6. On the singularities of analytic functions.

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Introduction.

1. In the theory of functions, it is of great importance to study the value distribution for analytic functions. Everyone knows that such a study of meromorphic functions in the domain $D: |z| < R \leq +\infty$ has had a wonderful development in the last thirty years. It would be natural to extend the known results on the distribution of meromorphic functions to the case of general analytic functions. In the present paper something in this direction will be contributed.

2. In Chapter 1, we shall investigate the behaviour of analytic functions uniform or multiform in the neighbourhood of its transcendental singularities and enunciate some results known or unknown. In § 1, we shall start with the definitions of Riemann surfaces and covering surfaces given by H. Weyl, T. Radó, S. Stoilow and L. Ahlfors⁽¹⁾ and in §§ 2-3, the definitions of analytic functions and its inverse analytic functions will be recalled. It is well-known that F. Iversen⁽²⁾ first studied systematically the inverse functions of meromorphic functions. After giving definitions of transcendental singularities of analytic functions and preparing some theorems in § 4, it is proposed to generalize a beautiful theorem of Iversen to the case of general analytic functions in § 5 (Cf. theorem 4). In § 6, another interpretation of the generalized Iversen's theorem obtained in § 5, will be given by using analytic continuations in H. Weyl's sense⁽³⁾. In § 7, two theorems concerning analytic functions having no transcendental singularity are to be discussed. In § 8, Iversen's property of multiform analytic functions will be studied and two important problems proposed which have not yet been solved completely. In the last

(1) H. Weyl: Die Idee der Riemannschen Fläche., 2. Aufl., Leipzig, 1923.

T. Radó: Über den Begriff der Riemannschen Fläche, Acta Szeged, II. 1925.

S. Stoilow: Leçons sur les principes topologiques de la théorie des fonctions analytiques, Paris, 1938.

L. Ahlfors: Geometrie der Riemannschen Flächen, Comptes Rendus du Congrès International des Mathématiciens, Oslo, 1936, t. 1.

(2) F. Iversen: Recherches sur les fonctions inverses des fonctions méromorphes, Thèse de Helsingfors, 1914.

(3) H. Weyl: Loc. cit.

section, § 9, of this chapter, some applications of Montel's theory of normal family to the investigations of the behaviour of an analytic function in the vicinity of its transcendental singularity will be considered.

3. Given an arbitrary domain D in the z -plane, we can find a uniform analytic function having D as its domain of existence by a classical theorem⁽⁴⁾. The investigation of the value distribution for uniform analytic functions with a general domain of existence has met some difficulties, coming from the irregularities of the natural boundary, and up to the present time no complete theory of the value distribution for such functions has yet been established. For convenience, let us call the aggregate of some results, obtained by Iversen⁽⁵⁾, Gross⁽⁶⁾ and others⁽⁷⁾ on the cluster sets of such analytic functions, the theory of the cluster sets of analytic functions. Then it seems to the present writer that the theory of the cluster sets of analytic functions plays an important part in the study of the behaviour (the growth, the value distribution) of such functions near the natural boundary.

In Chapter 2, we shall enunciate some results which contribute to the theory of the cluster sets. We will here state the main object of this chapter more precisely. Recently A. Beurling⁽⁸⁾ has obtained in his thesis a very useful theorem, which we shall state later, on the cluster sets of analytic functions as an application of his study of harmonic majoration. The present writer has already remarked that this theorem

(4) A very elementary and fine proof for this theorem has been given by J. Besse. J. Besse: Sur le domaine d'existence d'une fonction analytique, *Commentarii Mathematici, Helvetici*, vol. **10** (1937).

(5) F. Iversen: Loc. cit.

(6) W. Gross: Über die Singularitäten analytischen Funktionen, *Monatshefte für Math. u. Phys.*, Bd. **29** (1918), pp. 1-47.

(7) W. Seidel: On the cluster values of analytic functions, *Transactions of American Math. Soc.*, vol. **34** (1932), pp. 1-21.

J. L. Doob: On a theorem of Gross and Iversen, *Ann. of Math.*, vol. **33** (1932), pp. 753-757.

A. Beurling: Études sur un problème de majoration, Thèse de Upsal, 1933.

M. L. Cartwright: On the asymptotic values of functions with a non-enumerable set of singularities, *Journ. London Math. Soc.*, vol. **11** (1936), pp. 303-305.

S. Irie: Sur un théorème de M. Beurling, *Proc. Imp. Acad. Japan*, vol. **13** (1937), pp. 224-246.

U. Minami: An extension of Phragmen-Lindelöf's theorem, *Proc. Imp. Acad. Japan*, vol. **13** (1937), pp. 241-243. and Sur un théorème de Phragmén-Lindelöf, *Tohoku Math. Journ.*, vol. **43** (1937), pp. 85-93.

