20. *The relative differential geometry in affine space* (*).

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Introduction.

In this paper a convex closed curve or a convex closed surface is respectively taken as the unit curve or the unit surface as in euclidean space⁽¹⁾.

In the first chapter a convex closed curve $E(c)$ is taken as the unit curve and we make correspondence between a point P on a curve and a point Q , on the unit curve, so that the tangents at the points P and Q are parallel to each other. Denote by q the affine perpendicular distance to Q on E from the origin O . At first I will give a geometrical meaning of the affine perpendicular distance. By using it we define the r -affine arc length and try to make a study about the theory of plane curves.

In the second chapter a convex closed surface $E(c)$ is taken as the unit surface and we make correspondence between a point P on a space curve and a point Q , on the unit surface, so that the tangent plane at Q is parallel to the affine rectifying plane at P . At first the affine perpendicular distance to the osculating plane and to the affine rectifying plane from a point are defined and the affine perpendicular distance⁽²⁾ of a surface from a point is introduced by the above definition, and consequently a geometrical meaning of this distance is given. By using the affine perpendicular distance of the unit surface we define the r -affine arc length and try to make a study about the theory of space curves.

In the third chapter a convex closed surface $E(c)$ is also taken as the unit surface and we make correspondence between a point P on a

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(1) J. Hirakawa, The Euclidean Relative Differential Geometry; I. Theory of plane curves. Proc. of the Physico Math. Soc. of Japan, 17 (1935); II. Theory of space curves. Jap. Journ. of Math., XIII (1936); III. Theory of surface-strips. Loc. cit., XIV (1937); IV. Theory of surfaces. Loc. cit., XIV (1938).

(2) W. Blaschke, Differential Geometrie, II, p. 110, Affinentfernung.

surface and a point Q on the unit surface, such that the tangent planes are parallel to each other. At first a relation between the two points P and Q is investigated. In the next place, by using the affine perpendicular distance of $E(c)$, we define a vector ξ_{ik} by the elementary tensor g_{ik} , and try to make a study about the theory of surfaces.

In conclusion I express many thanks to *Prof. T. Kubota* for his remarks concerning this investigation.

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Chapter 1. Theory of plane curves.

§ 1. Affine perpendicular distances and relative affine arc length.

Take a point P on a plane curve C and denote by P' and P'' two points consecutive to the point P . Let us denote the radius vector of P from the origin O by $\mathbf{r}(t)$ as a function of a parameter t , and denote by $\mathbf{r}(t+\Delta t)$ and $\mathbf{r}(t+\Delta t+\Delta(t+\Delta t))$ respectively the radius vectors OP' and OP'' . Take any point $A(a)$ on this plane, then we have

$$\begin{aligned} (1) \quad \lim_{P', P'' \rightarrow P} \frac{2\Delta AP'P''}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} |\mathbf{a} - \mathbf{r}(t+\Delta t), \mathbf{a} - \mathbf{r}(t+\Delta t+\Delta(t+\Delta t))| \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} |\mathbf{a} - \mathbf{r}(t+\Delta t), -\dot{\mathbf{r}}(t+\Delta t)\Delta(t+\Delta t) - \dots| \\ &= |\mathbf{a} - \mathbf{r}(t), -\dot{\mathbf{r}}(t)| \equiv \omega \end{aligned}$$

$$\begin{aligned} (2) \quad \lim_{P', P'' \rightarrow P} \frac{2\Delta PP'P''}{\Delta t^3} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t^3} |\mathbf{r}(t+\Delta t) - \mathbf{r}(t), \mathbf{r}(t+\Delta t+\Delta(t+\Delta t)) - \mathbf{r}(t)| \\ &= |\dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t)| \equiv \sigma^3. \end{aligned}$$

Let us put

$$p = \frac{\omega}{\sigma},$$

then we have

$$(3) \quad p = \frac{\omega}{\sigma} = \left| \mathbf{r} - \mathbf{a}, \frac{d\mathbf{r}}{dS} \right| = (\mathbf{a} - \mathbf{r})\xi \cdot \left| \frac{d\mathbf{r}}{dS} \right| = p|\xi'| = p\bar{p}^{\frac{1}{3}},$$

where ξ , $\bar{S}$, p and $\bar{p}$ denote respectively the unit normal vector, the affine arc length, the perpendicular distance from A to the tangent line at P and the radius of curvature at P . Thus we find that p is the same as the "Affinenfernung" which has been introduced by Prof. W. Süss⁽³⁾. Hereafter we call p the *affine perpendicular distance*.

In §3 we will prove the following

Theorem 1. *If the affine perpendicular distance from a fixed point to any point on a given curve C be constant, the curve C is a conic.*

Let us make correspondence between a point P on a curve C and a point Q on the unit curve $E(e)$ so that the tangent at P and Q are

(3) W. Süss, Zur relativen-Differentialgeometrie I, Jap. Journ. of Math., IV (1927).

parallel to each other; denote by q the affine perpendicular distance from the origin O to the point Q , then we have

$$(4) \quad q = \frac{\omega_0}{\sigma_0} = (-e\xi) \left| \frac{de}{d\bar{s}} \right| = q\bar{\rho}_0^{\frac{1}{3}},$$

where $\omega_0 = |e(t), \dot{e}(t)|$, $\sigma_0^3 = |\dot{e}(t), \ddot{e}(t)|$ and $\bar{s}$, q and $\bar{\rho}_0$ denote respectively the affine arc length, the perpendicular distance and the radius of curvature at Q .

The *relative affine arc length* dS is defined as

$$(5) \quad dS = \frac{\omega_0}{\sigma_0} \sigma dt = q d\bar{S}.$$

Now we will prove the following relation:

$$(6) \quad \frac{dS}{ds} = \frac{d\bar{S}}{d\bar{s}} = \rho^{\frac{2}{3}},$$

where ρ denotes the radius of r -curvature.⁽⁴⁾

As the corresponding tangents at P and Q are parallel, we can put

$$(7) \quad \frac{dx}{d\bar{S}} = \lambda \frac{de}{d\bar{s}},$$

where λ is a positive scalar quantity.

By means of the relations

$$(8) \quad \left| \frac{dx}{d\bar{S}} \right| = \bar{\rho}(x)^{\frac{1}{3}} \quad \text{and} \quad \left| \frac{de}{d\bar{s}} \right| = \bar{\rho}(e)^{\frac{1}{3}},$$

we have

$$\lambda = \left(\frac{\bar{\rho}(x)}{\bar{\rho}(e)} \right)^{\frac{1}{3}} = \rho^{\frac{1}{3}},$$

then (7) becomes

$$(9) \quad \frac{dx}{d\bar{S}} = \rho^{\frac{1}{3}} \frac{de}{d\bar{s}}.$$

By differentiating (1) with respect to $\bar{S}$ and using the relation

$$\left| \frac{dx}{d\bar{S}}, \frac{d^2x}{d\bar{S}^2} \right| = 1, \quad \text{we obtain}$$

(4) W. Süss, Zur relativen Differentialgeometrie I, Jap. Journ. of Math., IV. (1927), and J. Hirakawa, loc. cit. (1).

$$(10) \quad \frac{dS}{ds} = \frac{d\bar{S}}{d\bar{s}} = \kappa^2 = \rho^{\frac{2}{3}}.$$

By means of (1), (2) and (9), we obtain the following relation

$$(11) \quad \frac{\sigma}{\sigma_0} = \rho^{\frac{2}{3}}.$$

Taking the r -affine arc length S as parameter, we have from (5)

$$(12) \quad \Sigma = \frac{\Sigma_0}{\Omega_0} = \frac{1}{q},$$

where Σ_0 , Ω_0 and Σ denote respectively the values of σ_0 , ω_0 and σ for the parameter $t=s$.

Therefore we have

$$(13) \quad \Omega_0 = \rho^{-\frac{2}{3}}.$$

From (10) by making correspondence to the unit circle we find

$$(14) \quad \bar{\rho}(\bar{\chi})^{-\frac{2}{3}} d\bar{S} = \bar{\rho}(\epsilon)^{-\frac{2}{3}} d\bar{s} = \text{invariant} = d\theta,^{(5)}$$

where θ denotes the angle which the tangent makes with x -axis.

§ 2. R-affine curvature and fundamental equations.

Let us put

$$(15) \quad \frac{d\chi}{dS} = \chi' = \xi_1, \quad \text{and} \quad \frac{d^2\chi}{dS^2} = \chi'' = \frac{\xi_2}{q^3},$$

then from (12) we have

$$(16) \quad (\chi', \chi'') = \frac{1}{q^3},$$

and

$$(17) \quad (\xi_1, \xi_2) = 1.$$

By differentiating (17) with respect to S , we have

$$(18) \quad (\xi_1, \xi'_2) = 0,$$

(5) This relation is taken from (153) and (157) in page 32, W. Blaschke, *Diff. Geom. II*, (1923).

then we obtain the *fundamental relations* :

$$(19) \quad \xi'_1 = \frac{\xi_2}{q^3},$$

and

$$(20) \quad \xi'_2 + k\xi_1 = 0,$$

where k is called the *r-affine curvature*.

By means of (17) and (19) k is found from the equation

$$(21) \quad k = (\xi_2, \xi'_2) = q^3(\xi_2, \xi''_1).$$

From (18) we have

$$(22) \quad (\xi'_1, \xi'_2) + (\xi_1, \xi''_2) = 0,$$

and

$$(23) \quad (\xi'_1, \xi'_2) = (q^3\xi''_1, \xi''_2) = (\xi_2, \xi''_1).$$

Thus the following relations are obtained :

$$(24) \quad (\xi_2, \xi''_1) = (\xi''_2, \xi_1),$$

and

$$(25) \quad k = (\xi_2, \xi'_2) = q^3(\xi_2, \xi''_1) = q^3(\xi''_2, \xi_1).$$

In the next place we will seek for a relation between the *r-affine curvature* k and elementary affine curvature $\bar{k}$.

From the relations

$$(26) \quad \begin{aligned} \xi'_1 &= \frac{d\xi_1}{d\bar{S}} \frac{d\bar{S}}{dS} = \frac{1}{q} \frac{d\xi_1}{d\bar{S}}, \\ \xi''_1 &= \frac{1}{q^2} \frac{d^2\xi_1}{d\bar{S}^2} - \frac{1}{q^3} \frac{dq}{d\bar{S}} \frac{d\xi_1}{d\bar{S}}, \end{aligned}$$

we have

$$(27) \quad \xi_2 = q \frac{d^2\xi_1}{d\bar{S}^2} - \frac{dq}{d\bar{S}} \frac{d\xi_1}{d\bar{S}},$$

and

$$(28) \quad \xi'_2 = \frac{d^3\xi_1}{d\bar{S}^3} - \frac{1}{q} \frac{d^2q}{d\bar{S}^2} \frac{d\xi_1}{d\bar{S}}.$$

By means of (20) and the elementary formulae, which is found in case where $q = 1$ in (20), $\frac{d^3\xi_1}{d\bar{S}^3} = -\bar{k} \frac{d\xi_1}{d\bar{S}}$ ⁽⁶⁾, we have

(6) W. Blaschke, loc. cit. p. 15.

$$(29) \quad k = q\bar{k} + \frac{d^2q}{d\bar{S}^2}$$

From (19) and (21) we have

$$\xi_1'' = \frac{\xi_2'}{q^3} - \frac{q'}{q^4} \xi_2,$$

and

$$\xi_2'' = -k\xi_1' - k'\xi_1 = -\frac{k}{q^3}\xi_2 - k'\xi_1,$$

consequently we obtain

$$(30) \quad \xi_1' \times \xi_1'' = \frac{k}{q^6},$$

and

$$(31) \quad \xi_2' \times \xi_2'' = \frac{k^2}{q^3}.$$

Now we will proceed to seek for *Bouquet's* formulae: Let us take a point $P_0(x_0)$ on a curve C as the origin and take the tangent and the r -affine normal as x_1 - and x_2 -axis respectively. The coordinates $x(S)$ of a point P sufficiently near to $P_0(S=0)$ can be expanded in power series of S , whose absolute value is sufficiently small:

$$x(S) = x_0(0) + x_0'(0)S + \frac{1}{2!}x_0''(0)S^2 + \frac{1}{3!}x_0'''(0)S^3 + \dots$$

Hence we have

$$x_0 = \{0, 0\}, \quad x_0' = \{1, 0\}, \quad x_0'' = \{0, 1\},$$

and from (15) and (20)

$$x_0''' = \left\{ -\frac{k_0}{q_0^3}, -\frac{3q_0'}{q_0} \right\}.$$

Hence follow the extension of *Bouquet's* formulae:

$$(32) \quad x_1(S) = S - \frac{k_0}{3!q_0^3}S^3 + \dots,$$

$$x_2(S) = * + \frac{1}{2}S^2 - \frac{3q_0'}{3!q_0}S^3 + \dots$$

§ 3. R-affine perpendicular distances.

Take the point $T(\zeta)$ and take a point $P(\xi)$ on a curve C , then we define the r -affine perpendicular distance r as follows,

$$(33) \quad r = \left| \xi - \zeta, \frac{d\xi}{dS} \right|.$$

As a consequence of (3) and (5) we have

$$(34) \quad r = \frac{1}{q} \left| \xi - \zeta, \frac{d\xi}{dS} \right| = \frac{p}{q}.$$

Now will prove the following.

Theorem 2. *If a point ζ lies on a r -affine normal at ξ on C , the r -affine perpendicular distance r from ζ to the tangent at ξ has a stationary value.*

Proof. By means of (32) and (33) we have

$$\begin{aligned} r &= \left| x_1 - z_1, \frac{dx_1}{dS} \right| \\ &\quad \left| x_2 - z_2, \frac{dx_2}{dS} \right| \\ &= \left| \left(s - \frac{k_0}{3! q_0^3} S^3 + \dots \right) - z_1, \quad 1 - \frac{k_0}{2 q_0^3} S^2 + \dots \right| \\ &\quad \left| \left(\frac{1}{2} S^2 - \frac{3 q_0'}{3! q_0} S^3 + \dots \right) - z_2, \quad s - \frac{3 q_0'}{2 q_0} S^2 + \dots \right| \\ &= -z_1 \left(S - \frac{3 q_0'}{2 q_0} S^2 + \dots \right) + z_2 \left(1 - \frac{k_0}{2 q_0^3} S^2 + \dots \right) \\ &\quad + S^2 \left(1 - \frac{k_0}{3! q_0^3} S^3 + \dots \right) \left(1 - \frac{3 q_0'}{q_0} S + \dots \right) \\ &\quad - S^2 \left(1 - \frac{k_0}{2! q_0^3} S^2 + \dots \right) \left(\frac{1}{2} S - \frac{3 q_0'}{3! q_0} S^2 + \dots \right), \end{aligned}$$

then

$$\lim_{s \rightarrow 0} \frac{dr}{dS} = z_1 = 0. \quad (\text{Q. E. D.})$$

§ 4. Indicatrices of Tangents and of Curvatures.

