|\lambda_n| \leq \sigma} \left( A_n \sigma^{-1} \lambda_n |e^{i\lambda_n x} + \sum_{|\lambda_n| > \sigma} A_n \text{sgn} \lambda_n e^{i\lambda_n x}$$

where we denote by $\Phi(x)$ a Bohr almost periodic function $i^{-1} \pi^{-1} \sigma^{-1} \varphi(x)$.

Case 2. All Fourier exponents lying in the closed interval $(-\sigma, \sigma)$ are negative numbers. Then we have

$$\Phi(x) \sim \sum_{|\lambda_n| \leq \sigma} (-A_n) \sigma^{-1} \lambda_n |e^{i\lambda_n x} + \sum_{|\lambda_n| > \sigma} A_n \text{sgn} \lambda_n e^{i\lambda_n x}.$$  

On the other hand, it is well known that the expression

$$S^*(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x + 2t) \frac{\sin^2 \sigma t}{\sigma t^2} \, dt \quad (\sigma > 0)$$

is a Bohr almost periodic function with the Fourier series

$$S^*(x) \sim \sum_{|\lambda_n| \leq \sigma} (1 - \sigma^{-1} |\lambda_n|) A_n e^{i\lambda_n x}.$$  

Thus we obtain the following results according to two cases above described.
Case 1. A Bohr almost periodic function \( \phi(x) + S^*(x) \) has the Fourier series

\[
\sum_{|\lambda_n| \leq \sigma} (A_n)e^{i\lambda_n x} + \sum_{|\lambda_n| > \sigma} A_n \text{sgn} \lambda_n e^{i\lambda_n x} = \sum A_n \text{sgn} \lambda_n e^{i\lambda_n x}.
\]

Case 2. A Bohr almost periodic function \( \phi(x) - S^*(x) \) has the Fourier series

\[
\sum_{|\lambda_n| \leq \sigma} (-A_n)e^{i\lambda_n x} + \sum_{|\lambda_n| > \sigma} A_n \text{sgn} \lambda_n e^{i\lambda_n x} = \sum A_n \text{sgn} \lambda_n e^{i\lambda_n x}.
\]

Consequently, in any case, the conjugate series

\[
\sum iA_n \text{sgn} \lambda_n e^{i\lambda_n x}
\]

is a Bohr almost periodic Fourier series. The proof of theorem VII is thus completed.

Part II. Summation of Fourier Series of Bohr Almost Periodic Functions by Partial Sums.

§ 1. A general theorem and its discussion.

1. First of all, we want to give a proof of the following theorem.

Theorem VIII. Let

\[ f(x) \sim \sum A_n e^{i\lambda_n x} \quad (-\infty < x < +\infty) \]

be a Bohr almost periodic function of a real variable such that

\[
(A) \quad \left| \int_0^\infty \frac{\sin \sigma t}{t} \left[ f(t+x) + f(-t + x) \right] dt \right| < +\infty
\]

for any positive \( \sigma \) uniformly in \( x \) and

\[
(B) \quad \lim_{\sigma \to \infty} \int_{-\varepsilon}^{\varepsilon} \frac{\sin \sigma t}{t} \left[ f(t+x) + f(-t + x) \right] dt = 0
\]

for any \( \varepsilon > 0 \) (also uniformly in \( x \)). Then the convergence of the partial sums \( S_n(x) \) of \( f(x) \)

\[ S_n(x) \sim \sum_{|\lambda_n| < \sigma} A_n e^{i\lambda_n x} \]
to \( f(x) \) as \( \sigma \to +\infty \), depends only upon the behaviour of \( f(x) \) in the neighbourhood of the point \( x \) (theorem of Riemann-Lebesgue type).

2. For the proof, consider the function defined by

\[
\varphi_A(x) = \frac{1}{\pi} \int_0^A \frac{\sin \sigma t}{t} \left[ f(t+x) + f(-t+x) \right] dt
\]

where \( A \) and \( \sigma \) are any positive numbers. Then \( \varphi_A(x) \) is a Bohr almost periodic function. Indeed, taking \( \tau \) as a translation number of \( f(x) \) belonging to \( \varepsilon \), we have

\[
|\varphi_A(x+\tau) - \varphi_A(x)| \leq \frac{2\varepsilon}{\pi} \int_0^A \frac{|\sin \sigma t|}{t} dt \leq \frac{2\varepsilon}{\pi} \int_0^A \frac{2\sigma}{1 + \sigma t} dt = 4\pi^{-1} \varepsilon \log(1 + \sigma A).
\]

Thus a translation number \( \tau \) of \( f(x) \) belonging to \( \varepsilon \) is a translation number of \( \varphi_A(x) \) belonging to \( 4\pi^{-1} \varepsilon \log(1 + \sigma A) \). Next consider a Fourier coefficient \( a_A(\lambda) \) of \( \varphi_A(x) \). We have

\[
a_A(\lambda) = 2\pi^{-1} a(\lambda) \int_0^A \frac{\sin \sigma t}{t} \cos \lambda t dt
\]

using the same method of proof as in Part I, paragraph 1, section 2. It immediately follows

\[
\varphi_A(x) \sim 2\pi^{-1} \sum \left\{ A_n \int_0^A \frac{\sin \sigma t}{t} \cos \lambda_n t dt \right\} e^{i\lambda_n x}.
\]

Now our assumption (A) give us the limit equation

\[
\lim_{A \to \infty} \varphi_A(x) = \varphi(x)
\]

uniformly in \( x \) and \( \varphi(x) \) is also a Bohr almost periodic function. Thus we have

\[
\varphi(x) \sim 2\pi^{-1} \sum \left\{ A_n \int_0^\infty \frac{\sin \sigma t}{t} \cos \lambda_n t dt \right\} e^{i\lambda_n x}.
\]

Since

\[
\frac{2}{\pi} \int_0^\infty \frac{\sin \sigma t}{t} \cos \lambda_n t dt = \begin{cases} 1 & \text{for } |\lambda_n| < \sigma \\ 0 & \text{for } |\lambda_n| > \sigma \end{cases}
\]

we finally obtain

\[
\varphi(x) \sim \sum_{|\lambda_n| < \sigma} A_n e^{i\lambda_n x}.
\]
Some new properties of Bohr almost periodic Fourier series.