K. Noshiro: On the theory of the cluster sets of analytic functions, *Journ. Fac. Sc. Hokkaido Imp. University. Ser. I*, vol. **6** (1933), pp. 217-231.

K. Kunugui: Sur un théorème de MM. Seidel-Beurling, *Proc. Imp. Acad. Japan*, vol. **15** (1939), pp. 27-32.

(8) A. Beurling: Loc. cit. See pp. 100-103.

has many applications to the theory of the cluster sets⁽⁹⁾. Beurling proved it by using another of his important and profound theorems and so it has been desirable to give an elementary direct proof for it. Recently K. Kunugui⁽¹⁰⁾ has improved Beurling's theorem in a complete form by excluding an unnecessary assumption and has given an elementary proof for it, by using some results obtained by S. Irie⁽¹¹⁾ and the present writer⁽¹²⁾. The main object of this chapter is, first, to give another elementary proof for Beurling-Kunugui's theorem, by modifying Kunugui's proof in several points (this method is not essentially distinct from Kunugui's but is simpler than his) and, next, to give some applications of Beurling-Kunugui's theorem, of which the most important results are concerned with an investigation of neighbourhoods of transcendental singularities of a univalent multiform analytic function.

4. Chapter 2 will start with some definitions of the cluster sets of a uniform analytic function in § 1 and then continue to three equivalent forms of Beurling-Kunugui's theorems in § 2. Two lemmas are prepared in § 3, and then a direct proof for Beurling-Kunugui's theorems can be given in § 4 and some applications of these theorems can be stated in § 5.

Chapter 1. On the Transcendental Singularities of Uniform and Multiform Analytic Functions.

§ 1. Riemann surface and covering surface.

1. It seems not unnecessary to remember the ideas of Riemann surfaces and covering surfaces (*Überlagerungsfläche*). Suppose that R is a connected Hausdorff's space determined by a system $\{U\}$ of neighbourhoods such that any neighborhood U is topologically equivalent to the interior of the unit-circle: $|\tau| < 1$. Then R is called a two-dimensional manifold. In order that a two-dimensional manifold R should be a surface, it is required that R should be triangulable (or separable)—it is an important remark of T. Radó that triangulability is equivalent to separability. We first state the definition of *Riemann surface* given by H. Weyl and T. Radó: let R be a two-dimensional manifold determined by a system $\{U\}$ of neighbourhoods and let $\{T\}$ be a family of topological transformations. Suppose that the system $\{U\}$ and the family $\{T\}$ satisfy the following conditions:

(9) K. Noshiro: Loc. cit.

(10) K. Kunugui: Loc. cit.

(11) S. Irie: Loc. cit.

(12) K. Noshiro: Loc. cit.

(1) with every neighbourhood U , a topological transformation $\tau = T(p)$ which transforms U into a circular domain, is associated and moreover.

(2) for any domain D inside the product $U_1 \cdot U_2$, the transformation $\tau_2 = T_2 T_1^{-1}(\tau_1)$ transforms D_1 into D_2 in a one-one conformal manner, where D_1 and D_2 are the images of D transformed by $\tau_1 = T_1(p)$ and $\tau_2 = T_2(p)$ which are associated with U_1 and U_2 respectively. Then the system $\{U\}$ and the family $\{T\}$ define a Riemann surface R and $\tau = T(p)$ is called a *local parameter*. As is well-known, a function $w = \psi(p)$, defined on a Riemann surface R , is said to be regular at a point $p = p_0$ on R , if, on mapping U on a circular domain by the local parameter $\tau = T(p)$ associated with U , U being an arbitrary neighbourhood containing p_0 , the function $w = \psi[p(\tau)]$, defined in a certain neighbourhood of $\tau_0 = T(p_0)$, becomes regular at $\tau = \tau_0$ in the ordinary sense. Similarly we may define a conformal mapping between two Riemann surfaces. Suppose first that there is a topological transformation $q = \psi(p)$ between the Riemann surfaces R and W . Denote the image of every point p_0 (on R) by q_0 (on W). Let U, V be arbitrary neighbourhoods of p_0, q_0 and $\tau = T(p), t = S(q)$ be the local parameters associated with U, V respectively. Suppose secondly that for any point p_0 on R , the function $t = S\psi T^{-1}(\tau)$, defined in a certain neighbourhood of $\tau_0 = T(p_0)$, is regular and univalent there. Then the mapping by means of $q = \psi(p)$ will be said to be conformal.