The *indicatrix of tangents* is defined as the curve C_1 described by the point ξ_1 . Let us denote by q_1 the perpendicular distance from a point at the corresponding point on $E(c)$ for the point ξ_1 on C_1 , then from (30) the r -arc length S_1 is given by the equation

$$(35) \quad S_1 = \int_0^s q_1(\xi'_1 \times \xi''_1)^{\frac{1}{3}} dS = \int_0^s \frac{q_1}{q^2} k^{\frac{1}{3}} dS.$$

The *indicatrix of curvatures* is defined as the curve C_2 described by the point $-\xi_2$. Let us denote by S_2 the r -arc length on C_2 , then from (31) S_2 is given by the equation

$$S_2 = \int_0^s q((- \xi'_2) \times (- \xi''_2))^{\frac{1}{3}} dS = \int_0^s k^{\frac{2}{3}} dS.$$

Assume that C is closed and denote by L , L_1 and L_2 respectively the perimeters of C , C_1 and C_2 , and let us put

$$\begin{aligned} \text{Max. } k &= \bar{k}, & \text{Min. } k &= \underline{k}, \\ \text{Max. } q_1 q^{-2} &= \bar{Q} & \text{and Min. } q_1 q^{-2} &= \underline{Q}, \end{aligned}$$

then we obtain the following inequalities

$$(36) \quad \begin{aligned} \underline{Q} \bar{k}^{\frac{1}{3}} L &< L_1 < \bar{Q} \bar{k}^{\frac{1}{3}} L, \\ \underline{k}^{\frac{2}{3}} L &< L_2 < \bar{k}^{\frac{2}{3}} L. \end{aligned}$$

Denote by p_2 and p respectively the affine perpendicular distance and the elementary one from the origin O to a point $-\xi_2$ on C_2 , then we have

$$(37) \quad p_2 = \frac{|-\xi_2, -\xi'_2|}{|-\xi'_2|} = \frac{|k|}{|k||\xi_1|} = q \bar{\rho}^{-\frac{1}{3}},$$

therefore

$$p = p \bar{\rho}^{\frac{1}{3}} = p \frac{q}{p_2},$$

that is

$$(38) \quad \frac{p}{q} = \frac{p}{p_2}.$$

Denote by $\bar{\rho}_2$ the radius of curvature at $-\xi_2$ on C_2 , then by means of (20) and (31) we obtain

$$(39) \quad \bar{\rho}_2 = \frac{(\xi_2'')^{\frac{3}{2}}}{|\xi_2' \times \xi_2''|} = \frac{|k^3|}{q^3} \bar{\rho} \cdot \frac{1}{\frac{|k^2|}{q^3}} = |k| \bar{\rho},$$

therefore we have

$$(40) \quad |k| = \frac{\bar{\rho}_2}{\bar{\rho}}.$$

In consequence of (37) and (39) we have

$$(41) \quad p_2 = p_1 \bar{\rho}_2^{\frac{1}{3}} = q \bar{\rho}^{-\frac{1}{3}} |k|^{\frac{1}{3}} \bar{\rho}^{\frac{1}{3}} = q |k|^{\frac{1}{3}},$$

then

$$(42) \quad |k| = \left(\frac{p_2}{q} \right)^3.$$

Let us denote by p_0 and r_0 respectively the affine perpendicular distance and the r -affine perpendicular distance from the origin O to a point x on C . When r_0 is constant, the curve C is called the *relative affine circle*.

If the curve C be a r -affine circle, from (38) both curves C and C_2 are similar and similarly situated, then from (40) the r -affine curvature k is constant. Therefore from (42) it is concluded that the C_2 is a r -affine circle.

Conversely, if a curve C be a r -affine circle, it is easily seen that the r -affine perpendicular distance from the origin to a point on C is necessarily constant.

Hence we obtain the following

Theorem 3. *A necessary and sufficient condition that a curve C be a r -affine circle is that its r -affine curvature is constant.*

Theorem 4. *If a curve C be a r -affine circle, its indicatrix of curvature is also a r -affine circle.*

In case where q is equal to the unit length, by the relation $k = \bar{k}$ Theorem 2 is reduced to Theorem 1 in § 1, because a conic is the curve whose affine curvature is constant.⁽⁷⁾

Theorem 5. *A necessary and sufficient condition that a curve C be a conic is that its affine perpendicular distance from the properly chosen point is constant.*

(7) W. Blaschke, Diff. Geom., II, p. 18.

Now assume that C is a r -affine circle, then by a symmetric transformation

$$(43) \quad r = c^{\frac{3}{4}} r^*,$$

we have

$$(44) \quad r^* = \frac{p^*}{q} = 1,$$

that is

$$(45) \quad p^* = q.$$

Hereafter we consider the case where $p=q$: Denote by ξ the elementary normal unit vector, and let us put

$$(46) \quad \begin{aligned} p &= (-r\xi), & \bar{p}_2 &= (\ddot{r}\xi), \\ q &= (-c\xi), & \bar{q}_2 &= (\ddot{c}\xi), \end{aligned}$$

where $\ddot{r}$ and $\ddot{c}$ mean respectively derivatives with respect to the elementary affine arc length.

Then from (45) we have

$$(47) \quad \frac{p}{\bar{p}_2} = \frac{q}{\bar{q}_2} \quad (8)$$

Let us put

$$(48) \quad \frac{p}{\bar{p}_2} = \frac{q}{\bar{q}_2} = \lambda,$$

then we have

$$(49) \quad r + \lambda \ddot{r} = \mu \dot{r}.$$

By differentiating (49) with respect to $\bar{S}$ we obtain

$$\dot{r} - \lambda \bar{k} \dot{r} + \lambda \ddot{\ddot{r}} = \dot{\mu} \dot{r} + \mu \ddot{\dot{r}},$$

then

$$(50) \quad \mu = \dot{\lambda} \quad \text{and} \quad \frac{1}{\lambda} - \frac{\ddot{\lambda}}{\lambda} = \bar{k},$$

consequently (49) becomes

$$(51) \quad r + \lambda \ddot{r} = \lambda \dot{r}.$$

(8) This is easily seen from (3) in this paper and (153) in W. Blaschke, loc. cit., p. 32.

Differentiating (7) with respect to $\bar{S}$ we have

$$\ddot{\mathfrak{x}} = \dot{\bar{S}}^{-\frac{1}{2}} \ddot{\mathfrak{e}} + \frac{1}{2} \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} \dot{\mathfrak{e}},$$

where $\ddot{\mathfrak{x}} = \frac{d^2 \mathfrak{x}}{d\bar{S}^2}$, $\ddot{\mathfrak{e}} = \frac{d^2 \mathfrak{e}}{d\bar{S}^2}$, $\dot{\mathfrak{e}} = \frac{d\mathfrak{e}}{d\bar{S}}$, $\ddot{\bar{S}} = \frac{d^2 \bar{S}}{d\bar{s}^2}$ and $\dot{\bar{S}} = \frac{d\bar{S}}{d\bar{s}}$.

Substituting in (51) we obtain

$$(52) \quad \mathfrak{x} + \lambda \left(\dot{\bar{S}}^{-\frac{1}{2}} \ddot{\mathfrak{e}} + \frac{1}{2} \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} \dot{\mathfrak{e}} \right) - \lambda \dot{\bar{S}}^{-\frac{1}{2}} \dot{\mathfrak{e}} = 0.$$

From (48) in the same manner as we have obtained (51) we have the equation

$$(53) \quad \mathfrak{e} + \lambda \ddot{\mathfrak{e}} = \lambda \dot{\mathfrak{e}}.$$

By means of (53) the equation (52) becomes

$$(54) \quad \mathfrak{x} - \dot{\bar{S}}^{-\frac{1}{2}} \mathfrak{e} + \frac{1}{2} \lambda \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} \dot{\mathfrak{e}} = 0.$$

Differentiating (54) with respect to $\bar{s}$, we have

$$\ddot{\bar{S}}^{\frac{3}{2}} \dot{\mathfrak{e}} - \dot{\bar{S}}^{-\frac{1}{2}} \dot{\mathfrak{e}} + \frac{1}{2} \lambda \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} \ddot{\mathfrak{e}} + \frac{1}{2} \frac{d}{d\bar{s}} \left(\lambda \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} \right) \dot{\mathfrak{e}} = 0,$$

and by substituting (53)

$$\left\{ \ddot{\bar{S}}^{\frac{3}{2}} - \dot{\bar{S}}^{-\frac{1}{2}} + \frac{1}{2} \lambda \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} + \frac{1}{2} \frac{d}{d\bar{s}} \left(\lambda \ddot{\bar{S}} \dot{\bar{S}}^{-\frac{3}{2}} \right) \right\} \dot{\mathfrak{e}} = 0.$$

Hence the affine arc length $\bar{S}$ must satisfy the following differential equation

$$(55) \quad \bar{S}^4 - \bar{S}^2 + \lambda \ddot{\bar{S}} \bar{S} + \frac{1}{2} \lambda \ddot{\bar{S}} \dot{\bar{S}} - \frac{3}{4} \lambda \ddot{\bar{S}}^2 = 0.$$

From the relation $q/\bar{q}_2 = \lambda$ in (43), λ is found as a function of the affine arc length $\bar{s}$ of the unit curve, and consequently if we define $\bar{S}$ as a function of $\bar{s}$ such that it satisfies the differential equation (55), the unit r -affine circle is determined as the solution of the differential equation (51). Thus we have the following

Theorem 6. *If the affine arc length $\bar{S}$ of a curve satisfies (55) and its radius vector $\mathfrak{x}$ of the point on the curve satisfies (51), the curve necessarily be an unit r -affine circle, and its converse is also true. And the*

r-affine circle is a curve such that it is reduced to an unit *r*-affine circle by a symmetric transformation.

§ 5. R-affine evolutes.

R-affine normals of a curve $C(\mathfrak{x})$ envelope a curve $C^*(\mathfrak{x}^*)$ which is called the *r*-affine evolute. A point $\mathfrak{x}^*$ on C^* is given as follows

$$(56) \quad \mathfrak{x}^* = \mathfrak{x} + \lambda \xi_2.$$

Differentiating (56) with respect to S we have

$$\frac{d\mathfrak{x}^*}{dS} = (1 - \lambda k) \xi_1 + \frac{d\lambda}{dS} \xi_2 \equiv \sigma \xi_2,$$

then

$$\lambda = \frac{1}{k},$$

therefore (56) becomes

$$(57) \quad \mathfrak{x}^* = \mathfrak{x} + \frac{1}{k} \xi_2.$$

The point $\mathfrak{x}^*$ is called the *centre of r-affine curvature* at a point $\mathfrak{x}$ on C .

Denote by $\mathfrak{p}^*$ and $\mathfrak{r}^*$ respectively the affine perpendicular distance and *r*-affine perpendicular distance from $\mathfrak{x}^*$ to $\mathfrak{x}$, then from (33) we obtain

$$(58) \quad \mathfrak{r}^* = \left| \mathfrak{x} - \mathfrak{x}^*, \frac{d\mathfrak{x}}{dS} \right| = \left| -\frac{1}{k} \xi_2, \xi_1 \right| = \frac{1}{k}.$$

Hence we obtain from (58) and (57):

Theorem 7. *A necessary and sufficient condition that a curve be a r-affine circle is that the r-affine normals pass through a fixed point.*

Denote by S^* the *r*-affine arc length on the evolute $\mathfrak{x}^*$ and denote by $\mathfrak{q}^*$ the affine distance at the corresponding point on E , then we obtain

$$(59) \quad \frac{dS^*}{dS} = \mathfrak{q}^* \frac{k'^{\frac{2}{3}}}{k}$$

§ 6. Natural equations.

Let C and C^* be two curves defined in terms of their respective *r*-affine arc length S , and let us make correspond points with the same

values of S on each curve. Let us assume, furthermore, that at corresponding points the r -affine curvature have the same value. We shall prove that C and C^* are affinely transformable to each other by the same method as in the proof given by *Prof. W. Blaschke*.⁽⁹⁾

From the assumption $\mathfrak{x}$ and $\mathfrak{x}^*$ must satisfy the differential equations (20)

$$\xi'_2 + k(S)\xi_1 = 0,$$

or

$$(60) \quad (q^3 \mathfrak{x}'')' + k(S)\mathfrak{x}' = 0.$$

At first we consider the case where both tangents at the corresponding points $\mathfrak{x}$ and $\mathfrak{x}^*$ are parallel. In this case x'_1 and x'_2 must be a system of solutions of the linear differential equation of the second order.

$$(61) \quad (q^3 y')' + ky = 0.$$

Take y_1 and y_2 as the linearly independent solutions of (61), then we have

$$(q^3 y'_2)' y_1 - (q^3 y'_1)' y_2 = 0,$$

and consequently

$$(q^3 y'_2)' y_1 - (q^3 y'_1)' y_2 = \text{constant}.$$

By multiplication of a suitable constant we can make the above constant equal to one so that

$$(q^3 y'_2)' y_1 - (q^3 y'_1)' y_2 = 1.$$

Now let us put

$$x'_1 = a_{11} y_1 + a_{12} y_2,$$

$$x'_2 = a_{21} y_1 + a_{22} y_2.$$

Then we have

$$(\mathfrak{x}', \mathfrak{x}'') = (a_{11}a_{22} - a_{12}a_{21})(y_1 y'_2 - y_2 y'_1) = \frac{a}{q^3}. \quad (|a_{ii}| \equiv a \neq 0)$$

Let us take $a=1$ and put

$$x_1^* = \int_0^s y_1 dS, \quad x_2^* = \int_0^s y_2 dS,$$

thus the general solutions of (60) are given by the following equations

$$x_1 = a_{10} + a_{11}x_1^* + a_{12}x_2^*,$$

$$x_2 = a_{20} + a_{21}x_1^* + a_{22}x_2^*,$$

(9) Loc. cit., p. 17.

where

$$(\mathfrak{x}', q^3 \mathfrak{x}'') = 1.$$

In case where both tangents at the corresponding point $\mathfrak{x}$ and $\mathfrak{x}^*$ are not parallel, if q^* be affinely equivalent to q , the curve must be a conic or an affine transform of a periodic curve.⁽¹⁰⁾ In this case we can make the corresponding tangents parallel by an equivalent affine transformation, consequently it is reduced to the preceding case.

Hence we have the following

Theorem 8. *If the r -affine curvature k is given as an analytic function of the r -affine arc length S , a curve whose r -affine curvature is $k(S)$ is uniquely determined in general, except for the translation. In case where the unit curve is a periodic curve such that $q^*=q$, the curve is uniquely determined except for translations and an equivalent affine transformation, and in case where the unit curve is a conic the curve is uniquely determined except for translations and equivalent affine transformations.*

We call $k = k(S)$ the natural equation.

§ 7. Curves of constant r -affine breadth.

I have made an investigation concerning the *affine breadth*.⁽¹¹⁾ Recently *L. Beretta* and *A. Maxia*⁽¹²⁾ have made a study of the same subject in his paper. I will here extend the affine breadth in our geometry: Take a convex closed plane curve $C(\mathfrak{x})$ and denote the points P_1 and P_2 such that the tangent lines at P_1 and P_2 are parallel to each other by $\mathfrak{x}_1$ and $\mathfrak{x}_2$. Then the r -affine breadth b is defined as follows

$$(62) \quad b = 2 \frac{(\mathfrak{x}_2 - \mathfrak{x}_1) \xi_1}{(\eta_1 \xi_1) + (\eta_2 \xi_2)}$$

where ξ_i are the unit normal vectors at $\mathfrak{x}_i$ and η_i denotes the r -affine normal vector at $\mathfrak{x}_i$.

From the definition we have the following theorems:⁽¹³⁾

Theorem 9. *Let us denote by t_i the r -affine perpendicular distance from $\mathfrak{x}_i$ to $\mathfrak{x}_k$ ($i, k=1, 2; i \neq k$), where $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are the points of contact of the parallel tangents, then b is equal to the harmonic mean of t_1 and t_2 and b is an affine invariant.*

⁽¹⁰⁾ J. Hirakawa, E. R. G., I, loc. cit., Theorem 2 in § 2.

⁽¹¹⁾ J. Hirakawa, The affine breadth and its relation to the relative breadth, Jap. Journ. of Math., XII (1935).