We call, for brevity's sake, \( \phi(x) \) a partial sum of \( f(x) \) and denote it by \( S_\sigma(x) \). Thus we have

\[
S_\sigma(x) = \frac{1}{\pi} \int_0^\infty \frac{\sin \sigma t}{t} \left[ f(t+x) + f(-t+x) \right] dt.
\]

Putting

\[
f(t+x) + f(-t+x) - 2f(x) = \phi(x)
\]

we can write

\[
S_\sigma(x) - f(x) = \frac{1}{\pi} \int_0^\infty \frac{\sin \sigma t}{t} \phi(t) dt.
\]

Since

\[
\lim_{\sigma \to \infty} \int_0^\infty \frac{\sin \sigma t}{t} \phi(t) dt = 0
\]

for any \( \varepsilon > 0 \), we get

\[
\lim_{\sigma \to \infty} \frac{1}{\pi} \int_0^\infty \frac{\sin \sigma t}{t} \phi(t) dt = 0
\]

on account of our assumption (B) uniformly in \( x \) and thence

\[
\lim_{\sigma \to \infty} \left\{ S_\sigma(x) - f(x) \right\} = \lim_{\sigma \to \infty} \frac{1}{\pi} \int_0^\infty \frac{\sin \sigma t}{t} \phi(t) dt.
\]

This last equality shows us the truth of our theorem VIII.

3. Now an important question arises concerning the above theorem. We do not know whether the conditions (A) and (B) are really a certain inner condition or not. But it appears highly improbable that these (A) and (B) are such conditions. We shall see in the following discussion a certain ground for the justification of our conjecture now in question. For that purpose, we use as a lemma the following well known theorem. Theorem (10). Let \( f(x) \) be a real or complex-valued function defined in \([\varepsilon, +\infty)\) such that the integral

\[
\int_\varepsilon^\infty |f(x)| dx
\]

is convergent, or such that \( f(x) \downarrow 0 \) as \( x \uparrow +\infty \) for \( x \geq T \), \( T \) being an arbitrary great fixed positive number, then the Fourier integral

\(\text{(10)}\) S. Bochner [6].
\[ \Phi(\sigma) = \int_{-\infty}^{\infty} f(x) \sin \sigma x \, dx \]

exists for any \( \sigma \) and \( \Phi(\sigma) \to 0 \) for \( \sigma \to \pm \infty \).

4. Now when the integral

\[ \int_{T}^{\infty} \frac{|f(t)|}{t} \, dt \]

is convergent for a Bohr almost periodic function \( f(x) \), \( (-\infty < x < +\infty) \), then putting

\[ F(x) = \int |f(x)| \, dx \]

we have

\[ \int_{T}^{\infty} \frac{|f(x+t)| - |f(t)|}{t} \, dt = \int_{T}^{\infty} \frac{F(x+t)-F(t)}{t} \, dt - \frac{F(x+T)-F(T)}{T} \]

Since

\[ F(x+t)-F(t) = \int_{t}^{t+T} |f(t)| \, dt \]

is a Bohr almost periodic function of a real variable \( t \) uniformly in \( x \), we obtain

\[ \int_{T}^{\infty} \frac{F(x+t)-F(t)}{t} \, dt < +\infty \]

uniformly in \( x \) and the integral

\[ \int_{T}^{\infty} \frac{|f(x+t)|}{t} \, dt \]

is convergent uniformly in \( x \). Similarly the convergence of the integral

\[ \int_{-\infty}^{T} \frac{|f(t)|}{t} \, dt \]

shows us the existence of the integral

\[ \int_{T}^{\infty} \frac{|f(x-t)|}{t} \, dt . \]
Some new properties of Bohr almost periodic Fourier series.

We also note

\[ \int_0^\infty \frac{\sin \sigma t}{t} \left[ f(t+x) + f(-t+x) \right] dt < 4M \log (1 + \sigma \varepsilon) < +\infty \]

where \( M \) is the upper bound of \( |f(x)| \) for \( -\infty < x < +\infty \).

Thus, on account of the above lemma, a Bohr almost periodic function for which the integrals

\[ \int_T^\infty \frac{|f(t)|}{t} dt \quad \text{and} \quad \int_{-\infty}^T \frac{|f(t)|}{t} dt \]

are convergent, satisfies the conditions (A) and (B) of theorem VIII. But it is easily to verify that a Bohr almost periodic function for which

\[ \int_T^\infty \frac{|f(t)|}{t} dt < +\infty \]

for any positive \( T \), is identically equal to zero.

5. Next let \( f(t) \) be a positive Bohr almost periodic function for which

\[ f(t+x) + f(-t+x) = \psi(t; x) \equiv \frac{\psi(t)}{t} \]

is monotone uniformly in \( x \) for \( t \geq T \). Since \( \psi(t) \) is almost periodic and therefore uniformly bounded, we have

\[ \frac{\psi(t)}{t} \to 0 \quad \text{for} \quad t \to +\infty \quad \text{uniformly in} \quad x. \]

Thus the conditions (A) and (B) are also fulfilled in this case.