2. We now pass to the definition, given by S. Stoilow⁽¹³⁾ and L. Ahlfors⁽¹⁴⁾, of covering surface: let W and W_0 be two given orientable surfaces and let $p_0 = S(p)$ be a uniform transformation, defined on W , by which every point p on W corresponds to a point P_0 on W_0 —it is often said that p lies above the point p_0 on W_0 or that p_0 is the projection (*Spurpunkt*) of the point p . If the transformation $p_0 = S(p)$ is an interior transformation in Stoilow's sense, in other words, if the transformation $p_0 = S(p)$ satisfies the following conditions:

- (1) $S(p)$ is continuous on w ,
- (2) For any open set G on W , the set G_0 of all points $p_0 = S(p)$ obtained by letting p vary throughout in G , is always open,
- (3) the set of all points p such that $S(p) = p_0$ (p_0 fixed) is isolated⁽¹⁵⁾, then W will be called a covering surface of W_0 and W_0 a basic surface of W . It is a remarkable result proved by Stoilow that the interior

⁽¹³⁾ S. Stoilow: Loc. cit.

⁽¹⁴⁾ L. Ahlfors: Loc. cit.

⁽¹⁵⁾ Stoilow replaced (3) by a mild condition that any continuum on W of more than one point cannot be transformed into a point on W_0 . See, Stoilow: loc. cit.

transformation $p_0 = S(p)$ is locally univalent on W except possible isolated branch-points on W and the behaviour of $p_0 = S(p)$ in the neighbourhood of a branch-point is entirely similar to that of the function $\omega = \zeta^n$ near $\zeta = 0$. In a particular case when W_0 is a Riemann surface, we may regard the covering surface W of W_0 as a Riemann surface, by introducing a suitable family of local parameters on W .

§ 2. Definition of analytic function.

1. As functional elements let us use regular elements, pole elements and algebraic elements. More precisely, any Puiseux's series

$$(1) \quad w = c_\mu(z-c)^{\frac{\mu}{k}} + c_{\mu+1}(z-c)^{\frac{\mu+1}{k}} + \cdots + c_{\mu+\nu}(z-c)^{\frac{\mu+\nu}{k}} + \cdots$$

$$(\mu \geq 0, \quad k \geq 1 \text{ integer}),$$

with non-vanishing radius of convergence, will be said to define a functional element with center c which we will denote, after Y. Yosida, by $P(z; c)$ ⁽¹⁶⁾. Hereafter it will, without loss of generality, be supposed that there are exactly k distinct direct analytic continuations of the element (1) such that they have a centre lying within the domain: $0 < |z-c| < \rho$ in common, where ρ denotes the radius of convergence of $P(z; c)$. If $k=1$, $P(z; c)$ is a regular or pole element; that is, it has a rational character. On the contrary, if $k > 1$, $P(z; c)$ is an algebraic (or rather ramified) element, and the number $k-1$ is called the order of ramification. It is well-known that an element $P(z; \infty)$ with centre at $z = \infty$ has such a property that by means of the transformation $\zeta = \frac{1}{z}$, it is transformed into an element (with centre at the origin) of the form (1). If given a continuous curve

$$(2) \quad C: z = z(\lambda) \quad (0 \leq \lambda \leq 1)$$

in the z -plane and a system of functional elements (rational or algebraic)

$$(3) \quad P(z; z(\lambda)) \quad (0 \leq \lambda \leq 1),$$

we may find, for any value λ_0 of λ , a positive number ε such that all the elements $P(z; z(\lambda))$ for $|\lambda - \lambda_0| \leq \varepsilon$ are direct analytic continuations ⁽¹⁷⁾

⁽¹⁶⁾ For this symbol, we are indebted to Y. Yosida. Cf. Y. Yosida: *Theory of Functions*, (1938), (in Japanese). It is worthy of notice that the theory of analytic continuation is stated very clearly and strictly there.

⁽¹⁷⁾ Consider any element $P(z; a)$ itself as its direct continuation.

of $P(z; z(\lambda_0))$, then the system (3) will be said to define an analytic continuation with algebraic characters along the curve C ; in this case, we also say that the initial element $P(z; z(0))$ can be continued (in a certain manner) along the curve C and at the end-point $z = z(1)$ the element $P(z; z(1))$ can be obtained or it may be said that the element $P(z; z(0))$ can be connected analytically with the element $P(z; z(1))$ along the curve C . It is remarkable that on keeping the initial element (supposed to be rational) and the curve fixed, analytic continuation with rational characters can be performed uniquely; whereas the same does not hold for analytic continuation with algebraic characters. Hereafter, without any statement, only analytic continuations with algebraic characters will be considered. The analytic function $w = f(z)$ determined by a given element $P(z; a)$ is defined as the set of all elements (rational or algebraic) obtained by continuing $P(z; a)$ along all paths with the end-point $z = a$, in so far as analytic continuations are possible. For any element $P(z; c)$ of $w = f(z)$, the complex number $P(c; c)$ is called a value of the analytic function at $z = c$. In a particular case where the function $w = f(z)$ is uniform, the set of all centres c of the elements $P(z, c)$ of $w = f(z)$ is a connected (schlicht) domain which is called the domain of existence for $w = f(z)$. Consider now the general case where $w = f(z)$ is uniform or multiform. For every element $p \equiv P(z; c)$ of $w = f(z)$, an analytic neighbourhood $U(p)$ of an element p may be defined as the set of the element $p \equiv P(z; c)$ and all its direct continuations with centres within $|z - c| < r$, where r denotes any number less than ρ and ρ is the radius of convergence of $P(z; c)$ ⁽¹⁸⁾.