⁽¹²⁾ L. Beretta e A. Maxia, Insiemi convessi e orbiformi, Memoria estratta dai Rendiconti di Matematica e delle sue applicazioni, Serie V. Volume 1-Fascicolo 1-1940.

⁽¹³⁾ We can prove in the same way as in the preceding literature (11).

Theorem 10. *A necessary and sufficient condition that the r -affine breadth of ξ be constant is that the two vectors $\xi_1 - \xi_2$ and $\eta_1 - \eta_2$ are parallel.*

Theorem 11. *If b is constant, then the following relation holds*

$$(63) \quad b = 2 \frac{\bar{\rho}_1 + \bar{\rho}_2}{k_1 \bar{\rho}_1 + k_2 \bar{\rho}_2},$$

where k_i denotes the r -affine curvature at ξ_i .

Theorem 12. *If the r -affine normals at the points of contact of two parallel tangents are r -affine double normal of an oval, the oval has the following properties:*

- 1°. *The oval is a curve of constant r -affine breadth.*
- 2°. *The sum of two radii of r -affine curvature at ξ_1 and ξ_2 is constant ($=b$).*
- 3°. *The indicatrix of r -affine curvature is central.*
- 4°. *The r -affine perpendicular distance between two points ξ_1 and ξ_2 is constant ($=b$).*
- 5°. *The sum of the two r -affine perpendicular distances from the origin to ξ_1 and ξ_2 is constant ($=b$).*

Proof. From the assumption it can be put

$$(64) \quad \eta_2 = -\sigma \eta_1,$$

where σ is a positive scalar quantity.

Then we have

$$(65) \quad -k_2 \xi'_2 = \sigma k_1 \xi'_1 - \sigma' \eta_1,$$

consequently $\sigma' = 0$ or $\sigma = \text{constant}$.

According to (64) we have $\eta_1 = -\sigma \eta_2$, therefore $\sigma = 1$. then

$$(66) \quad \eta_1 + \eta_2 = 0.$$

Hence we conclude that the indicatrix of r -affine curvature is central.

From the assumption and *Theorem 10* it is evident that the r -affine breadth b is constant.

By the definition of the r -affine breadth and the assumption it can be put

$$(67) \quad \xi_1 - \xi_2 + \frac{b}{2}(\eta_1 - \eta_2) = 0.$$

From (66) we have

$$(68) \quad \xi_1 - \xi_2 = -b\eta_1,$$

Let us form the vector product of both members of the above equation with ξ'_1 , then from $|\xi'_1, \eta_1| = 1$ we have

$$(69) \quad |\xi'_1, \xi_1 - \xi_2| = -b,$$

consequently

$$(70) \quad |\xi_1, \xi'_1| + |\xi_2, \xi'_2|\lambda = b,$$

Let r_i denote the r -affine perpendicular distance from the origin to ξ_i , then (70) becomes

$$(71) \quad r_1 + r_2\lambda = b,$$

consequently in the same manner we obtain

$$(72) \quad r_2 + r_1 \frac{1}{\lambda} = b.$$

By means of (70) and (71) we have $\lambda = 1$, then

$$(73) \quad \xi'_1 = -\xi'_2.$$

Therefore we obtain

$$(74) \quad r_1 + r_2 = b = \text{constant}.$$

By means of (69) and (73) we have

$$(75) \quad t_1 = |\xi_1 - \xi_2, \xi'_1| = |\xi_2 - r_1, \xi'_2| = t_2 = b = \text{constant}.$$

Differentiating (66) with respect to S_1 , we have

$$\eta'_1 + \eta'_2 \frac{dS_2}{dS_1} = 0,$$

from (20)

$$-k_1 \xi'_1 - k_2 \xi'_2 \frac{dS_2}{dS_1} = 0,$$

then

$$(76) \quad \frac{k_1}{k_2} = \frac{dS_2}{dS_1}.$$

Differentiating (68) with respect to S_1 and substituting (76), we have

$$(77) \quad b = \frac{1}{k_1} + \frac{1}{k_2}.$$

Chapter II. Theory of space curves.

§ 1. Affine perpendicular distances.

Take a point $P(\mathfrak{x}(t))$ on a space curve C and take three consecutive points $P'(\mathfrak{x}(t_1))$, $P''(\mathfrak{x}(t_2))$ and $P'''(\mathfrak{x}(t_3))$ on it, where $t_1 = t + \Delta t$, $t_2 = t_1 + \Delta t_1$ and $t_3 = t_2 + \Delta t_2$.

Let $A(a)$ be a fixed point in space, and denote by V_1 and V_2 respectively the volumes of two tetrahedrons $A-P_1P_2P_3$ and $P-P_1P_2P_3$, then we consider the following quantities:

$$\begin{aligned}
 (1) \quad \Omega &= \lim_{\Delta t \rightarrow 0} \frac{6 V_1}{\Delta t^3} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t^3} \left| a - \mathfrak{x}(t_1), \mathfrak{x}(t_1 + \Delta t_1) - \mathfrak{x}(t_1), \mathfrak{x}(t_2 + \Delta t_2) - \mathfrak{x}(t_2) \right| \\
 &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t^3} \left| a - \mathfrak{x}(t_1), \dot{\mathfrak{x}}(t_1) \Delta t_1 + \frac{\ddot{\mathfrak{x}}(t_1)}{2!} \Delta t_1^2 + \cdots, \dot{\mathfrak{x}}(t_2) \Delta t_2 + \frac{\ddot{\mathfrak{x}}(t_2)}{2!} \Delta t_2^2 + \cdots \right| \\
 &= |\alpha - \dot{\mathfrak{x}}, \ddot{\mathfrak{x}}, \ddot{\mathfrak{x}}|,
 \end{aligned}$$

and

$$\begin{aligned}
 (2) \quad \Sigma^2 &= \lim_{\Delta t \rightarrow 0} \frac{6 V_2}{\Delta t^6} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t^6} \left| \mathfrak{x}(t + \Delta t) - \mathfrak{x}(t), \mathfrak{x}(t_1 + \Delta t_1) - \mathfrak{x}(t_1), \mathfrak{x}(t_2 + \Delta t_2) - \mathfrak{x}(t_2) \right| \\
 &= |\ddot{\mathfrak{x}}, \ddot{\mathfrak{x}}, \ddot{\mathfrak{x}}|.
 \end{aligned}$$

Let us put

$$(3) \quad \mathfrak{p}_1 = \frac{\Omega}{\Sigma},$$

then we have

$$(4) \quad \mathfrak{p}_1 = \frac{|\alpha - \dot{\mathfrak{x}}, \ddot{\mathfrak{x}}, \ddot{\mathfrak{x}}|}{|\ddot{\mathfrak{x}}, \ddot{\mathfrak{x}}, \ddot{\mathfrak{x}}|^{\frac{1}{2}}} = \frac{|\alpha - \dot{\mathfrak{x}}, d\mathfrak{x}, d^2\mathfrak{x}|}{(dS)^3} = \left| \alpha - \dot{\mathfrak{x}}, \frac{d\mathfrak{x}}{dS}, \frac{d^2\mathfrak{x}}{dS^2} \right|,$$

where $d\bar{S}$ denotes the affine linear element.

$\mathfrak{p}_1$ is called the *affine perpendicular distance from A to the osculating plane at a point $\mathfrak{x}$ on C .*

It is evident that $\mathfrak{p}_1$ is an affine invariant and an invariant for parameter-transformations.

Let us denote the elementary perpendicular distance by p_1 , and denote the radius of curvature by ρ and denote the radius of torsion by τ , then we have

$$(5) \quad |p_1| = p_1 \rho^{-1} |\tau|^{-\frac{1}{2}}.$$

Now we consider the affine perpendicular distance ρ_1 concerning an asymptotic curve C on a surface $F(x(u, v))$. Take the asymptotic lines as parameteric curves, and let C be a parameteric curve $v = \text{constant}$, then we have

$$(6) \quad \begin{aligned} x_{uu} &= \alpha x_u + \beta x_v, \\ x_{vv} &= \gamma x_u + \delta x_v, \end{aligned}$$

and then

$$(7) \quad \begin{aligned} x_{uuu} &= \alpha x_{uu} + \beta x_{uv} + \alpha_u x_u + \beta_u x_v \\ &= \alpha(\alpha x_u + \beta x_v) + \alpha_u x_u + \beta_u x_v + \beta x_{uv} \\ &= (\alpha^2 + \alpha_u) x_u + (\alpha\beta + \beta_u) x_v + \beta x_{uv}. \end{aligned}$$

Therefore, from (4), (6) and (7), we obtain

$$(8) \quad p_1 = \frac{|\alpha - x, x_u, x_{uu}|}{|x_u, x_{uu}, x_{uuu}|^{\frac{1}{2}}} = \frac{|\alpha - x, x_u, x_v|}{|x_u, x_v, x_{uv}|^{\frac{1}{2}}},$$

then it is seen that ρ_1 is equal to "Affinentfernung" ⁽¹⁴⁾ defined by Prof. W. Blaschke.

Consequently a geometrical meaning of the affine perpendicular distance p_1 from a point A to a point P on a surface F is given as follows:

Theorem 1. If we denote by V_2 and V_1 respectively the volumes of the tetrahedron formed by the point P and the three consecutive points on an asymptotic line passing through P and the tetrahedron formed by the point A and the same three consecutive points of P , the affine perpendicular distance p from A to P is equal to $\lim_{\Delta t \rightarrow 0} \sqrt[3]{6} V_1 / \sqrt[3]{V_2}$.

Now let us put

$$(9) \quad p_2 = \left| x - \alpha, \frac{dx}{dS}, \frac{d^2x}{dS^2} \right|,$$

(14) Loc. cit. Diff. Geom. II., p. 110.

then we call p_2 the *affine perpendicular distance* from a point α to the *affine rectifying plane* at a point ξ on C . Hence p_2 is evidently an *affine invariant*.

§ 2. R-affine arc length.

Take a closed convex surface $E(e)$ as the unit surface and let the point $P(\xi)$ on a space curve C so correspond to the point $Q(e)$ on the unit surface E that the tangent plane at Q is parallel to the affine rectifying plane at P on C . The locus of Q is called the unit curve, $C_0(e)$. Let us denote the affine perpendicular distance from the origin to Q on E by q , then we have

$$(10) \quad q = \frac{|e_u, e_v, -e|}{|ln - m^2|^{\frac{1}{2}}},$$

where $l = |e_u, e_v, e_{uu}|$, $m = |e_u, e_v, e_{uv}|$ and $n = |e_u, e_v, e_{vv}|$.

The *relative affine arc length* S is defined as

$$(11) \quad dS = q \Sigma^{\frac{1}{3}} dt = q |\dot{\xi}, \ddot{\xi}, \ddot{\xi}|^{\frac{1}{6}} dt = q d\bar{S}.$$

If $t = S$, we have

$$(12) \quad q^6 |\dot{\xi}', \ddot{\xi}'', \ddot{\xi}''| = 1.$$

Now let us put

$$(13) \quad \dot{\xi}' = \xi_1, \quad \ddot{\xi}'' = \xi_2, \quad q^6 \ddot{\xi}''' = \xi_3,$$

then we get

$$(14) \quad |\xi_1, \xi_2, \xi_3| = 1,$$

and consequently

$$(15) \quad |\xi_1, \xi_2, \xi_3'| = 0,$$

Therefore we have the *fundamental equation*:

$$(16) \quad \xi_3' + k_1 \xi_2 + k_2 \xi_1 = 0,$$

where k_1 and k_2 are respectively called the *relative affine curvature* and the *relative affine torsion*. Moreover ξ_1 , ξ_2 and ξ_3 are called respectively the *tangential*, *principal normal*- and *binormal-unit vector*. Hence we have the extension of *Frenet-Serret* formulae:

$$(17) \quad \xi'_1 = \xi_2, \quad \xi'_2 = \frac{\xi_2}{q^6}, \quad \xi'_3 = -k_1 \xi_2 - k_2 \xi_1.$$

k_1 and k_2 are found from the following

$$(18) \quad k_1 = |\xi_1, \xi_3, \xi'_3|, \quad k_2 = -|\xi_2, \xi_3, \xi'_3|.$$

In the next place we will proceed to seek for *Bouquet's* formulae: Let us take a point $P_0(\xi_0)$ on a curve C as the origin and take the tangent, the r -affine principal normal and the r -affine binormal respectively as x_1 -, x_2 - and x_3 -axis. The coordinates $\chi(S)$ of a point P sufficiently near to $P_0(S=0)$ can be expanded in power series of S , whose absolute value is sufficiently small.

$$\chi(S) = \chi_0 + \chi'_0 S + \frac{\chi''_0}{2!} S^2 + \dots$$

However we find

$$\chi_0 = \{0, 0, 0\}, \quad \chi'_0 = \{1, 0, 0\}, \quad \chi''_0 = \{0, 1, 0\}, \quad \chi'''_0 = \{0, 0, 1\},$$

and from (12)

$$\begin{aligned} \chi_0^{\text{IV}} &= \left\{ -\frac{k_{20}}{Q_0}, -\frac{k_{10}}{Q_0}, -\frac{Q'_0}{Q_0^2} \right\}, \\ \chi_0^{\text{V}} &= \left\{ \frac{1}{Q^2} (2Q'_0 k_{20} - Q_0 k'_{20}), \frac{1}{Q_0^2} (2k_{10} Q'_0 - Q_0 k'_{10} - Q_0 k_{20}), \right. \\ &\quad \left. \frac{1}{Q_0^3} (2Q'_0 - Q_0 Q''_0 - Q_0 k_{10}) \right\}, \end{aligned}$$

where k_{10} , k_{20} and $Q_0 = q_0^6$ denote respectively the values of corresponding quantities k_1 , k_2 and $Q = q^6$ for $S = 0$.

Thus we get the extension of *Bouquet's* formulae:

$$\begin{aligned} x_1(S) &= S * * - \frac{k_{20}}{4! Q_0} S^4 + \frac{1}{5! Q_0^2} (2Q'_0 k_{20} - Q_0 k'_{20}) S^5 + \dots, \\ (19) \quad x_2(S) &= * \frac{1}{2!} S^2 * - \frac{k_{10}}{4! Q_0} S^4 + \frac{1}{5! Q_0^2} (2k_{10} Q'_0 - Q_0 k'_{10} - Q_0 k_{20}) S^5 + \dots, \\ x_3(S) &= * * \frac{1}{3! Q_0^2} S^3 - \frac{Q'_0}{4! Q_0^2} S^4 + \frac{1}{5! Q_0^3} (2Q_0'^2 - Q_0 Q_0'' - Q_0 k_{10}) S^5 + \dots \end{aligned}$$

Now from our correspondence between two points $\mathfrak{r}$ and $\mathfrak{e}$ we can put

$$(20) \quad \mathfrak{e}_i = a_i^1 \bar{\xi}_1 + a_i^3 \bar{\xi}_3, \quad (i = 1, 2)$$

where $\mathfrak{e}$ is the radius vector of the point on the unit curve C_0 and $\mathfrak{e}_i = \frac{\partial \mathfrak{e}}{\partial u^i}$. Then $\mathfrak{e}$ can be considered as a function of $\bar{S}$. Consequently differentiating (20) with respect to $\bar{S}$ we have

$$(21) \quad \begin{aligned} c_{ij} \frac{d\bar{u}^j}{d\bar{S}} &= \frac{da_i^1}{d\bar{S}} \bar{\xi}_1 + \frac{da_i^3}{d\bar{S}} \bar{\xi}_3 + a_i^1 \bar{\xi}_2 - a_i^3 (\bar{k}_1 \bar{\xi}_2 + \bar{k}_2 \bar{\xi}_1) \\ &= \left(\frac{da_i^1}{d\bar{S}} - a_i^3 \bar{k}_2 \right) \bar{\xi}_1 + (a_i^1 - \bar{k}_1 a_i^3) \bar{\xi}_2 + \frac{da_i^3}{d\bar{S}} \bar{\xi}_3, \end{aligned}$$

then

$$(22) \quad l_{ij} \frac{d\bar{u}^j}{d\bar{S}} = \left| \mathfrak{e}_1, \mathfrak{e}_2, c_{ij} \frac{d\bar{u}^j}{d\bar{S}} \right| = a(\bar{k}_1 a_i^3 - a_i^1),$$

and

$$(23) \quad \frac{d\bar{u}^j}{d\bar{S}} = a(\bar{k}_1 a_i^3 - a_i^1) l^{ij}$$

If we take the asymptotic curves as parameteric curves, then we obtain

$$(24) \quad \begin{aligned} \frac{d\bar{u}^1}{d\bar{S}} &= a(\bar{k}_1 a_2^3 - a_2^1) l^{21} = \frac{1}{l_{12}} (\bar{k}_1 a_2^3 - a_2^1), \\ \frac{d\bar{u}^2}{d\bar{S}} &= a(\bar{k}_1 a_1^3 - a_1^1) l^{12} = \frac{1}{l_{12}} (\bar{k}_1 a_1^3 - a_1^1). \end{aligned}$$

§ 3. R-affine perpendicular distances.