However a positive Bohr almost periodic function \( \psi(t) \) for which \( \psi(t)t^{-1} \) is non-decreasing uniformly in \( x \) for \( t \geq T \) must be a constant. This will be proved in the following way.

Let \( \psi(t) \) be a non-constant positive Bohr almost periodic function with above described properties. From the almost-periodicity of \( \psi(t) \) there must exist two points \( t_0 \) and \( t_1 \) (\( t_0 < t_1; \; t_0, t_1 \geq T \)) such that

\[ \psi(t_1) > \psi(t_0) > 0. \]

We write \( \psi(t_0) = \alpha, \; \psi(t_1) = \beta \) and let \( \varepsilon \) be any positive number such that

\[ \operatorname{Min} \left( \frac{\beta - \alpha}{2}, \alpha \right) > \varepsilon. \]
Now two inequalities
\[ |\psi(t_0 + \tau) - \psi(t_0)| \leq \varepsilon, \quad |\psi(t_1 + \tau) - \psi(t_1)| \leq \varepsilon, \]
where \( \tau = \tau(\varepsilon) \) is a translation number of \( \psi(t) \) belonging to \( \varepsilon \), give us the inequalities
\[ a - \varepsilon \leq \psi(t_0 + \tau) \leq a + \varepsilon, \quad \beta - \varepsilon \leq \psi(t_1 + \tau) \leq \beta + \varepsilon. \]
Therefore we have
\[ \frac{\beta - \varepsilon}{t_1 + \tau} \leq \frac{\psi(t_1 + \tau)}{t_1 + \tau} \leq \frac{\alpha + \varepsilon}{t_0 + \tau} \]
from which
\[ 1 < \frac{\beta - \varepsilon}{\alpha + \varepsilon} \leq \frac{t_1 + \tau(\varepsilon)}{t_0 + \tau(\varepsilon)}.
\]
Since there exists an arbitrary great positive translation number \( \tau(\varepsilon) \) for any fixed \( \varepsilon > 0 \), the last inequality is obviously absurd. Thus \( \psi(t) \) and also \( f(t) \) must be a constant.

6. On account of the argument in the preceding section, it seems there exists no non-constant Bohr almost periodic function for which the conditions (A) and (B) of theorem VIII are simultaneously satisfied. But these conditions are really fulfilled in some special class of Bohr almost periodic functions for which the distribution of Fourier exponents and the behaviour of the function itself are restricted in some manner. We shall treat this problem in the following paragraph.

§ 2. Some theorems concerning quasi-regular functions.

1. In this paragraph, we write the Fourier series of a Bohr almost periodic function \( f(x) \) in the form
\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} (a(\lambda_n)e^{i\lambda_n x} + a(-\lambda_n)e^{-i\lambda_n x}) \]
where the exponents \( \lambda_n \) are positive. We say that \( f(x) \) has an interval \((\mu, \nu)\), if none of the \( \lambda_n \) lie in the interval \((0 \leq \mu \leq \lambda \leq \nu)\), and we call \( l = \nu - \mu \) the length of the interval. Now our special hypothesis will be that \( f(x) \) has an infinite number of intervals whose lengths may tend to zero in some way, in which case we call it a quasi-regular function. In what follows, we note that we do not need any assumption about the density of the exponents outside the intervals.
2. Before stating our theorems, we begin with some lemmas.

**Lemma 1.** Consider a general trigonometric polynomial
\[
p(t) = \sum_{n=1}^{N} (a_n \cos \lambda_n t + b_n \sin \lambda_n t)
\]
where \( \lambda_1, \lambda_2, \ldots, \lambda_N \geq A > 0 \). If we denote \( K = \text{u. b. } |p(t)| \), then
\[
\int_{-\infty}^{+\infty} p(u)du \leq \frac{CK}{A}
\]
where \( C \) is an absolute constant.

This beautiful inequality is due to H. Bohr \(^{11} \).

**Lemma 2.** Let
\[
f(x) \sim \sum_{n=1}^{\infty} (a(\lambda_n)e^{i\lambda_n x} + a(-\lambda_n)e^{-i\lambda_n x}) \quad (\lambda_n \geq 0)
\]
be a Bohr almost periodic function whose Fourier exponents \( \lambda_n \) have not \( \sigma \) as a limiting point, i.e.
\[(1) \quad |\lambda_n - \sigma| \geq A > 0\]
\( \sigma \) being a fixed positive number greater than \( A \). Moreover let
\[
\chi_\sigma(x; t) = \chi_\sigma(t) = \sin \sigma[t(f(x + t) + f(x - t))]
\]
Then we have
\[
\int_{-\infty}^{+\infty} \chi_\sigma(u)du \leq \frac{4CM}{A}
\]
where \( M = \text{u. b. } |f(x)| \) and \( C \) is an absolute constant.