The *r-affine perpendicular distance* r_1 , from a point $\mathfrak{z}$ to the osculating plane at $\mathfrak{r}$ is defined as (3) in § 1 by the equation

$$(25) \quad r_1 = q^2 \left| \mathfrak{z} - \mathfrak{r}, \frac{d\mathfrak{r}}{dS}, \frac{d^2\mathfrak{r}}{dS^2} \right|.$$

Thus we obtain easily the following equation

$$(26) \quad r_1 = \frac{p_1}{q}.$$

In the next place the *r-affine perpendicular distance* r_2 from a point $\mathfrak{z}$ to the rectifying plane at $\mathfrak{x}$ is defined analogous to (9) by the expression

$$(27) \quad r_2 = \left| \mathfrak{x} - \mathfrak{z}, \frac{d\mathfrak{x}}{dS}, \frac{d^3\mathfrak{x}}{dS^3} \right| q^6.$$

From the definition we have

$$(28) \quad r_2 = (\mathfrak{x} - \mathfrak{z}) (\xi_1 \times \xi_3) = | \mathfrak{x} - \mathfrak{z}, \xi_1, \xi_3 |.$$

By means of the moving trihedral $(\mathfrak{x}) - (\xi_1)(\xi_2)(\xi_3)$ any point $\mathfrak{z}$ in space can be represented by

$$(29) \quad \mathfrak{z} = \mathfrak{x} + a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3.$$

Substituting in (25) we have

$$(30) \quad r_1 = a_3,$$

Substituting in (27) we have

$$(30') \quad r_2 = a_2.$$

Now we prove that if a point $\mathfrak{z}$ lies on the *r-affine rectifying plane* of a space curve, then the *r-affine perpendicular distance* is stationary.

By means of (19) and (25) we obtain

$$r_1 = q^2 \begin{vmatrix} z_1 - \left(S - \frac{k_{20}}{4! Q_0} S^4 + \dots \right) & 1 - \frac{k_{20}}{3! Q_0} S^3 + \dots & -\frac{k_{20}}{2Q_0} S^2 + \dots \\ z_2 - \left(\frac{1}{2!} S^2 - \frac{k_{10}}{4! Q_0} S^4 + \dots \right) & S - \frac{k_{10}}{3! Q_0} S^3 + \dots & 1 - \frac{k_{10}}{2Q_0} S^2 + \dots \\ z_3 - \left(\frac{1}{3! Q_0} S^3 - \frac{Q'_0}{4! Q_0^2} S^4 + \dots \right) & \frac{1}{2Q_0} S^2 - \frac{Q'_0}{3! Q_0} S^3 + \dots & \frac{1}{Q_0} S - \frac{Q'_0}{2Q_0^2} S^2 + \dots \end{vmatrix},$$

then

$$\begin{aligned} \frac{dx_1}{ds} = q^2 & \begin{vmatrix} z_1 - \left(S - \frac{k_{20}}{4! Q_0} S^4 + \dots \right) & 1 - \frac{k_{20}}{3! Q_0} S^3 + \dots & -\frac{k_{20}}{Q_0} S + \dots \\ z_2 - \left(\frac{1}{2} S^2 - \frac{k_{10}}{4! Q_0} S^4 + \dots \right) & S - \frac{k_{10}}{3! Q_0} S^3 + \dots & -\frac{k_{10}}{Q_0} S + \dots \\ z_3 - \left(\frac{1}{3! Q_0} S^3 - \frac{Q'_0}{4! Q_0^2} S^4 + \dots \right) & \frac{1}{2Q_0} S^2 - \frac{Q'_0}{3! Q_0^2} S^3 + \dots & \frac{1}{Q_0} - \frac{Q'_0}{Q_0^2} S + \dots \end{vmatrix} \\ + 2q \frac{dq}{ds} & \begin{vmatrix} z_1 - \left(S - \frac{k_{20}}{4! Q_0} S^4 + \dots \right) & 1 - \frac{k_{20}}{3! Q_0} S^3 + \dots & -\frac{h_{20}}{2Q_0} S^2 + \dots \\ z_2 - \left(\frac{1}{2!} S^2 - \frac{k_{10}}{4! Q_0} S^4 + \dots \right) & S - \frac{k_{10}}{3! Q_0} S^3 + \dots & 1 - \frac{k_{10}}{2Q_0} S^2 + \dots \\ z_3 - \left(\frac{1}{3! Q_0} S^3 - \frac{Q'_0}{4! Q_0^2} S^4 + \dots \right) & \frac{1}{2Q_0} S^2 - \frac{Q'_0}{3! Q_0^2} S^3 + \dots & \frac{1}{Q_0} S - \frac{Q'_0}{2Q_0^2} S^2 + \dots \end{vmatrix}, \end{aligned}$$

therefore

$$\begin{aligned} (31) \quad \lim_{s \rightarrow 0} \frac{dx_1}{dS} &= q^2 \begin{vmatrix} z_1 & 1 & 0 \\ z_2 & 0 & 0 \\ z_3 & 0 & \frac{1}{Q_0} \end{vmatrix} + 2q \lim_{s \rightarrow 0} \frac{dq}{dS} \begin{vmatrix} z_1 & 1 & 0 \\ z_2 & 0 & 1 \\ z_3 & 0 & 0 \end{vmatrix} \\ &= -q^2 \frac{z_2}{Q_0} + 2q z_3 \lim_{s \rightarrow 0} \frac{dq}{dS}. \end{aligned}$$

However we have

$$\frac{dq}{dS} = \frac{\partial q}{\partial u^i} \frac{du^i}{dS}. \quad (i = 1, 2)$$

If the affine normal at Q passing through the origin, then we have

$$(32) \quad \lim_{s \rightarrow 0} \frac{\partial q}{\partial u^i} = 0. \quad (i = 1, 2) \quad (15)$$

The values of $\lim_{s \rightarrow 0} \frac{du^1}{dS}$ and $\lim_{s \rightarrow 0} \frac{du^2}{dS}$ are different from zero in general. Thus we get

Theorem 2. *The r -affine perpendicular distance from a point z to the osculating plane at a point x on C , has a stationary value, when z lies on*

(15) W. Blaschke, loc. cit., p. 111.

the r -affine rectifying plane and the affine surface normal at the corresponding point on E passes through the origin.

Cor. The affine perpendicular distance p_1 has a stationary value, when z lies on the affine rectifying plane.

By the same method we have

$$(33) \quad \lim_{s \rightarrow 0} \frac{dx_2}{dS} = q^6 \left(\frac{z_1}{Q_0} + \frac{Q'_0}{Q_0^2} z_2 - \frac{k_{10}}{Q_0} z_3 \right) - \frac{z_2}{Q_0} 6 q^5 \lim_{s \rightarrow 0} \frac{dq}{dS}.$$

Consequently we get the following

Theorem 3. The r -affine perpendicular distance from a point z to the r -affine rectifying plane at a point x on C , has a stationary value, when z lies on the r -affine principal normal, the affine surface normal at the corresponding point on E passes through the origin and $Q'_0 = 0$.

Cor. The affine perpendicular distance p_2 has a stationary value, when z lies on the affine principal normal.

§ 4. Indicatrices of tangents, r -curvatures and r -covariant curvatures.

The curves described by the end points of the vectors, from the origin, ξ_1 , $-\xi_2$ and $\xi_3 + k_1 \xi_1$, at a point P on a curve $C(x)$ are called respectively the *indicatrix of r -tangents*, the *indicatrix of r -principal normals* and the *indicatrix of covariant r -curvatures* in affine space. We will here discuss the relations between a curve $C(x)$ and its indicatrices.

Denote by η_1 the coordinates of a point on the indicatrix C_1 of r -tangent and denote by S_1 the arc length of C_1 , then we have

$$\begin{aligned} dS_1 &= q_1 \left| \eta'_1, \eta''_1, \eta'''_1 \right|^{\frac{1}{6}} dS \\ &= q_1 \left| \xi_2, \frac{\xi_3}{Q} \cdot \frac{1}{Q} (-k_1 \xi_2 - k_2 \xi_1) \right|^{\frac{1}{6}} dS, \end{aligned}$$

therefore

$$(34) \quad dS_1 = \frac{q_1}{q^2} (-k_2)^{\frac{1}{6}} dS.$$

Consequently k_2 is given by

$$(35) \quad k_2 = - \left(\frac{q d\bar{S}_1}{d\bar{S}} \right)^6.$$

Therefore by means of (2) and (8) in § 1, a geometrical meaning of k_2 is given :

Theorem 4. *If we denote by V_2' the corresponding quantity on C_1 to V_2 on C and denote by V_i^0 the corresponding quantities of the asymptotic curve on the unit surface to V_i on C , then we have*

$$(36) \quad k_2 = -\lim_{\Delta t \rightarrow 0} \frac{\sqrt{6} V_1' V_1^0}{V_2' \sqrt{V_2^0}}.$$

Denote the point coordinates on the indicatrix C_2 of r -principal normals by η_2 and its arc length by s_2 , then we have

$$\begin{aligned} dS_2 &= q_2 |\eta_2', \eta_2'', \eta_2'''|^{\frac{1}{6}} dS \\ &= q_2 |\xi''', \xi^{IV}, \xi^V|^{\frac{1}{6}} dS \\ &= \frac{q_2}{\sqrt{Q}} \left| \xi_3, k_1 \xi_2 + k_2 \xi_1, \left(k_1' + k_2 + \frac{2Q'}{Q} k_1 \right) \xi_2 + \left(k_2' + \frac{2Q'}{Q} k_2 \right) \xi_1 \right|^{\frac{1}{6}} dS, \end{aligned}$$

consequently

$$(37) \quad dS_2 = \frac{q_2}{q^3} (k_2^2 + k_1' k_2 - k_1 k_2')^{\frac{1}{6}} dS.$$

If we denote the point coordinates on the indicatrix of covariant r -curvatures by η_3 and its r -arc length by S_3 , then we have

$$(38) \quad dS_3 = q_3 |\eta_3', \eta_3'', \eta_3'''|^{\frac{1}{6}} dS,$$

now

$$\begin{aligned} \eta_3 &= \xi_3 + k_1 \xi_1, \\ \eta_3' &= -(k_1 \xi_2 + k_2 \xi_1) + k_1 \xi_1' + k_1' \xi_1 = (k_1' - k_2) \xi_1, \\ \eta_3'' &= (k_1' - k_2) \xi_2 + (k_1'' - k_2') \xi_1, \\ \eta_3''' &= \frac{1}{Q} (k_1' - k_2) \xi_3 + 2(k_1'' - k_2') \xi_2 + (k_1''' - k_2'') \xi_1. \end{aligned} \quad (39)$$

Substituting (39) in (38) we get

$$\begin{aligned} (40) \quad dS_3 &= q_3 \left| (k_1' - k_2) \xi_1, (k_1' - k_2) \xi_2, (k_1' - k_2) \frac{\xi_3}{Q} \right|^{\frac{1}{6}} dS \\ &= \frac{q_3}{q} (k_1 - k_2)^{\frac{1}{2}} dS. \end{aligned}$$

From this fact we get the following

Theorem 5. *If the straight line parallel to the vector $\xi_3 + k_1 \xi_1$ and passing through the point $\mathfrak{x}$ on a space curve, passes through a fixed point, then we have $\bar{S}_3 = a + b\bar{S}$, where a and b are constants, and the two curves C and C_3 are parallel curves.*

Proof. Denote the fixed point by $\mathfrak{z}$, then we have

$$\mathfrak{z} = \mathfrak{x} + \lambda(\xi_3 + k_1 \xi_1),$$

$$\mathfrak{z}' = (1 - \lambda k_2 + \lambda k_1' + \lambda' k_1) \xi_1 + \lambda' \xi_3 = 0,$$

consequently

$$\lambda' = 0 \quad \text{or} \quad \lambda = \text{constant},$$

and

$$k_2 - k_1' = \frac{1}{\lambda} = \text{constant}.$$

Thus from (40) the result follows.

§ 5. A geometrical meaning of the r -affine principal normal.

Denote the coordinates of a point on the tangent surface of a space curve $C(\mathfrak{x})$ by $\mathfrak{X} = \{X_1, X_2, X_3\}$, then we have

$$\mathfrak{X} = \mathfrak{x} + \tau \mathfrak{x}',$$

where τ is the parameter and $\mathfrak{x}' = \frac{d\mathfrak{x}}{dS}$.

By using the formulae (19).

$$\begin{aligned} X_1 &= x_1 + \tau x_1' = S - \frac{k_{20}}{4! Q_0} S^4 + \dots + \tau \left(1 - \frac{k_{20}}{3! Q_0} S^3 + \dots \right), \\ (41) \quad X_2 &= x_2 + \tau x_2' = \frac{1}{2} S^2 - \frac{k_{10}}{4! Q_0} S^4 + \dots + \tau \left(S - \frac{k_{10}}{3! Q_0} S^3 + \dots \right), \\ X_3 &= x_3 + \tau x_3' = \frac{1}{3! Q_0} S^3 - \frac{Q_0'}{4! Q_0} S^4 + \frac{A_0}{5! Q_0^3} S^5 + \dots \\ &\quad + \tau \left(\frac{1}{2 Q_0} S^2 - \frac{Q_0'}{3! Q_0^2} S^3 + \frac{A_0}{4! Q_0^3} S^4 + \dots \right), \end{aligned}$$

where $A_0 \equiv 2 Q_0' - Q_0 Q_0'' - Q_0 k_{10}$.

As the point $\mathfrak{X}$ lies on the r -osculating plane,

$$X_3 = 0,$$

and we have

$$\begin{aligned}
 (42) \quad \tau &= -\frac{\frac{1}{3!} S^3 - \frac{Q'_0}{4! Q_0^2} S^4 + \frac{A_0}{5! Q_0^3} S^5 + \dots}{\frac{1}{2Q_0} S^2 - \frac{Q'_0}{3! Q_0^2} S^3 + \frac{A_0}{4! Q_0^3} S^4 + \dots} \\
 &= -\frac{S}{3} \left(1 - \frac{Q'_0}{4Q_0} S + \frac{A_0}{20Q_0^2} S^2 + \dots \right) \left(1 - \frac{Q'_0}{3Q_0} S + \frac{A_0}{12Q_0} S^2 + \dots \right)^{-1} \\
 &= -\frac{S}{3} \left(1 - \frac{Q'_0}{4Q_0} S + \frac{A_0}{20Q_0^2} S^2 + \dots \right) \left(1 + \frac{Q'_0}{3Q_0} S - \frac{A_0}{12Q_0^2} S^2 + \frac{Q_0'^2}{9Q_0^2} S^2 + \dots \right) \\
 &= -\frac{S}{3} \left(1 + \frac{Q'_0}{12Q_0} S + \frac{5Q_0'^2 - 6A_0}{180Q_0^2} S^2 + \dots \right).
 \end{aligned}$$

Substituting (42) in X_1 and X_2 of (41) we have

$$\begin{aligned}
 X_1 &= S - \frac{k_{20}}{4! Q_0} S^4 + \dots - \frac{S}{3} \left(1 + \frac{Q'_0}{12Q_0} S + \frac{5Q_0'^2 - 6A_0}{180Q_0^2} S^2 + \dots \right) \left(1 - \frac{k_{20}}{3! Q_0} S^3 + \dots \right) \\
 &= \frac{2}{3} S - \frac{Q'_0}{36Q_0} S^2 - \frac{5Q_0'^2 - 6A_0}{540Q_0^2} S^3 + \dots, \\
 X_2 &= \frac{1}{2} S^2 - \frac{k_{10}}{4! Q_0} S^4 + \dots - \frac{S}{3} \left(1 + \frac{Q'_0}{12Q_0} S + \frac{5Q_0'^2 - 6A_0}{180Q_0^2} S^2 + \dots \right) \\
 &\quad \times \left(S - \frac{k_{10}}{3! Q_0} S^3 + \dots \right) = \frac{1}{6} S^2 - \frac{Q'_0}{36Q_0} S^3 + \dots,
 \end{aligned}$$

and

$$\begin{aligned}
 (43) \quad \left(\frac{dX_1}{dS} \right)_{s=0} &= \frac{2}{3}, \quad \left(\frac{d^2X_1}{dS^2} \right)_{s=0} = -\frac{Q'_0}{18Q_0}, \quad \left(\frac{d^3X_1}{dS^3} \right)_{s=0} = -\frac{5Q_0'^2 - 6A_0}{90Q_0^2}, \\
 \left(\frac{dX_2}{dS} \right)_{s=0} &= 0, \quad \left(\frac{d^2X_2}{dS^2} \right)_{s=0} = \frac{1}{3}, \quad \left(\frac{d^3X_2}{dS^3} \right)_{s=0} = -\frac{Q'_0}{6Q_0},
 \end{aligned}$$

therefore

$$\left(\begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix} \right)_{s=0} = \frac{2}{9}, \quad \left(\begin{vmatrix} X'_1 & X'''_1 \\ X'_2 & X'''_2 \end{vmatrix} \right)_{s=0} = -\frac{Q'_0}{9Q_0}.$$

Denote by S^* the r -affine arc length of the curve C^* , which is obtained as the osculating plane section of the tangent surface of C , then we have

$$\left(\frac{dS^*}{dS}\right)_{s=0} = \left(q^* \begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix} \right)^{\frac{1}{3}}_{s=0} = q^* \left(\frac{2}{9}\right)^{\frac{1}{3}},$$

and

$$\begin{aligned} \frac{d^2 X_1}{dS^{*2}} &= \frac{d}{dS^*} \left(\frac{dX_1}{dS} \frac{dS}{dS^*} \right) = \frac{d}{dS^*} \left(\frac{1}{q^*} X'_1 \begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix}^{-\frac{1}{3}} \right) \\ &= -\frac{q^{*'}}{q^{*3}} X'_1 \begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix}^{-\frac{2}{3}} + \frac{X''_1}{q^{*2}} \begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix}^{-\frac{2}{3}} \\ &\quad - \frac{1}{3} \frac{X'_1}{q^{*2}} \begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix}^{-\frac{5}{3}} \begin{vmatrix} X'_1 & X''_1 \\ X'_2 & X''_2 \end{vmatrix}^{-\frac{1}{3}}, \end{aligned}$$

then

$$\begin{aligned} (44) \quad \left(\frac{d^2 X_1}{dS^{*2}}\right)_{s=0} &= -\frac{q^{*'}}{q^{*3}} \frac{2}{3} \left(\frac{9}{2}\right)^{\frac{2}{3}} + \frac{1}{q^{*2}} \left(-\frac{Q'_0}{18Q_0}\right) \left(\frac{9}{2}\right)^{\frac{2}{3}} \\ &\quad - \frac{1}{3q^{*2}} \left(\frac{2}{3}\right) \left(\frac{9}{2}\right)^{\frac{5}{3}} \left(-\frac{Q'_0}{9Q_0}\right) \\ &= -\frac{2}{3} \left(\frac{9}{2}\right)^{\frac{2}{3}} \frac{q^{*'}}{q^{*3}} + \frac{1}{18q^{*2}} \left(\frac{9}{2}\right)^{\frac{2}{3}} \frac{Q'_0}{Q_0} \\ &= \frac{1}{18q^{*2}} \left(\frac{9}{2}\right)^{\frac{2}{3}} \left(\frac{Q'_0}{Q_0} - 12 \frac{q^{*'}}{q^*}\right) \\ &= \frac{1}{3q^{*2}} \left(\frac{9}{2}\right)^{\frac{2}{3}} \left(\frac{q'_0}{q_0} - \frac{2q^{*'}}{q^*}\right). \end{aligned}$$

Hence if $q = cq^{*2}$ where c is a constant of integration, we have

$$\left(\frac{d^2 X_1}{dS^{*2}}\right)_{s=0} = 0.$$

Consequently in this case the r -affine principal normal of C coincides with the r -affine normal of C^* . Thus we have an extension of *R. Weitzenböck's theorem*:⁽¹⁶⁾

Theorem 6. *For the plane curve C^* determined by the plane section of the tangent surface of a space curve C with the osculating plane at a*

⁽¹⁶⁾ R. Weitzenböck, Wiener Sitzungsberichte, 127 (1918), pp. 969-997.

point on C , if we take as the unit curve the plane section of the unit surface with the plane, which is parallel to the osculating plane at ξ , passes through the corresponding point e on E , and the origin is taken as $q^* = c\sqrt{q}$, the r -affine principal normal of C coincides with the r -affine normal of C^* .

§ 6. Natural equations.

Let C and C^* be two curves defined in terms of their respective r -affine arc length S , and let point upon each with the same values of S correspond to each other. Let us assume, furthermore, that at corresponding points the r -affine curvature and the r -affine torsion have the same value. We will prove that C and C^* are transformable by affine transformation: From the assumption ξ and ξ^* must satisfy the differential equation (16).

$$\xi_3' + k_1 \xi_2 + k_2 \xi_1 = 0,$$

or

$$(45) \quad (Q\xi''')' + k_1 \xi'' + k_2 \xi' = 0.$$

At first we consider the case where both r -rectifying planes at the corresponding points ξ and ξ^* are parallel. In this case $x_i' (i = 1, 2, 3)$ must be a system of solutions of the linear differential equation of the third order.

$$(46) \quad (Qy'')' + k_1 y' + k_2 y = 0,$$

Take $y_i (i = 1, 2, 3)$ as the linearly independent solutions of (46), then we have

$$(47) \quad \begin{vmatrix} (Qy_1'')' & y_1' & y_1 \\ (Qy_2'')' & y_2' & y_2 \\ (Qy_3'')' & y_3' & y_3 \end{vmatrix} = 0,$$

therefore

$$\begin{vmatrix} y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \\ y_3'' & y_3' & y_3 \end{vmatrix} = \frac{C}{Q},$$

where c is a constant of integration.

By multiplication of a suitable constant we can make the above constant equal to one so that

$$(48) \quad \begin{vmatrix} y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \\ y_3'' & y_3' & y_3 \end{vmatrix} = \frac{1}{Q}.$$

Now let us put

$$(49) \quad x_i' = a_{i1}y_1 + a_{i2}y_2 + a_{i3}y_3, \quad (i = 1, 2, 3)$$

then we have

$$(50) \quad |\xi', \xi'', \xi'''| = a |y, y', y''| = \frac{a}{Q},$$

where $a \equiv |a_{ik}| \neq 0$.

Let us take $a = 1$ and put

$$(51) \quad x_i^* = \int_0^s y_i dS, \quad (i = 1, 2, 3)$$

thus the general solutions of (45) are given by the following equations

$$(52) \quad x_i = a_{i0} + a_{i1}x_1^* + a_{i2}x_2^* + a_{i3}x_3^*, \quad (i = 1, 2, 3)$$

where $|\xi', \xi'', \xi'''| = \frac{1}{q^6} = \frac{1}{Q}$.

In case where both r -rectifying planes at the corresponding points ξ and ξ^* are not parallel, if ξ^* be affinely equivalent to q , the surface must be a quadric or an affine transform of a periodic surface.⁽¹⁷⁾ In this case we can make the corresponding r -rectifying planes parallel by an equivalent affine transformation, consequently it is reduced to the preceding case. Hence we have the following.

Theorem 7. *If the r -affine curvature k_1 and the r -affine torsion k_2 are given as analytic functions of the r -affine arc length S , a curve which has k_1 as the r -affine curvature and k_2 as the r -affine torsion, is uniquely determined except for the translation in general. In case where the unit surface is a periodic surface such that $q^* = q$, the curve is uniquely determined except for translation and a rotation, and in case where the unit surface is a quadric, the curve is uniquely determined except for the equivalent affine transformations and translation.*

We call $k_1 = k_1(S)$ and $k_2 = k_2(S)$ the natural equations.

⁽¹⁷⁾ J. Hirakawa, E. R. G. II, loc. cit., Theorem 1 in §1, p. 86.

§ 7. Special Curves.

I. Curves of constant r -curvature.

The centre of r -curvature is defined as follows:

$$\eta = \xi + \frac{1}{k_1} \xi''.$$

Denote by r_2 the r -affine perpendicular distance from the centre of r -curvature to the r -rectifying plane at a point ξ on a space curve C , then we have

$$r_2 = \left| -\frac{1}{k_1} \xi'', \xi_1, \xi_3 \right| = \frac{1}{k_1}.$$

If the relative curvature is constant, then we have

$$\eta' = \xi' + \frac{1}{k_1} \xi'''.$$

Thus we know the tangent at η on the curve described by the centre of r -curvature is parallel to the r -rectifying plane at ξ .

II. Curves of constant torsion.

The following theorem is an extension of that given by E. Čech:

Theorem 8. *A necessary and sufficient condition that the r -affine principal normals of a space curve $C(\xi)$ be parallel to a fixed plane is that k_2 is identically zero.*

Proof. From (16) we have

$$\xi'_3 + k_1 \xi'' = 0. \quad (i)$$

Denote by α the normal unit vector of a plane, which satisfies the following relations

$$(\alpha \xi'') = (\alpha \xi''') = (\alpha \xi_3) = 0, \quad (ii)$$

then by means of (i) and (ii) we have

$$(\alpha \xi'_3) = 0, \quad (iii)$$

and consequently

$$(\alpha' \xi'') = (\alpha' \xi_3) = 0. \quad (iv)$$

From (ii) and (iv) we get

$$a' = \lambda a ,$$

by virtue of

$$\xi'' + \sigma \xi_3 \neq 0 .$$

However from $a^2 = 1$, we have $aa' = 0$, from which follows

$$\lambda = 0 ,$$

and consequently

$$a' = 0 , \quad \text{or} \quad a = a \text{ constant vector.}$$

Hence the r -affine principal normals are always parallel to a fixed plane.

Conversely, if the r -affine principal normals are always parallel to a fixed plane, it is easily seen that the r -affine torsion is constant.

III. R -affine Bertrand curves.

The curves such that the r -affine principal normals of a curve are the r -affine principal normals of another curve, are called r -affine Bertrand curves. We will seek for a necessary and sufficient condition that the two curves $C(\xi)$ and $C^*(\xi^*)$ belong to the same Bertrand curves. By assumption the following relations hold :

$$(53) \quad \xi^* = \xi + \lambda \xi'' ,$$

$$(54) \quad \xi^{*''} = a \xi'' .$$

Put

$$\frac{dS^*}{dS} = \varphi ,$$

then we have

$$(55) \quad \xi^{*'} \varphi = \xi' + \lambda' \xi'' + \lambda \xi''' ,$$

$$(56) \quad \xi^{*''} \varphi^2 + \xi^{*'} \varphi' = \xi'' (1 + \lambda'') + 2 \lambda' \xi''' + \lambda \xi^{IV} .$$

From (16) we obtain

$$(57) \quad \xi^{IV} = -\frac{1}{Q} (Q' \xi''' + k_1 \xi'' + k_2 \xi') .$$

Substituting (57) in (56) we have

$$(58) \quad \mathfrak{x}^{*''} \varphi^2 + \mathfrak{x}^{*'} \varphi' = -\frac{\lambda k_2}{Q} \mathfrak{x}' + \left(1 + \lambda'' - \frac{\lambda k_1}{Q}\right) \mathfrak{x}'' + \left(2\lambda' - \frac{\lambda Q'}{Q}\right) \mathfrak{x}''' ,$$

substituting (54) and (55)

$$(59) \quad \alpha \varphi^2 \mathfrak{x}'' + \frac{\varphi'}{\varphi} (\mathfrak{x}' + \lambda' \mathfrak{x}'' + \lambda \mathfrak{x}''') \\ = -\frac{\lambda k_2}{Q} \mathfrak{x}' + \left(1 + \lambda'' - \frac{\lambda k_1}{Q}\right) \mathfrak{x}'' + \left(2\lambda' - \frac{\lambda Q'}{Q}\right) \mathfrak{x}''' .$$

Consequently from the fact that $\mathfrak{x}'$, $\mathfrak{x}''$ and $\mathfrak{x}'''$ are linearly independent we have

$$(60) \quad \frac{\varphi'}{\varphi} + \frac{\lambda k_2}{Q} = 0 ,$$

$$(61) \quad \alpha \varphi^2 + \frac{\lambda' \varphi'}{\varphi} = 1 + \lambda'' - \frac{\lambda k_1}{Q} ,$$

$$(62) \quad \frac{\lambda \varphi'}{\varphi} = 2\lambda' - \frac{\lambda Q'}{Q} .$$

From (62) we have

$$(63) \quad \varphi = C \frac{\lambda^2}{Q} .$$

Differentiating (54) we have

$$(64) \quad \mathfrak{x}^{*'''} \varphi = a' \mathfrak{x}'' + \alpha \mathfrak{x}''' ,$$

then

$$|\varphi \mathfrak{x}^{*'}, \mathfrak{x}^{*''}, \varphi \mathfrak{x}^{*'''}| = \frac{\varphi^2}{Q^*} ,$$

therefore by means of (54), (55) and (64)

$$\frac{\varphi^2}{Q^*} = |\mathfrak{x}' + \lambda' \mathfrak{x}'' + \lambda \mathfrak{x}''' , a \mathfrak{x}'' , a' \mathfrak{x}'' + \alpha \mathfrak{x}'''| = \frac{a^2}{Q} ,$$

then

$$(65) \quad \varphi = \pm \sqrt{\frac{Q^*}{Q}} a .$$

Eliminating φ'/φ from (60) and (62)

$$(66) \quad k_2 = \frac{Q'}{\lambda} - \frac{2Q\lambda'}{\lambda^2} = \lambda \left(\frac{Q}{\lambda^2} \right)' = \lambda \left(\frac{C}{\varphi} \right)' = -\frac{C\lambda\varphi'}{\varphi^2}.$$

Now we can put

$$\xi = \xi^* + \lambda^* \xi^{*''},$$

then by means of (53) and (54) we have

$$(67) \quad \lambda^* = -\frac{\lambda}{\alpha}.$$

The following relation can be seen in the same way as (66) was obtained:

$$(68) \quad \begin{aligned} k_2^* &= \lambda^* \left(\frac{Q^*}{\lambda^{*2}} \right)' = C^* \lambda^* \frac{\varphi'}{\varphi} = -\frac{C^* \lambda}{\alpha \varphi} \varphi' \\ &= -C^* \left(\pm \frac{1}{\varphi^2} \sqrt{\frac{Q^*}{Q}} \lambda \varphi' \right) = \pm \frac{C^*}{C} \sqrt{\frac{Q^*}{Q}} k_2. \end{aligned}$$

However we have

$$\lambda^* = -\frac{\lambda}{\alpha} = -\lambda \left(\pm \frac{1}{\varphi} \sqrt{\frac{Q^*}{Q}} \right) = -\lambda \left(\pm \frac{Q}{C\lambda^2} \sqrt{\frac{Q^*}{Q}} \right) = \mp \frac{\sqrt{QQ^*}}{C\lambda},$$

then

$$(69) \quad \lambda\lambda^* = \mp \frac{\sqrt{QQ^*}}{C},$$

therefore

$$|C| = |C^*|.$$

Thus we have the conditions

$$(70) \quad \frac{k_2^*}{\sqrt{Q^*}} = \pm \frac{k_2}{\sqrt{Q}},$$

that is

$$(71) \quad \frac{k_2^*}{q^{*3}} = \pm \frac{k_2}{q^3}.$$

Hence we have following

Theorem 9. *A necessary and sufficient condition that two curves belong to the same Bertrand curves is that the relation*

$$\frac{k_2^*}{q^{*3}} = \pm \frac{k_2}{q^3}$$

holds.