**Proof of lemma 2.** Separating the Fourier series
\[
\sum_{n=1}^{\infty} (a(\lambda_n)e^{i\lambda_n x} + a(-\lambda_n)e^{-i\lambda_n x}) \quad (\lambda_n \geq 0)
\]
in a real and an imaginary part, we can write
\[
f(x) \sim \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) + i \sum_{n=1}^{\infty} (c_n \cos \lambda_n x + d_n \sin \lambda_n x)
\]

\(^{11}\) H. Bohr [1], [2], [3]. J. Favard [5]. The essential part of Bohr's inequality is the fact that the exact value of \( C \) is equal to \( \frac{\pi}{2} \). But we do not use this true value in the course of our investigations.
where \(a_n, b_n, c_n, d_n\) are all real numbers. Then we have

\[
f(x + t) + f(x - t) \sim \sum_{n=1}^{\infty} 2(a_n \cos \lambda_n x + b_n \sin \lambda_n x) \cos \lambda_n t + i \sum_{n=1}^{\infty} 2(c_n \cos \lambda_n x + d_n \sin \lambda_n x) \cos \lambda_n t = \sum_{n=1}^{\infty} 2A_n(x) \cos \lambda_n t + i \sum_{n=1}^{\infty} 2B_n(x) \cos \lambda_n t
\]

and

\[
\chi_n(t) \sim \sum_{n=1}^{\infty} 2A_n(x) \{ \sin (\sigma + \lambda_n) t + \varepsilon_n \sin |\sigma - \lambda_n| t \} + i \sum_{n=1}^{\infty} 2B_n(x) \{ \sin (\sigma + \lambda_n) t + \varepsilon_n \sin |\sigma - \lambda_n| t \}
\]

where \(\varepsilon_n = \text{sgn} (\sigma - \lambda_n)\). Putting \(\chi_n(t) = \chi_n^{(1)}(t) + i \chi_n^{(2)}(t)\), where \(\chi_n^{(1)}(t)\) and \(\chi_n^{(2)}(t)\) are obviously a Bohr almost periodic function, we get

\[
\chi_n^{(1)}(t) \sim \sum_{n=1}^{\infty} \left\{ 2A_n(x) \sin (\sigma + \lambda_n) t + 2 \varepsilon_n A_n(x) \sin |\sigma - \lambda_n| t \right\},
\]

\[
\chi_n^{(2)}(t) \sim \sum_{n=1}^{\infty} \left\{ 2B_n(x) \sin (\sigma + \lambda_n) t + 2 \varepsilon_n B_n(x) \sin |\sigma - \lambda_n| t \right\}.
\]

Since (1) holds and \(\sigma \geq A\), the exponents of \(\chi_n^{(1)}(t)\) and \(\chi_n^{(2)}(t)\) are positive and not less than \(A\). Now let

\[
p_n(t) = \sum_{n=1}^{B} a_n^{(v)} \sin \lambda_n^{(v)} t \quad \text{and} \quad q_n(t) = \sum_{n=1}^{B} \beta_n^{(v)} \sin \lambda_n^{(v)} t
\]

be Bochner-Fejér polynomials uniformly convergent to \(\chi_n^{(1)}(t)\) and \(\chi_n^{(2)}(t)\) respectively. Since \(\lambda_n^{(v)}\) is a number in the sequences \(\{\lambda_n + \sigma\}, \{|\sigma - \lambda_n|\}\), we have \(\lambda_n^{(v)} \geq A > 0\). Thus by lemma 1, we have

\[
\left| \int_0^t p_v(u) du \right| \leq \frac{C}{A} \text{ u. b. } |p_v(t)|,
\]

\[
\left| \int_0^t q_v(u) du \right| \leq \frac{C}{A} \text{ u. b. } |q_v(t)|.
\]

Letting \(v\) tend to infinity, we get

\[
\left| \int_0^t \chi_n^{(v)}(u) du \right| \leq \frac{C}{A} \text{ u. b. } |\chi_n^{(1)}(t)| + \frac{C}{A} \text{ u. b. } |\chi_n^{(2)}(t)|
\]

\[
\leq \frac{2C}{A} \text{ u. b. } |\chi_n(t)| \leq \frac{4CM}{A}.
\]
Lemma 3. Under the assumption in the preceding lemma,

$$S_\sigma(x) = \frac{1}{\pi} \int_0^\infty \frac{\sin \sigma t}{t} \left[ f(x+t) + f(x-t) \right] dt$$

becomes a Bohr almost periodic function and its Fourier series is

$$S_\sigma(x) \sim \sum_{\lambda_n < \sigma} (a(\lambda_n)e^{i\lambda_n x} + a(-\lambda_n)e^{-i\lambda_n x}) \quad (\lambda_n \geq 0).$$

Proof of lemma 3. By the preceding lemma,

$$\int_0^A \sin \sigma t \left[ f(x+t) + f(x-t) \right] dt = \int_0^A \chi_\sigma(x; t) dt$$

is uniformly bounded with respect to $A$ and $x$ and has $4CM A^{-1}$ as its upper bound. Now let $0 < X < X'$. Integration by parts shows

$$\int_X^{X'} \frac{\chi_\sigma(x; t)}{t} dt = \left[ \frac{1}{t} \int_0^X \chi_\sigma(x; u) du \right]_X^{X'} + \int_X^{X'} \frac{dt}{t^2} \int_0^X \chi_\sigma(x; u) du$$

$$= \frac{1}{X'} \int_0^{X'} \chi_\sigma(x; t) dt - \frac{1}{X} \int_0^{X} \chi_\sigma(x; t) dt + \int_X^{X'} \frac{dt}{t^2} \int_0^{X} \chi_\sigma(x; u) du.$$

Thus we have

$$\left(2\right) \int_X^{X'} \frac{\chi_\sigma(x; t)}{t} dt \leq \frac{4CM}{AX'} + \frac{4CM}{AX} + \frac{4CM}{A} \int_X^{X'} \frac{dt}{t^2}$$

which tends uniformly to zero as $X \to +\infty$. Therefore

$$\varphi_A(x) = \frac{1}{\pi} \int_0^A \chi_\sigma(x; t) dt$$

tends uniformly to the limit function $S_\sigma(x)$ as $A \to +\infty$ and

$$S_\sigma(x) = \frac{1}{\pi} \int_0^\infty \chi_\sigma(x; t) dt$$

is a Bohr almost periodic function. The fact that the Fourier series of $S_\sigma(x)$ can put in the form

$$S_\sigma(x) \sim \sum_{\lambda_n < \sigma} (a(\lambda_n)e^{i\lambda_n x} + a(-\lambda_n)e^{-i\lambda_n x})$$

has already been proved in the preceding paragraph, section 2.
Lemma 4. Under the assumptions in the lemma 2, we have

\[(I = \text{u. b. } |f(x)|)\]

where \(K\) is some absolute constant and \(L\), an arbitrary positive number.