Remarks. When $q = \text{constant}$, this is reduced to the theorem given by Mr. S. Ogiwara.⁽¹⁸⁾

Chapter III. Theory of surfaces.

§ 1. Relations between two surfaces with their parallel tangent planes.

Let us make correspondence between a point $P(x)$ on a surface $F(x)$ and a point $Q(c)$ on a surface $E(c)$ such that the tangent planes at P and Q are parallel to each other. Then we can put

$$(1) \quad c_i = -a_i^\sigma x_\sigma,$$

where $i, \sigma = 1, 2$ and in euclidean space⁽¹⁹⁾ we have

$$(2) \quad a_i^k = -D_{x_i} g^{ak},$$

$$(3) \quad a \equiv a_1^1 a_2^2 - a_1^2 a_2^1 = \frac{D}{g} = K^* = \frac{\bar{K}(x)}{\bar{K}(c)}.$$

Now we will seek for another form of a_i^k in affine space. Differentiate (1) with respect to u^k , then we have

$$(4) \quad l_{ik} = \left| c_1, c_2, \frac{\partial^2 c}{\partial u^i \partial u^k} \right| = - \left| a_1^\sigma x_\sigma, a_2^\lambda x_\lambda, a_i^\mu \frac{\partial^2 x}{\partial u^\mu \partial u^k} \right| = -a a_i^\mu L_{\mu k},$$

$$l_{ik} L^{kj} = -a a_i^\mu \delta_\mu^j = -a a_i^j,$$

then

$$(5) \quad a_i^j = -\frac{1}{a} l_{i\alpha} L^{\alpha j},$$

⁽¹⁸⁾ Ueber affine Bertrandsche Kurven, Jap. Journ. of Math. IV (1927), pp. 13-99.

⁽¹⁹⁾ J. Hirakawa, E. R. G., IV, loc. cit., (61), (66) and (67) in § 8, p. 117.

therefore

$$a \equiv a_1^1 a_2^2 - a_1^2 a_2^1 = \frac{l}{a^2 L},$$

where $l = l_{11}l_{22} - l_{12}^2$, $L = L_{11}L_{22} - L_{12}^2$.

Consequently we have

$$(6) \quad \frac{l}{L} = a^3 = K^{*3}.$$

Now let us denote the affine areal element by dO , then by means of $dO(\epsilon) = |l|^{\frac{1}{2}} du dv$ and $dO(\chi) = |L|^{\frac{1}{2}} du dv$, we obtain

$$(7) \quad \frac{dO(\epsilon)}{dO(\chi)} = K^{*\frac{3}{2}},$$

and

$$(8) \quad \bar{K}^{\frac{3}{2}}(\epsilon) dO(\epsilon) = \bar{K}^{\frac{3}{2}} dO(\chi) = d\omega,$$

where $d\omega$ is the areal element on the unit sphere.

Consequently if the surface $F(\chi)$ be a convex closed surface, then we have

$$(9) \quad \oint \bar{K}^{\frac{3}{2}}(\chi) dO(\chi) = \oint d\omega = 4\pi.$$

§ 2. Fundamental quantities and fundamental equations.

We take a convex closed surface $E(\epsilon)$ as the unit surface and we will discuss the relative geometric properties of the surface $F(\chi)$ for the unit surface $E(\epsilon)$.

Let us denote the affine perpendicular distance from the origin to the tangent plane at the point $Q(\epsilon)$ on $E(\epsilon)$ by q , and consider the following covariant vector

$$(10) \quad \xi_{ik} = q \chi_{ik} - q_i \chi_{k1} - q_k \chi_{1i},$$

where χ_{ik} denotes a covariant differentiation with respect to the fundamental tensor

$$(11) \quad g_{ik} = \left(\bar{x} \frac{\partial^2 x}{\partial u^i \partial u^k} \right),$$

where $\bar{x} = \frac{x_1 \times x_2}{|L_{11}L_{22} - L_{12}^2|^{\frac{1}{2}}}$ and $L_{ik} = \left| x_1, x_2, \frac{\partial^2 x}{\partial u^i \partial u^k} \right|$.

Let us put

$$(12) \quad \mathfrak{X} = \frac{\mathfrak{r}_1 \times \mathfrak{r}_2}{q |L|^{\frac{1}{4}}},$$

then we have

$$(13) \quad g_{ik} = \mathfrak{X} \xi_{ik} = \frac{|\mathfrak{r}_1, \mathfrak{r}_2, \xi_{ik}|}{q |L|^{\frac{1}{4}}} = \left| \frac{l}{L} \right|^{\frac{1}{4}} \frac{|\mathfrak{r}_1, \mathfrak{r}_2, \xi_{ik}|}{|e_1, e_2, -e|} \\ = K^{\frac{3}{4}} \frac{|\mathfrak{r}_1, \mathfrak{r}_2, \xi_{ik}|}{|e_1, e_2, -e|} = g_{ki}.$$

The *r-affine normal vector* is defined by the vector

$$(14) \quad \mathfrak{y} = \frac{1}{2} g^{\alpha\beta} \xi_{\alpha\beta}.$$

which is covariant for the affine transformation and the parameter transformation.

Then we have

$$(15) \quad \mathfrak{y} \mathfrak{X} = 1.$$

By covariant differentiation of (14) and (15) we get

$$(16) \quad \mathfrak{y}_l = \frac{1}{2} g^{\alpha\beta} \xi_{\alpha\beta, l},$$

$$(17) \quad \mathfrak{y}_l \mathfrak{X} + \mathfrak{y} \mathfrak{X}_l = 0.$$

Let us put

$$(18) \quad A_{ik, l} = \mathfrak{X} \xi_{ik, l}, \quad (i, k, l = 1, 2)$$

then we take $A_{ik, l}$ as the quantities of the third order.

By means of (16) and (17) we obtain

$$(19) \quad \mathfrak{y}_l \mathfrak{X} = \frac{1}{2} g^{\alpha\beta} \xi_{\alpha\beta, l} \mathfrak{X} = \frac{1}{2} g^{\alpha\beta} A_{\alpha\beta, l} = -\mathfrak{y} \mathfrak{X}_l = -\frac{1}{2} V_l.$$

From (13) by means of $g_{ik, l} = 0$ we have

$$(20) \quad A_{ik, l} = \mathfrak{X} \xi_{ik, l} = -\mathfrak{X}_l \xi_{ik} = A_{ki, l}.$$

Now we prove that V_l is equal to zero: Take the asymptotic curves as parametric curves, then we have

$$\begin{aligned}\eta_l &= g^{12} \xi_{12, l} \\ &= g^{12} \left(\frac{\partial \xi_{12}}{\partial u^l} - \Gamma_{1l}^\lambda \xi_{\lambda 2} - \Gamma_{l2}^\lambda \xi_{1\lambda} \right) \\ &= g^{12} (q \xi_{12l} + q_l \xi_{12} - q_1 \xi_{2l} - q_2 \xi_{1l} - q_{1l} \xi_2 - q_{2l} \xi_1 - \Gamma_{1l}^\lambda \xi_{\lambda 2} - \Gamma_{l2}^\lambda \xi_{1\lambda}), \\ \frac{1}{2} V_l &= \mathfrak{X} \eta_l = g^{12} \left\{ \frac{|\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{12l}|}{|L_{12}|^{\frac{1}{2}}} + q_l g_{12} - q_1 g_{2l} - q_2 g_{1l} - (\Gamma_{1l}^1 + \Gamma_{l2}^2) g_{12} \right\},\end{aligned}$$

where $\Gamma_{1l}^1 = \Gamma_{1l, \mu} g^{\mu 1} = \Gamma_{1l, 2} g^{21} = \frac{1}{2} \left(\frac{\partial g_{12}}{\partial u^l} + \frac{\partial g_{l2}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2} \right) g^{12},$

$$\Gamma_{l2}^2 = \Gamma_{l2, \mu} g^{\mu 2} = \Gamma_{l2, 1} g^{12} = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^l} - \frac{\partial g_{l2}}{\partial u^1} \right) g^{12},$$

and $q_l g_{12} - q_1 g_{2l} - q_2 g_{1l} = 0.$

Consequently we get

$$\begin{aligned}\frac{1}{2} V_l &= \mathfrak{X} \eta_l = g^{12} \left(\frac{|\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{12l}|}{|L_{12}|^{\frac{1}{2}}} - \frac{\partial g_{12}}{\partial u^l} g^{12} \right) \\ &= g^{12} \left(\frac{1}{2 |L_{12}|^{\frac{1}{2}}} \frac{\partial L_{12}}{\partial u^l} - \frac{1}{2} \frac{\partial L_{12}}{\partial u^l} \frac{1}{|L_{12}|^{\frac{1}{2}}} \right),\end{aligned}$$

therefore

$$(21) \quad V_l = 0,$$

that is

$$(22) \quad g^{\alpha\beta} A_{\alpha\beta i} = 0. \quad (i = 1, 2)$$

These equations are the conditions of apolarity.

Let us put

$$\xi_{ik} = a_{ik}^\alpha \xi_\alpha + b_{ik} \eta,$$

then by multiplication of $\mathfrak{X}$ and $-\mathfrak{X}_l$ we have

$$b_{ik} = g_{ik},$$

and

$$(23) \quad a_{ik}^{\alpha} = q A_{ik}^{\alpha} \equiv A_{ik}^{*\alpha}.$$

Therefore we have

$$(24) \quad \xi_{ik} = A_{ik}^{*\alpha} \xi_{\alpha} + g_{ik} \eta = q A_{ik}^{\alpha} \xi_{\alpha} + g_{ik} \eta.$$

Let us put

$$(25) \quad \eta_i = b_i^{\alpha} \xi_{\alpha} + C_i \eta,$$

then by means of $\xi \eta_i = \xi \xi_{\alpha} = 0$ and $\xi \eta = 1$ we have $C_i = 0$.

Differentiating (25) covariantly with respect to u^k and putting

$$(26) \quad \xi \eta_{ik} = B_{ik} \equiv \frac{B_{ik}^*}{q},$$

we have

$$(27)$$

$$\eta_i = q B_i^{\alpha} \xi_{\alpha} = B_i^{*\alpha} \xi_{\alpha},$$

where $B_i^k = B_{i\sigma} g^{\sigma k}$.

From (17) and (21) we have

$$(28) \quad B_{ik} = \eta_{ik} \xi = -\eta_i \xi_k = \eta \xi_{ki} = B_{ki}.$$

It is easily seen that the vectors ξ , ξ_1 and ξ_2 are linearly independent of the following relations

$$(29) \quad |\xi, \xi_1, \xi_2| |\eta, \xi_1, \xi_2| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\frac{g_{11}}{q} & -\frac{g_{12}}{q} \\ 0 & -\frac{g_{12}}{q} & -\frac{g_{22}}{q} \end{vmatrix} = \frac{g}{q^2},$$

then

$$(30) \quad |\xi, \xi_1, \xi_2| = \frac{g}{q^3 |g|^{\frac{1}{2}}}.$$

Consequently we can put

$$(31) \quad \xi_{ik} = a_{ik}^{\alpha} \xi_{\alpha} + b_{ik} \xi.$$

Let us form the scalar product of (31) by η we have

$$b_{ik} = B_{ik}.$$

By the definition of $\mathfrak{X}$ we get

$$(32) \quad \mathfrak{X} \mathfrak{X}_l = 0,$$

and differentiating covariantly

$$(33) \quad \mathfrak{X}_i \mathfrak{X}_l = -\mathfrak{X} \mathfrak{X}_{li} = -\frac{g_{li}}{q},$$

$$(34) \quad \mathfrak{X}_{ik} \mathfrak{X}_l = \frac{q_k}{q^2} g_{li} - \mathfrak{X}_i \mathfrak{X}_{lk}.$$

Hence from (20) we obtain

$$(35) \quad \begin{aligned} -\mathfrak{X}_i \mathfrak{X}_{lk} &= -\frac{\mathfrak{X}_i}{q} (\xi_{kl} + q_l \mathfrak{X}_k + q_k \mathfrak{X}_l) \\ &= \frac{1}{q} \left(A_{kl, i} + \frac{q_l}{q} g_{ik} + \frac{q_k}{q} g_{il} \right), \end{aligned}$$

and substituting in (34)

$$(36) \quad \mathfrak{X}_{ik} \mathfrak{X}_l = \frac{1}{q} \left(A_{kli} + \frac{q_l}{q} g_{ik} + \frac{2q_k}{q} g_{il} \right).$$

Consequently forming the scalar product of (31) by $\mathfrak{X}_l$, we have

$$(37) \quad a_{ik}^\alpha g_{\alpha l} = - \left(A_{kli} + \frac{q_l}{q} g_{ik} + \frac{2q_k}{q} g_{il} \right) \equiv -E_{ilk}.$$

Thus the equation (31) becomes

$$(38) \quad \boxed{\mathfrak{X}_{ik} = -E_{ik}^\alpha \mathfrak{X}_\alpha + B_{ik} \mathfrak{X} .}$$

The equations (24), (27) and (38) are the *fundamental equations*.

§ 3. R-affine lines of curvature and r-affine principal curvatures.

The *r-affine line of curvature* is such a curve that the *r*-affine surface normal along the curve describes a developable surface.