Proof of lemma 4. Since

\[\int_{0}^{\infty} \frac{\chi_{\sigma}(x; t)}{t} dt = \int_{0}^{L} \frac{\chi_{\sigma}(x; t)}{t} dt + \int_{L}^{\infty} \frac{\chi_{\sigma}(x; t)}{t} dt.\]

\[\int_{0}^{L} \frac{\chi_{\sigma}(x; t)}{t} dt \leq \int_{0}^{L} \frac{2\pi \cdot 2\Gamma}{1 + \sigma t} dt = 4\Gamma \log (1 + \sigma L)\]

and by (2), we obtain

\[\int_{L}^{\infty} \frac{\chi_{\sigma}(x; t)}{t} dt \leq \frac{4C\Gamma}{AL} + \frac{4C\Gamma}{A} \int_{L}^{\infty} dt = \frac{8C\Gamma}{AL}.\]

Thus we get

\[|S_{\sigma}(x)| \leq 4\pi^{-1} \Gamma \log (1 + \sigma L) + 8C\pi^{-1} \Gamma \Lambda^{-1} L^{-1}\]

and thence

\[|f(x) - S_{\sigma}(x)| \leq KL \left\{ 1 + \Lambda^{-1} L^{-1} + \log (1 + \sigma L) \right\}.\]

We shall here write for brevity's sake

\[J_{1}(f; x; \delta_{1}) = f(x + \delta_{1}) - f(x)\]

\[J_{r}(f; x; \delta_{1}, \delta_{2}, \ldots, \delta_{r}) = J_{r-1}(f(x + \delta_{1}) - f(x); x; \delta_{1}, \delta_{2}, \ldots, \delta_{r-1})\]

\[\omega_{r}(f; \delta) = \text{Max} \left| J_{r}(f; x; \delta_{1}, \delta_{2}, \ldots, \delta_{r}) \right|,\]

Now our final lemma is as follows.

Lemma 5. Let \(f(x)\) be a Bohr almost periodic function with the Fourier series

\[f(x) \sim a_{0} + \sum_{n=1}^{\infty} (a(\lambda_{n}) e^{i\lambda_{n}x} + a(-\lambda_{n}) e^{-i\lambda_{n}x}) \quad (\lambda_{n} > 0)\]
where \( f(x) \) has an interval \((\mu, \mu + \delta), (\mu > 0, \delta > 0)\). Then we have

\[
|f(x) - S_\mu(x)| \leq M_r \left\{ 1 + \delta^{-1} L^{-1} \log (1 + \mu L + \delta L) \right\} \omega_r \left( f; \frac{1}{\mu} \right)
\]

\((r = 1, 2, \ldots)\)

where \( M_r \) is some absolute constant and \( L \), an arbitrary positive number.

**Proof of lemma 5.** It has been proved by Bochner (12) that for a Bohr almost periodic function \( g(x) \) with Fourier exponents greater than or equal to \( \Lambda \), the inequality

\[
|g(x)| \leq A_r \omega_r \left( g; \frac{1}{\Lambda} \right) \quad (r = 1, 2, \ldots)
\]

always holds for some absolute constant \( A_r \). Since \( S_\mu(x) \) is almost periodic, the inequality (5) give us

\[
|f(x) - S_\mu(x)| \leq A_r \omega_r \left( f - S_\mu; \frac{1}{\mu} \right) \quad (r = 1, 2, \ldots).
\]

Thus, by use of the inequality (3) in lemma 4 and (6), we have

\[
|f(x) - S_\mu(x)| \leq A_r K \left\{ 1 + 2\delta^{-1} L^{-1} \log (1 + \mu L + 2^{-1}\delta L) \right\} \times \\
\omega_r \left( f; \frac{1}{\mu} \right)
\]

\[
\leq M_r \left\{ 1 + \delta^{-1} L^{-1} \log (1 + \mu L + \delta L) \right\} \omega_r \left( f; \frac{1}{\mu} \right)
\]

\((r = 1, 2, \ldots)\).

3. We are now to proceed to prove the following theorems.

**Theorem IX** (13). Let \( f(x) \) be a Bohr almost periodic function with the Fourier series

\[
f(x) \sim a_0 + \sum_{n=-\infty}^{\infty} \left( a(\lambda_n) e^{i\lambda_n x} + a(-\lambda_n) e^{-i\lambda_n x} \right) (\lambda_n \geq 0).
\]

Let \( 0 < \mu_1 < \mu_2 < \cdots \rightarrow + \infty \) and suppose that \( f(x) \) has an infinite number of intervals \((\mu_i, \mu_i + \delta_i), (i = 1, 2, \ldots)\) and that

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(12) S. Bochner [2].