From the definition, putting,

$$(39) \quad \mathfrak{z} = \mathfrak{x} + R\mathfrak{y},$$

we have

$$d\mathfrak{z} = d\mathfrak{x} + Rd\mathfrak{y} + dR\mathfrak{y} \equiv \sigma\mathfrak{y},$$

Substituting (27)

$$(40) \quad \begin{aligned} \mathfrak{x}_i du^i + RB_i^{*\alpha} \mathfrak{x}_\alpha du^i + (dR - \sigma)\mathfrak{y} &= 0, \\ (\mathfrak{x}_i + RB_i^{*\alpha} \mathfrak{x}_\alpha) du^i + (dR - \sigma)\mathfrak{y} &= 0. \end{aligned}$$

From (40) eliminating du^1, du^2 and $dR - \sigma$ we have

$$|\mathfrak{x}_1 + R(B_1^{*1} \mathfrak{x}_1 + B_1^{*2} \mathfrak{x}_2), \mathfrak{x}_2 + R(B_2^{*1} \mathfrak{x}_1 + B_2^{*2} \mathfrak{x}_2), \mathfrak{y}| = 0,$$

then

$$\begin{aligned} \frac{1}{R^2} |\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{y}| + \frac{1}{R} \{ |\mathfrak{x}_1, B_2^{*1} \mathfrak{x}_1 + B_2^{*2} \mathfrak{x}_2, \mathfrak{y}| + |B_1^{*1} \mathfrak{x}_1 + B_1^{*2} \mathfrak{x}_2, \mathfrak{x}_2, \mathfrak{y}| \} \\ + |B_1^{*1} \mathfrak{x}_1 + B_1^{*2} \mathfrak{x}_2, B_2^{*1} \mathfrak{x}_1 + B_2^{*2} \mathfrak{x}_2, \mathfrak{y}| = 0, \end{aligned}$$

consequently

$$(41) \quad \frac{1}{R^2} + \frac{1}{R} (B_1^{*1} + B_2^{*2}) + (B_1^{*1} B_2^{*2} - B_1^{*2} B_2^{*1}) = 0.$$

Denoting the roots of this equation by $\frac{1}{R_1}$ and $\frac{1}{R_2}$, we obtain

$$(42) \quad H \equiv \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = -\frac{B_1^{*1} + B_2^{*2}}{2} = -\frac{B_i^{*i}}{2},$$

and

$$(43) \quad K \equiv \frac{1}{R_1 R_2} = B_1^{*1} B_2^{*2} - B_1^{*2} B_2^{*1}.$$

$1/R$ are called the *r-affine principal curvatures* and R_i are called the *radii of the r-affine principal curvatures*.

The point

$$(44) \quad \mathfrak{z}_i = \mathfrak{x} + R_i \mathfrak{y} \quad (i = 1, 2)$$

are called the *centres of the r-affine principal curvatures*.

Eliminating R and $dR - \sigma$ from (40), then we have

$$|\xi_i du^i, B_k^\alpha \xi_\alpha du^k, \eta| = 0,$$

then

$$(45) \quad |\xi_i, B_k^\alpha \xi_\alpha, \eta| du^i du^k = 0,$$

therefore

$$(46) \quad B_1^2 du^1 du^2 + (B_2^2 - B_1^1) du^1 du^2 - B_2^1 du^2 du^1 = 0.$$

The equation (46) gives the parametric equation of the r -affine lines of curvature.

Thus we have the following.

Theorem 1. *A necessary and sufficient condition that the r -affine lines of curvature be parametric is that the conditions.*

$$(47) \quad B_1^2 = B_2^1 = 0, \quad B_2^2 \neq B_1^1$$

hold, and R_i are defined as

$$R_i = -\frac{1}{B_i^1}.$$

Denote the affine perpendicular distance from the centre $\mathfrak{z}_i$ of the r -affine normal curvature to ξ by $\mathfrak{p}_i$ and denote the r -affine perpendicular distance of ξ from $\mathfrak{z}_i$ by $\mathfrak{r}_i$, then we have

$$(48) \quad \mathfrak{r}_i = \frac{\mathfrak{p}_i}{q} = \frac{|\xi_1, \xi_2, \mathfrak{z}_i - \xi|}{q |L|^{\frac{1}{2}}} = \frac{R_i |\xi_1, \xi_2, \eta|}{q |L|^{\frac{1}{2}}} = R_i.$$

This gives a geometrical meaning of R_i

§ 4. Conditions of integrability.

From (10) and (24) we have

$$(49) \quad \begin{aligned} \xi_{ik} &= A_{ik}^\alpha \xi_\alpha + \frac{q_i}{q} \xi_k + \frac{q_k}{q} \xi_i + \frac{g_{ik}}{q} \eta \\ &= \left(A_{ik}^\alpha + \frac{q_i}{q} \delta_k^\alpha + \frac{q_k}{q} \delta_i^\alpha \right) \xi_\alpha + \frac{g_{ik}}{q} \eta, \end{aligned}$$

putting

$$(50) \quad T_{ik}^\alpha = A_{ik}^\alpha + \frac{q_i}{q} \delta_k^\alpha + \frac{q_k}{q} \delta_i^\alpha,$$

or

$$(50') \quad T_{ik,l}^{\alpha} = A_{ik,l} + \frac{q_i}{q} g_{kl} + \frac{q_k}{q} g_{il} ,$$

we obtain

$$(51) \quad \boxed{\xi_{ik} = T_{ik}^{\alpha} \xi_{\alpha} + \frac{g_{ik}}{q} \eta .}$$

By differentiating (51) covariantly with respect to u^l we have

$$\begin{aligned} \xi_{ik,l} &= T_{ik}^{\alpha} \xi_{\alpha l} + T_{ik,l}^{\alpha} \xi_{\alpha} + \frac{q_{ik}}{q} \eta_l - \frac{g_{ik} q_l}{q^2} \eta \\ &= T_{ik}^{\alpha} (T_{\alpha l}^{\beta} \xi_{\beta} + \frac{g_{\alpha l}}{q} \eta) + T_{ik,l}^{\alpha} \xi_{\alpha} + \frac{g_{ik}}{q} q B_l^{\alpha} \xi_{\alpha} - \frac{g_{ik} q_l}{q^2} \eta , \end{aligned}$$

then

$$(52) \quad \xi_{ik,l} = (T_{ik}^{\sigma} T_{\sigma l}^{\alpha} + T_{ik,l}^{\alpha} + g_{ik} B_l^{\alpha}) \xi_{\alpha} + \frac{1}{q^2} (q T_{ik}^{\alpha} g_{\alpha l} - g_{ik} q_l) \eta .$$

From the equation

$$(53) \quad \xi_{ik,l} = \xi_{il,k}$$

we have

$$(54) \quad T_{ik}^{\sigma} T_{\sigma l}^{\alpha} + T_{ik,l}^{\alpha} + g_{ik} B_l^{\alpha} = T_{il}^{\sigma} T_{\sigma k}^{\alpha} + T_{il,k}^{\alpha} + g_{il} B_k^{\alpha}$$

and

$$(55) \quad q T_{ik,l} - g_{ik} q_l = q T_{il,k} - g_{il} q_k .$$

By means of (50') the relation (55) becomes

$$(56) \quad \boxed{q A_{ik,l} - 2 g_{ik} q_l = q A_{il,k} - 2 g_{il} q_k .}$$

By multiplying (54) by g^{ik} , we have

$$(57) \quad \boxed{B_l^{\alpha} = g^{ik} (T_{il}^{\sigma} T_{\sigma k}^{\alpha} + T_{il,k}^{\alpha} - T_{ik}^{\sigma} T_{\sigma l}^{\alpha} - T_{ik,l}^{\alpha}) .}$$

As a consequence from the conditions of apolarity

$$g^{ik} A_{ik,l} = 0 ,$$

and

$$g^{ik}, m = 0 ,$$

we obtain

$$(58) \quad g^{ik} A_{ik, lm} = 0 ,$$

then

$$(59) \quad g^{ik} T_{ik, l} = \frac{2q_l}{q} ,$$

and

$$(60) \quad g^{ik} T_{ik, lm} = 2 \left(\frac{q_l}{q} \right)_m .$$

By means of (59) and (60) the relation (57) becomes

$$(61) \quad B_{lm} = g^{ik} T_{il}^{\alpha} T_{\alpha k, m} + g^{ik} T_{il, mk} - \frac{2q_m}{q} g^{n\sigma} T_{\sigma l, m} - 2 \left(\frac{q_m}{q} \right)_l .$$

By covariant differentiation of (27) with respect to u^k we have

$$(62) \quad \begin{aligned} \eta_{ikl} &= q B_i^{\alpha} \zeta_{\alpha k} + (q B_i^{\alpha})_k \zeta_{\alpha} \\ &= B_i^{\alpha} (\xi_{\alpha k} + q_{\alpha} \zeta_k + q_k \zeta_{\alpha}) + (q B_i^{\alpha})_k \zeta_{\alpha} \\ &= B_i^{\alpha} (q A_{\alpha k}^{\beta} \zeta_{\beta} + g_{\alpha k} \eta) + q_{\alpha} \zeta_k + q_k \zeta_{\alpha} + (q B_i^{\alpha})_k \zeta_{\alpha} \\ &= \{ q B_i^{\alpha} A_{\alpha k}^{\beta} + q_{\alpha} B_i^{\alpha} \delta_k^{\beta} + q_k B_i^{\beta} + (q B_i^{\beta})_k \} \zeta_{\beta} + B_{ik} \eta , \end{aligned}$$

Then as the condition of integrability of (27) we obtain

$$(63) \quad \begin{aligned} &q B_i^{\alpha} A_{\alpha k}^{\beta} + q_{\alpha} B_i^{\alpha} \delta_k^{\beta} + 2q_k B_i^{\beta} + q B_{i, k}^{\beta} \\ &= q B_k^{\alpha} A_{\alpha i}^{\beta} + q_{\alpha} B_k^{\alpha} \delta_i^{\beta} + 2q_i B_k^{\beta} + q B_{k, i}^{\beta} . \end{aligned}$$

or

$$(63') \quad \begin{aligned} &q B_i^{\alpha} A_{\alpha k, l} + q_{\alpha} B_i^{\alpha} g_{kl} + 2q_k B_{il} + q B_{i, lk} \\ &= q B_k^{\alpha} A_{\alpha i, l} + q_{\alpha} B_k^{\alpha} g_{il} + 2q_i B_{kl} + q B_{k, li} . \end{aligned}$$

We can find that the equation (38) is integrable by virtue of the conditions (56), (57) (or (61)) and (63) (or (63')). Thus the conditions of in-

tegrability for the fundamental equations (24), (27) and (38) are given by the equations (56), (57) (or (61)) and (63) (or (63')). Hence as in the euclidean space⁽²⁰⁾ we have the fundamental

Theorem 2. *When g_{ik} and $A_{ik,l}$ are given as the functions of parameters of u^1 and u^2 of the unit surface, satisfying equations (22), (56), (57) (or (61)) and (63) (or (63')) and these functions are differentiable up to the second order, there exist a surface for which these functions are the coefficients of the fundamental forms $g_{ik} du^i du^k$ and $A_{ik,l} du^i du^k du^l$, except for the translation with respect to the unit surface defined by a given function $E(u, v)$, unless the unit surface is a periodic surface or a surface of revolution or a quadric, and if the unit surface is a periodic surface or a surface of revolution or a quadric, there exists one and only one surface, except for the equivalent affine transformation.*

§ 5. R-Affine sphere.

When the r -affine normals of a surface F pass through a fixed point $\mathfrak{z}$, the surface F is called the r -affine sphere. If the surface is a r -affine sphere, its r -affine normals envelope a fixed point $\mathfrak{z}$. Then from (44) we have

$$(64) \quad \mathfrak{z} = \mathfrak{x} + R_1 \mathfrak{y},$$

where $R_1 = R_2 = \text{constant} = R (\neq \pm \infty)$. Consequently we obtain

$$(65) \quad H = \frac{1}{R} = \text{constant}, \quad K = \frac{1}{R^2} = \text{constant}.$$

Take the point $\mathfrak{z}$ as the origin O , then from (6) and (48) we have

$$(66) \quad r = \frac{p}{q} = \frac{|\mathfrak{x}_1, \mathfrak{x}_2, -\mathfrak{x}|}{|e_1, e_2, -e|} |K^*|^{\frac{3}{4}} = R = \text{constant},$$

Substituting (1)

$$(67) \quad r = \frac{p}{q} = \frac{|\mathfrak{x}_1, \mathfrak{x}_2, -\mathfrak{x}|}{|a_1^T \mathfrak{x}_0, a_2^T \mathfrak{x}_2, -e|} |K^*|^{\frac{3}{4}} = \frac{|\mathfrak{x}_1, \mathfrak{x}_2, -\mathfrak{x}|}{a |\mathfrak{x}_1, \mathfrak{x}_2, -e|} |K^*|^{\frac{3}{4}} \\ = \frac{p}{q} \frac{|K^*|^{\frac{3}{4}}}{K^*} = R,$$

(20) Loc. cit.

where p and q denote respectively the perpendicular distance from O to the corresponding tangent planes.

In case where $K^* > 0$, from (3) we have

$$(68) \quad r = \frac{p}{q} = RK^{*\frac{1}{3}} = R\left(\frac{D}{g}\right)^{\frac{1}{4}},$$

and by a symmetric transformation

$$(69) \quad \bar{x} = R^2 \bar{x}$$

we can make so that

$$(70) \quad p = q \quad \text{or} \quad r = \frac{p}{q} = K^{*\frac{1}{4}} = \left(\frac{\bar{K}(\bar{x})}{\bar{K}(e)}\right)^{\frac{1}{4}} = \frac{P}{Q},$$

where P and Q denote respectively the perpendicular distances from the origin to the tangent planes at the corresponding points on the indicatrices of their affine curvatures.⁽²¹⁾

Let us put

$$(71) \quad \frac{p}{P} = \frac{q}{Q} = \lambda,$$

then we have

$$(-\bar{x}\xi) = \lambda(\bar{y}\xi),$$

where ξ denotes the surface normal vector and $\bar{y}$ denotes the affine normal vector at $\bar{x}$.

Consequently we can put

$$(72) \quad \bar{x} + \lambda \bar{y} = b^\sigma \bar{x}_\sigma,$$

then by covariant differentiation with respect to u^i and by means of the elementary fundamental equations we have

$$(73) \quad \bar{x}_i + \lambda \bar{B}_i^\sigma \bar{x}_\sigma + \lambda_i \bar{y} = b^\sigma (\bar{A}_{\sigma i}^\alpha \bar{x}_\alpha + g_{\sigma i} \bar{y}) + b_i^\sigma \bar{x}_\sigma,$$

then

$$(74) \quad b^\sigma g_{\sigma i} = \lambda_i,$$

⁽²¹⁾ The relation $P = |K(\bar{x})|^{\frac{1}{4}}$ is easily seen from the equation (186) in §64, W. Blaschke, *Diff. Geom.* II, p. 166.

and

$$\begin{aligned}
 (75) \quad & 1 + \lambda \bar{B}_1^1 = b^\sigma \bar{A}_{\sigma 1}^1 + b_{,1}^1, \\
 & \lambda \bar{B}_1^2 = b^\sigma \bar{A}_{\sigma 1}^2 + b_{,1}^2, \\
 & 1 + \lambda \bar{B}_2^2 = b^\sigma \bar{A}_{\sigma 2}^2 + b_{,2}^2, \\
 & \lambda \bar{B}_2^1 = b^\sigma \bar{A}_{\sigma 2}^1 + b_{,2}^1.
 \end{aligned}$$

From (74) we have

$$(76) \quad b^i = \lambda_\alpha g^{\alpha i}.$$

By means of (76) and $\bar{\eta} = \frac{1}{2} g^{\alpha\beta} \chi_{\alpha\beta}$ the equation (73) becomes

$$(77) \quad \chi + \lambda \eta - \lambda_\alpha \chi_\beta g^{\alpha\beta} = 0.$$

From (1) we have

$$(1') \quad \chi_\alpha = -\bar{a}_\alpha^i e_i,$$

where $\bar{a}_i^j = (\text{cofactor of } a_i^j) / a = -\frac{1}{\bar{a}} L_{i\alpha} l^{\alpha j}$.