(13) A special case \( G = 1 \) has already been treated by T. Kawata and S. Takahashi by means of a direct computation. See T. Kawata—S. Takahashi [1].
\[ \delta_i = \varepsilon \mu_i^G \quad (i = 1, 2, \ldots) \]

where \( G \) is any fixed great positive number and \( \varepsilon \), any fixed small positive number. Further assume that \( f(x) \) satisfies the Dini-Lipschitz condition

\[ \omega_1(\delta) \log \frac{1}{\delta} = 0(1), \quad (\delta \to 0) \]

Then a quasi-regular sequence \( S_{\nu_i}(x) \)

\[ S_{\nu_i}(x) \sim \alpha_0 + \sum_{\lambda_n < \mu_i} (a(\lambda_n)e^{\lambda_n x} + a(-\lambda_n)e^{-\lambda_n x}), \quad (\lambda_n > 0) \]

of partial sums of \( f(x) \) converges uniformly to \( f(x) \) as \( i \to +\infty \) in every finite interval.

**Theorem X.** Let \( 0 < \mu_1 < \mu_2 < \cdots \to +\infty \) and assume that \( f(x) \) has an infinite number of intervals \( (\mu_i, \mu_i + \delta_i), \ (i = 1, 2, \ldots) \) and that \( f(x) \) satisfies the Lipschitz condition of order \( a(0 < a \leq 1) \). Further suppose that

\[ \delta_i = \varepsilon \exp(-G \mu_i^{\alpha}) \quad (0 < \alpha < 1; \ i = 1, 2, \ldots) \]

where \( G \) is any fixed great positive number and \( \varepsilon \), any fixed small positive number. Then the conclusion of theorem IX is also valid.

**Theorem XI.** Let \( 0 < \mu_1 < \mu_2 < \cdots \to +\infty \) and assume that \( f(x) \) has an infinite number of intervals \( (\mu_i, \mu_i + \delta_i), \ (i = 1, 2, \ldots) \) and that \( f(x) \) has an almost periodic \( m \)-th derivative. Further suppose that

\[ \delta_i = \varepsilon \exp(-G \mu_i^m) \quad (i = 1, 2, \ldots) \]

where \( G \) is any fixed great positive number and \( \varepsilon \), any fixed small positive number. Then the conclusion of theorem IX is also valid.

4. The proof of the above three theorems follows immediately by use of the inequality (4) in lemma 5 which corresponds to the special case \( L = \delta^{-1} \). Then we have

\[
(7) \quad |f(x) - S_\nu(x)| \leq M_\nu \left\{ 2 + \log \left( 2 + \nu \delta^{-1} \right) \right\} \omega_r \left( f; \frac{1}{\mu} \right) \\
\leq M_\nu^* \left( 1 + \log \frac{\mu}{\delta} \right) \omega_r \left( f; \frac{1}{\mu} \right) \quad (r = 1, 2, \ldots)
\]

where \( M_\nu^* \) is some absolute constant.
Proof of theorem IX. From (7) we have

$$|f(x) - S_{\nu_i}(x)| \leq M_i^* \left(1 + \log \frac{\mu_i}{\delta_i}\right) \omega_1(f; \frac{1}{\mu_i})$$

$$= M_i^* \left(1 + (G + 1) \log \mu_i + \log \epsilon^{-1}\right) \omega_1(f; \frac{1}{\mu_i}).$$

Since

$$\omega_1(f; \frac{1}{\mu_i}) \leq \rho \left(\frac{1}{\mu_i}\right) \log \mu_i$$

with $$\rho \left(\frac{1}{\mu_i}\right) \rightarrow 0$$ as $$i \rightarrow + \infty$$, we obtain

$$|f(x) - S_{\nu_i}(x)| \leq M_i^* \frac{1 + (G + 1) \log \mu_i + \log \epsilon^{-1}}{\log \mu_i} \cdot \rho \left(\frac{1}{\mu_i}\right)$$

$$= O(1) \cdot \rho \left(\frac{1}{\mu_i}\right)$$

$$= O(1)$$ as $$i \rightarrow + \infty$$.

Proof of theorem X. From (7) we have

$$|f(x) - S_{\nu_i}(x)| \leq M_i^* (1 + G \mu_i^{a'} + \log \mu_i + \log \epsilon^{-1}) \omega_1(f; \frac{1}{\mu_i})$$

and since

$$\omega_1(f; \frac{1}{\mu_i}) \leq K \frac{1}{\mu_i^a} \quad (0 < a \leq 1)$$

where $$K$$ is a positive constant, we obtain

$$|f(x) - S_{\nu_i}(x)| \leq M_i^* K \frac{1 + G \mu_i^{a'} + \log \mu_i + \log \epsilon^{-1}}{\mu_i^a} \quad (0 < a' < a)$$

$$= O(1)$$ as $$i \rightarrow + \infty$$.

Proof of theorem XI. From (7) we have

$$|f(x) - S_{\nu_i}(x)| \leq M_{m+1}^* \left(1 + \log \frac{\mu_i}{\delta_i}\right) \omega_{m+1}(f; \frac{1}{\mu_i})$$

and since $$f(x)$$ has an almost periodic $$m$$-th derivative, it is easily seen

$$\omega_{m+1}(f; \frac{1}{\mu_i}) \leq N_m \omega_1(f^{(m)}; \frac{1}{\mu_i^{a'}})$$
where $N_m$ is some absolute constant. Thus we get

$$|f(x) - S_{\mu_i}(x)| \leq M_m^{\infty} N_m \frac{1 + G\mu_i^m + \log \mu_i + \log \epsilon^{-1}}{\mu_i^m} \omega_1 \left( f^{(m)}; \frac{1}{\mu_i} \right)$$

$$= O(1) \cdot \omega_1 \left( f^{(m)}; \frac{1}{\mu_i} \right)$$

$$= O(1) \quad \text{as} \quad i \to + \infty.$$ 

5. Our next object is to prove the following theorems analogous to that given in section 3 under somewhat different conditions.

**Theorem XII.** Let $f(x)$ be a real Bohr almost periodic function with the Fourier series

$$f(x) = P_1(x) - P_2(x)$$

where

and assume that $f(x)$ satisfies the Dirichlet-Jordan condition

$$f(x) = P_1(x) - P_2(x)$$

where

$$P_{k+1}(x) - P_k(x) \geq 0 \quad (k = 1, 2; \ x_1 \leq x_2)$$

$$\omega_{P_k}(\delta) = O(1) \quad (\delta \to 0; \ k = 1, 2)$$

$$P_k(x) = O(|x|^\alpha) \quad (\alpha > 1; \ k = 1, 2).$$

Further suppose that $f(x)$ has an infinite number of intervals $(\mu_i - \epsilon, \mu_i + \epsilon)$, $(0 < \mu_1 < \mu_2 < \cdots \to + \infty ; \ i = 1, 2, \cdots)$ and that

$$\epsilon = \epsilon \mu_i^{-K} \quad (0 < K < (a-1)^{-1}; \ i = 1, 2, \cdots)$$

where $\epsilon$ is any fixed small positive number. Then a quasi-regular sequence $S_{\mu_i}(x)$

$$S_{\mu_i}(x) \sim a_0 + \sum_{\lambda_n < \mu_i} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) \quad (\lambda_n > 0)$$

of partial sums of $f(x)$ converges uniformly to $f(x)$ as $i \to + \infty$ in every finite interval.

**Theorem XIII.** Let $0 < \mu_1 < \mu_2 < \cdots \to + \infty$ and assume that $f(x)$ has an infinite number of intervals $(\mu_i - \epsilon, \mu_i + \epsilon)$, $(i = 1, 2, \cdots)$ and that $f(x)$ satisfies the Dirichlet-Jordan condition with
Some new properties of Bohr almost periodic Fourier series.

$$P_0(x) = O(\lvert x \rvert) \quad (k = 1, 2)$$

$$\delta_i = \varepsilon \exp (-G \mu_i^a) \quad (0 < \alpha < 1; \ i = 1, 2, \ldots)$$

where $G$ is any fixed great positive number and $\varepsilon$, any fixed small positive number. Then the conclusion of theorem XII is also valid.

**Theorem XIV.** Let $0 < \mu_1 < \mu_2 < \cdots \to +\infty$ and assume that $f(x)$ has an infinite number of intervals $(\mu_i - \delta_i, \mu_i + \delta_i)$, $(i = 1, 2, \ldots)$ and that $f(x)$ satisfies the Dirichlet-Jordan condition with

$$P_0(x) = O(\lvert x \rvert^a) \quad (0 < \alpha < 1; \ i = 1, 2, \ldots).$$

Further let $\delta_i (i = 1, 2, \ldots)$ tend to zero in any way. Then the conclusion of theorem XII is also valid.

6. **Proof of theorem XII.** Putting.

$$f(x+t) + f(x-t) - 2f(x) = \phi(t),$$

we have

$$S_{\nu_2}(x) - f(x) = \frac{1}{\pi} \int_0^{\infty} \sin \frac{\mu_i t}{t} \phi(t)dt$$

$$= \frac{1}{\pi} \int_0^{\mu_i^a \delta_i^{-1}} + \frac{1}{\pi} \int_{\mu_i^a \delta_i^{-1}}^0 + \frac{1}{\pi} \int_0^{\mu_i^a \delta_i^{-1}}$$

$$= I_1 + I_2 + I_3$$

where we put $0 < \rho < 1; \ 0 < \lambda < (\alpha - 1)^{-1} - K$.

Now we have

$$\lvert I_1 \rvert \leq 2 \left\{ \omega P_1 (\mu_i^\rho) + \omega P_2 (\mu_i^\rho) \right\} \max_{0 \leq \nu \leq \mu_i^\rho} \left\lvert \sin \frac{\mu_i t}{t} \phi(t)dt \right\lvert$$

$$= 0 (1) \quad \text{as} \quad i \to +\infty.$$  

Next, by integration by parts, we get

$$I_3 = -\frac{1}{\pi} \frac{\delta_i}{\mu_i^a} \int_0^{\mu_i^a \delta_i^{-1}} \sin \frac{\mu_i t}{t} \phi(t)dt + \frac{1}{\pi} \int_0^{\mu_i^a \delta_i^{-1}} \frac{dt}{t^2} \int_0^{\mu_i u} \sin \mu_i u \phi(u)du$$

and lemma 2 in section 2 give us, using former notations,
\[
\left| \int_0^T \sin \mu_i t \phi(t) \, dt \right| \leq \left| \int_0^T \chi_{\eta_i}(x; t) \, dt \right| + 2 |f(x)| \left| \int_0^T \sin \mu_i t \, dt \right| \\
\leq \frac{4CM}{\delta_i} + \frac{4M}{\mu_i}.
\]

Therefore we obtain
\[
|I_3| \leq \frac{4CM}{\pi \mu_i^\lambda} + \frac{4M\delta_i}{\pi \mu_i \mu_i^\lambda} + \left( \frac{4CM}{\pi \delta_i} + \frac{4M}{\pi \mu_i} \right) \int_{\mu_i^{-\lambda} t}^{\infty} \frac{dt}{t^2}.
\]

\[(2)' \]
\[
= \frac{8CM}{\pi \mu_i^\lambda} + \frac{8M\delta_i}{\pi \mu_i \mu_i^\lambda}
\]
\[
= O(\mu_i^{-\lambda})
\]
uniformly in \(x\).