By differentiating covariantly (1') with respect to u^β we obtain

$$(78) \quad \chi_{\alpha\beta} = -\bar{a}_{\alpha,\beta}^i e_i - \bar{a}_{\alpha,\beta}^i e_i,$$

and substituting (1') and (78) in $\bar{\eta}$ and $\bar{\eta}_\tau$

$$(79) \quad \bar{\eta} = -\frac{1}{2} g^{\alpha\beta} (\bar{a}_{\alpha,\beta}^i e_i + \bar{a}_{\alpha,\beta}^i e_i),$$

$$(80) \quad \bar{\eta}_\tau = \bar{B}_\tau^j \chi_j = -\bar{B}_\tau^j \bar{a}_j^i e_i.$$

Differentiate covariantly (77) with respect to u^τ , and substitute (1'), (79) and (80), then we have

$$\begin{aligned}
 (81) \quad & \left(\bar{a}_\tau^i + \lambda \bar{B}_\tau^j \bar{a}_j^i + \frac{1}{2} \lambda_\tau g^{\alpha\beta} \bar{a}_{\alpha,\beta}^i - \lambda_{\alpha\tau} g^{\alpha\beta} \bar{a}_\beta^i - \lambda_\alpha g^{\alpha\beta} \bar{a}_{\beta,\tau}^i \right) e_i \\
 & + \frac{1}{2} \lambda_\tau g^{\tau\mu} \bar{a}_{\tau,\mu}^i e_i - \lambda_\alpha g^{\alpha\beta} \bar{a}_\beta^i e_{i,\tau} = 0.
 \end{aligned}$$

From (71) in the same manner as we have obtained (77) we have the equation

$$(82) \quad e + \lambda \bar{\eta} - \lambda_\alpha \bar{g}^{\alpha\beta} e_\beta = 0.$$

By differentiating covariantly (82) with respect to u^i we have

$$(83) \quad \frac{1}{2} \lambda_i \bar{g}^{\tau\mu} e_{\tau\mu} - \lambda_\alpha \bar{g}^{\alpha\beta} e_{\beta i} = -e_i - \lambda b_i^\sigma e_\sigma + \lambda_{\alpha i} \bar{g}^{\alpha\sigma} e_\sigma,$$

where b_i^k denote the corresponding quantities on $E(e)$ to $\bar{B}_i^k$ on $F(\xi)$. The relation between $e_{i,k}$ and e_{ik} is given by the equation

$$(84) \quad e_{i,k} = e_{ik} + (\bar{\Gamma}_{ik}^\sigma - \Gamma_{ik}^\sigma) e_\sigma,$$

where $\bar{\Gamma}_{ik}^\sigma$ and Γ_{ik}^σ denote respectively Christoffel's symbol concerning respectively $\bar{g}_{ik}$ and g_{ik} .

Substituting (83) and (84) in (81) we have

$$(85) \quad g_{kl} - \bar{g}_{kl} + \lambda(\bar{B}_{kl} - b_{kl}) + \frac{1}{2} \lambda_k \left(\frac{|K^*|_{,l}}{|K^*|} + \Gamma_{\mu l}^\mu - \bar{\Gamma}_{\mu l}^\mu \right) \\ + \frac{1}{4} \lambda_l \frac{|K^*|_{,k}}{|K^*|} + 2\lambda_\alpha (\Gamma_{kl}^\alpha - \bar{\Gamma}_{kl}^\alpha) = 0.$$

Hence we have the following

Theorem 3. *If the fundamental tensor g_{ij} satisfy the equation (85) and the provided fundamental equations, and the radius vector ξ of the point on a curve is defined as a solution of the differential equation (77), the curve is necessarily an unit r -affine sphere, and its converse is also true. The r -affine sphere is a symmetric transformation of an unit r -affine sphere.*

§ 6. Indicatrix of r -principal curvatures.

The *indicatrix of r -principal curvatures* is defined as the surface described the end points of the r -affine normal vector $-\eta$ from the origin.

From (27) we have the following

Theorem 4. *The tangent planes at ξ and η are parallel to each other.*

Denote the surface normal by ξ , then we have

$$(86) \quad (\xi\eta) = q | \bar{K} |^{\frac{1}{4}}.$$

According to the correspondence of the parallel tangents by means of (1) and (3) we have

$$(87) \quad K = B_1^1 B_2^2 - B_1^2 B_2^1 = \frac{\bar{K}(\mathfrak{x})}{K(\mathfrak{y})}.$$

Hence a geometrical meaning of the r -affine total curvature is obtained as follows:

Theorem 5. *The r -affine total curvature means the ratio of the elementary total curvatures at a point $\mathfrak{x}$, on a surface $F(\mathfrak{x})$, and a point $\mathfrak{y}$, on its indicatrix of r -affine curvatures.*

Differentiating (27) covariantly with respect to u^k we have

$$(88) \quad \mathfrak{y}_{ik} = B_i^{*\sigma} \mathfrak{x}_{\sigma k} + B_{i.k}^{*\sigma} \mathfrak{x}_\sigma,$$

then putting

$$(89) \quad \begin{aligned} L_{0,ik} &= |-\mathfrak{y}_{ik}, -\mathfrak{y}_1, -\mathfrak{y}_2| \\ &= -|B_i^{*\sigma} \mathfrak{x}_{\sigma k}, B_1^{*\tau} \mathfrak{x}_\tau, B_2^{*\mu} \mathfrak{x}_\mu| \\ &= -KB_i^{*\sigma} L_{\sigma k}, \end{aligned}$$

consequently

$$(90) \quad L_0 = L_{0,11} L_{0,22} - L_{0,12}^2 = K^3 L.$$

Let r_0 denote the r -perpendicular distance from the origin to $\mathfrak{y}$, then we have

$$(91) \quad r_0 = \frac{|\mathfrak{y}, \mathfrak{y}_u, \mathfrak{y}_v|}{q |L_0|^{\frac{1}{4}}} = \frac{|\mathfrak{y}, B_1^{*\lambda} \mathfrak{x}_\lambda, B_1^{*\mu} \mathfrak{x}_\mu|}{q |K|^{\frac{3}{4}} |L|^{\frac{1}{4}}} = \frac{K}{|K|^{\frac{3}{4}}} = \pm |K|^{\frac{1}{4}}.$$

Thus we get the following

Theorem 6. *A necessary and sufficient condition that the r -affine total curvature be constant is that the indicatrix of r -affine curvatures is a r -affine sphere.*

Let $dO(\mathfrak{x})$ and $dO(\mathfrak{y})$ denote respectively the affine areal elements of the surface $F(\mathfrak{x})$ and its indicatrix of r -affine curvatures, then from (90) we have

$$(92) \quad \frac{dO(\mathfrak{y})}{dO(\mathfrak{x})} = |K|^{\frac{3}{4}},$$

and

$$(93) \quad O(\mathfrak{y}) = \int |K|^{\frac{3}{4}} dO(\mathfrak{x}).$$

Assume that the indicatrix of r -affine curvatures is a convex closed surface and denote its volume by V_0 , then we have

$$(94) \quad V_0 = \frac{1}{3} \iint |-\eta, -\eta_u, -\eta_v| du dv = \frac{1}{3} \iint (-q) K |L|^{\frac{1}{4}} du dv.$$

Let Ω denote the relative surface area (=Relativobegfläche)⁽²²⁾ between the unit surface and the indicatrix of r -affine curvature, then we have

$$d\Omega = \frac{1}{3} |e, \eta_u, \eta_v| du dv = \frac{1}{3} |e, \xi_u, \xi_v| K du dv,$$

substituting (1)

$$\begin{aligned} (95) \quad d\Omega &= \frac{1}{3} \left| e, -\frac{1}{a}(a_2^2 e_1 - a_1^2 e_2), -\frac{1}{a}(a_1^2 e_2 - a_2^2 e_1) \right| K du dv \\ &= \frac{1}{3} |e, e_1, e_2| \frac{K}{a} du dv \\ &= \frac{1}{3} (-q) |l|^{\frac{1}{4}} \frac{K}{K^*} du dv. \end{aligned}$$

By means of (94) and (95) we obtain

$$(96) \quad V_0 = \int K^* \left| \frac{L}{l} \right|^{\frac{1}{4}} d\Omega = \int K^{*\frac{1}{4}} d\Omega.$$

Using *Cauchy's* inequality, we have the following

$$V_0^4 = \left(\int K^{*\frac{1}{4}} d\Omega \right)^4 \leq \left(\int K^{*\frac{1}{2}} d\Omega \right) \left(\int d\Omega \right)^2 \leq \left(\int K^* d\Omega \right) \left(\int d\Omega \right)^3$$

then

$$(97) \quad V_0^4 \leq \Omega^3 \int K^* d\Omega = \Omega^3 \int K \frac{\bar{K}(\eta)}{K(e)} d\Omega = \Omega^3 \int K dV(e).$$

§ 7. Surfaces of constant r -affine breadth.

We will make here an extension of the *affine breadth*⁽²³⁾ as in § 7, chapter I: Take a convex closed surface $F(\eta)$ and denote the points P_1 and P_2 , such that the tangent planes at P_1 and P_2 are parallel to each other by ξ_1 and ξ_2 . And denote the surface normal and the r -affine surface normal at P_i by ξ_i and η_i , then the r -affine breadth b is defined as follows

⁽²²⁾ H. Minikowski, Volumen und Oberflächen, Math. Ann., 57 (1903).

⁽²³⁾ J. Hirakawa, The affine breadth and its relation to the relative breadth, Jap. Journ. of Math., XII (1935).

$$(98) \quad b = 2 \frac{(\xi_2 - \xi_1)\xi_1}{(\eta_1\xi_1) + (\eta_2\xi_2)}$$

However by the definition of the r -affine normal we have

$$(99) \quad \begin{aligned} \eta\xi &= \frac{1}{2}g^{\alpha\beta}\xi_{\alpha\beta}\xi = \frac{1}{2}g^{\alpha\beta}(q\xi_{\alpha\beta} - q_{\alpha}\xi_{\beta} - q_{\beta}\xi_{\alpha})\xi \\ &= q(\bar{\eta}\xi) \\ &= q|\bar{K}|^{\frac{1}{2}}, \end{aligned}$$

where $\bar{\eta}$ denotes the elementary affine surface normal.

Let us denote the function of supporting lines at ξ_i by p_i , then the r -affine breadth is also given by the following expression

$$(100) \quad b = \frac{2(p_1 + p_2)}{q_1\bar{K}_1^{\frac{1}{2}} + q_2\bar{K}_2^{\frac{1}{2}}}$$

Let p_i and t_i denote the affine perpendicular distance and the relative one from ξ_i to ξ_k ($i, k = 1, 2; i \neq k$), then the following relation is obtained:

$$(101) \quad \frac{2}{b} = \frac{1}{\frac{(\xi_2 - \xi_1)\xi_1}{q_1\bar{K}_1^{\frac{1}{2}}}} + \frac{1}{\frac{(\xi_1 - \xi_2)\xi_2}{q_2\bar{K}_2^{\frac{1}{2}}}} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{t_1} + \frac{1}{t_2}.$$

Hence we have the following

Theorem 7. *The r -affine breadth b is equal to the harmonic mean of the r -affine perpendicular distance t_1 and t_2 , and then b is an affine invariant.*

Theorem 8. *A necessary and sufficient condition that the affine breadth of an ovaloid E be constant, is that the two vectors $\xi_1 - \xi_2$ and $\eta_1 - \eta_2$ are parallel.*

The proof is given by means of the fundamental equation (27).

Theorem 9. *If the affine normals at the points of contact of the parallel tangent planes on an ovaloid are a r -affine double normal, this surface has the following properties:*

- 1°. *The ovaloid is a surface of constant r -affine breadth.*
- 2°. *The indicatrix of r -affine curvature is central.*

- 3°. The r -affine perpendicular distance between two points ξ_1 and ξ_2 is constant ($= b$).
- 4°. The sum of the two radii $R_{1,i}$ and $R_{2,i}$ of r -affine principal curvatures is constant ($= b$).
- 5°. The sum of the r -affine perpendicular distances r_1 and r_2 from the origin to ξ_1 and ξ_2 is constant ($= b$).

Proof. By the assumption and Theorem 8 we have

$$b = \text{constant},$$

and

$$(102) \quad \xi_1 - \xi_2 + \frac{b}{2}(\eta_1 - \eta_2) = 0.$$

From the assumption we can put

$$\eta_1 = -\sigma \eta_2.$$

By differentiation we find that $\sigma = \text{constant} = 1$, so that we have

$$(103) \quad \eta_1 + \eta_2 = 0.$$

Therefore the indicatrix of r -principal curvatures is central. Now we have

$$(104) \quad t_1 = \frac{(\xi_2 - \xi_1)\xi_1}{q_1 \bar{K}_1^{\frac{1}{2}}} = \frac{(\xi_1 - \xi_2)\xi_2}{q_2 \bar{K}_2^{\frac{1}{2}}} = t_2 = b = \text{constant}.$$

By means of (102) and (103) we obtain

$$(105) \quad \xi_2 = \xi_1 + b\eta_1, \quad \xi_1 = \xi_2 + b\eta_2.$$

By differentiation of (105) with respect to respectively u and v , and using (27) the following equations are obtained:

$$(106) \quad \begin{cases} \xi_{2,u} = (1 + bB_{1,1}^{*1})\xi_{1,u} + bB_{1,1}^{*2}\xi_{1,v}, \\ \xi_{2,v} = bB_{1,2}^{*1}\xi_{1,u} + (1 + bB_{1,2}^{*2})\xi_{1,v}, \\ \xi_{1,u} = (1 + bB_{2,1}^{*1})\xi_{2,u} + bB_{2,1}^{*2}\xi_{2,v}, \\ \xi_{1,v} = bB_{2,2}^{*1}\xi_{2,u} + (1 + bB_{2,2}^{*2})\xi_{2,v}, \end{cases}$$

Take the r -affine lines of curvature as parametric curves, then (106) becomes

$$(107) \quad \begin{cases} \xi_{2,u} = (1 + bB_{1,1}^{*1}) \xi_{1,u} , \\ \xi_{2,v} = (1 + bB_{1,2}^{*2}) \xi_{1,v} , \\ \xi_{1,u} = (1 + bB_{2,1}^{*1}) \xi_{2,u} , \\ \xi_{1,v} = (1 + bB_{2,2}^{*2}) \xi_{2,v} , \end{cases}$$

Hence by *Theorem 1* we have

$$(108) \quad \begin{aligned} b &= -\left(\frac{1}{B_{1,1}^{*1}} + \frac{1}{B_{2,1}^{*1}}\right) = R_{1,1} + R_{2,1} , \\ b &= -\left(\frac{1}{B_{1,2}^{*2}} + \frac{1}{B_{2,2}^{*2}}\right) = R_{1,2} + R_{2,2} . \end{aligned}$$

Let r_i denote the r -affine perpendicular distance from the origin to ξ_i , then we have

$$r_1 = \frac{(-\xi_1, \xi_1)}{(\eta_1 \xi_1)} = \frac{-(-\xi_2 - b\eta_2)\xi_2}{(\eta_2 \xi_2)} = -\frac{(-\xi_2)\xi_2}{(\eta_2 \xi_2)} + b\frac{(\eta_2 \xi_2)}{(\eta_2 \xi_2)} = -r_2 + b ,$$

therefore

$$(109) \quad r_1 + r_2 = b = \text{constant}.$$

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