To estimate \(I_2\), we make a further application of integration by parts. Then

\[
I_2 = \frac{1}{\pi} \frac{\delta_i}{\mu_i^\lambda} \int_{\mu_i^{-\lambda} t}^{\infty} \sin \mu_i t \phi(t) \, dt + \frac{1}{\pi} \int_{\mu_i^{-\lambda} t}^{\infty} \frac{dt}{t^2} \int_{\mu_i^{-\lambda} t}^{\infty} \sin \mu_i u \phi(u) \, du.
\]

As in the estimation of \(I_3\), we obtain
\[(3)' \]
\[
|J_1| = O(\mu_i^{-\lambda})
\]
uniformly in \(x\).

Next, by the second mean value theorem in the integral calculus, we have
\[
\left| \int_{\mu_i^{-\lambda} t}^{\infty} \sin \mu_i u P_k(x + u) \, du \right|
\]
\[
= \left| P_k(x + t) \right| \left| \int_{\mu_i^{-\lambda} t}^{\infty} \sin \mu_i t \, dt \right| \leq \frac{2 |P_k(x + t)|}{\mu_i} \quad \left( \mu_i^{-\lambda} \leq \eta_k \leq t; \ k = 1, 2 \right),
\]
and similarly
\[
\left| \int_{\mu_i^{-\lambda} t}^{\infty} \sin \mu_i u P_k(x - u) \, du \right| \leq \frac{2 |P_k(x - t)|}{\mu_i} \quad (k = 1, 2).
\]
Thus

$$|J_2| \leq \frac{2}{\pi \mu_i} \sum_{k=1}^{L_2} \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \int t^a \left( |P_k(x+t)| + |P_k(x-t)| \right) \frac{dt}{t^2} + \frac{4M}{\pi \mu_i^{1-a}}$$

$$\leq \frac{4}{\pi \mu_i} O\left( \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \frac{|x+t|^a + |x-t|^a}{t^2} dt \right) + O(\mu_i^{-1+a}) \quad (a > 1)$$

$$\leq \frac{4}{\pi \mu_i} O\left( 2^a \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \frac{|x|^a}{t^2} dt + 2^a \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \frac{t}{t^2} dt \right) + O(\mu_i^{-1+a})$$

$$= O(\mu_i^{-1+a} + (\alpha - 1)^{-1} \mu_i^{-1} \mu_i^{1-a} (x-1) + O(\mu_i^{-1+a}))$$

$$= O(\mu_i^{-1+a} + \varepsilon^{-(x-1)} \mu_i^{-1} \mu_i^{1-a} (x-1) + O(\mu_i^{-1+a})$$

$$= O(\mu_i^{-1+a} + \mu_i^{(a+K)(x-1)} + O(\mu_i^{-1+a})$$

for any finite $x$ and since $0 < (\lambda + K)(\alpha - 1) < 1$, we get

$$|J_2| = O(\mu_i^{-x''}) \quad (\varepsilon' > 0)$$

uniformly in $x$. Combining this with $(3)'$, we have

$$|I_2| = O(\mu_i^{-x''}) \quad (\varepsilon'' > 0)$$

uniformly in $x$. From $(1)'$, $(2)'$, $(4)'$, we finally obtain

$$\lim_{x \to \infty} \{ S_{\mu_i}(x) - f(x) \} = 0$$

uniformly in every finite interval.

**Proof of theorem XIII.** We only need to estimate $J_2$ in the proof of theorem XII. The other parts remain unaltered. Putting $\lambda = 1$, we have now

$$|J_2| \leq \frac{4}{\pi \mu_i} O\left( \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \frac{|x+t|^a}{t^2} dt + \frac{4M}{\pi \mu_i^{1-a}} \right)$$

$$\leq \frac{4}{\pi \mu_i} O\left( \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \frac{|x|^a}{t^2} dt + \int_{\frac{\mu_i k}{L_2}}^{\frac{\mu_i k+1}{L_2}} \frac{dt}{t} \right) + O(\mu_i^{-1+a})$$

$$= O(\mu_i^{-1+a} + \mu_i^{-1} \log \mu_i + \mu_i^{-1} \log \varepsilon^{-1} + \mu_i^{-1} G \mu_i^a + \mu_i^{-1} p \log \mu_i)$$

$$+ O(\mu_i^{-1+a}) \quad (0 < a < 1)$$

$$= O\left( \frac{\log \mu_i}{\mu_i} \right) + O(\mu_i^{-x''}) \quad (\varepsilon' > 0)$$
uniformly in every finite interval.

Proof of theorem XIV. We also simply need to estimate $J_2$. In this case, we have

$$|J_2| = O\left(\mu_i^{-1} \int_{\mu_i^{-1}}^{\mu_i^{1-p}} \left| \frac{x+t}{t^a} + \frac{|x-t|}{t^a} \right| dt \right) + O(\mu_i^{1-p^*}), \quad (0 \leq a < 1)$$

$$= O\left(\mu_i^{-1} \int_{\mu_i^{-1}}^{\infty} \frac{|x|^p + t^p}{t^a} dt \right) + O(\mu_i^{1-p^*})$$

$$= O(\mu_i^{p+1} + \mu_i^{-1} \mu_i^{p(1-a)}) + O(\mu_i^{1-p^*})$$

uniformly in every finite interval.

7. Lastly we add here few remarks. The three theorems in section 3 in this paragraph show us an important fact that if we assume more strict regularity of a given Bohr almost periodic quasi-regular function, the lengths of the intervals may tend to zero more rapidly. Moreover the lengths of the intervals in the theorem XIV may tend to zero as rapid as we please. We believe that these theorems will throw some light on the second problem referred to in the introduction.

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