2. Since the system $\{U(p)\}$ of neighbourhoods which has just been defined above satisfies the so-called four axioms of neighbourhoods, the analytic function $w = f(z)$ may be regarded as a connected Hausdorff's space. Any element $P(z; c)$ can be represented by the pair

$$(1) \quad z = c + t^k, \quad w = S(t)$$

or

$$(2) \quad z = t^{-k}, \quad w = S(t)$$

according as c is finite or infinite, where $S(t)$ is Laurent's series whose principal part contains only a finite number of terms and further $k-1$ denotes the order of ramification of $P(z; c)$. To fix ideas, let c be finite. If ρ and σ denote the radii of convergence of $P(z; c)$ and $S(t)$ respectively, it is evident that $\rho = \sigma^k$. Expand $z = c + t^k$ and $w = S(t)$ into Taylor

⁽¹⁸⁾ In the case where c is infinite, we may define analytic neighbourhoods of $P(z; \infty)$ in an entirely similar manner.

series: $z = H(\tau)$ and $w = K(\tau)$ ($\tau = t - t_0$) in a certain neighbourhood of $t = t_0$ ($0 < |t_0| < \sigma$), and eliminate τ between $z = H(\tau)$ and $w = K(\tau)$, then we shall obtain an element p_{t_0} identical with a direct continuation $P(z; z_0)$ with centre at $z_0 = c + t_0^k$ of $P(z; c)$. We see without difficulty that the converse of the above argument remains valid. It is clear that the transformation $t \rightarrow p_t$, which transforms every point t for $|t| < \sigma$ into an element p_t (p_t denoting $P(z; c)$ itself when $t = 0$) belonging to the analytic ρ -neighbourhood $U(p)$, defined for $|z - c| < \rho$, of $p \equiv P(z; c)$, becomes a topological transformation between the circle $|t| < \sigma$ and the analytic neighbourhood⁽¹⁹⁾. Therefore the analytic function $w = f(z)$ may be considered as not only a connected Hausdorff's space but also a two-dimensional manifold; it will be denoted by F . If the parameter t be taken as a local parameter associated with every element "point" $p \equiv p(z; c)$ on F , then it is easy to prove that F becomes a Riemann surface. From the fact that both z and w , of (1) and (2), are regular within $|t| < \sigma$ except possible poles at $t = 0$, it follows that the uniform functions $z = c(p)$ and $w = f(p)$, defined on the Riemann surface F by means of letting the centre $z = c$ and the value $w = P(c; c)$ correspond to every point $p \equiv P(z; c)$ respectively, are regular at all points on F except possible poles. Thus it has been proved that if, for every point $p \equiv P(z; c)$, we consider the point $z = c$ as its projection, then F becomes a covering surface of the whole z -plane and at the same time it is proved that the value w of the analytic function $w = f(z)$ may be considered as a uniform meromorphic function $w = f(p)$ on the Riemann covering surface F . Hereafter, given an analytic function $w = f(z)$, let us call F the Riemann (covering) surface associated with $w = f(z)$ and denote by $w = f(p)$ the function when it is considered on the surface F .

3. An inverse analytic function.

1. We shall start with the definition of the inverse element. Consider first the simplest case in which a given element $P(z; c)$ is regular, simple and of the form:

$$w = a + c_1(z - c) + c_2(z - c)^2 + \dots, \quad c_1 \neq 0.$$

Take a positive number r to be sufficiently small, so that the element $P(z; c)$ is convergent and univalent for $|z - c| < r$ and maps $|z - c| < r$ conformally upon a Jordan domain $\mathcal{A}$ in the w -plane, and expand the inverse function which maps $\mathcal{A}$ conformally on $|z - c| < r$, into Taylor

(19) An entirely similar argument holds good when c is infinite.

series near $w = \alpha$, then we shall obtain the inverse element $Q(w; \alpha)$ of $P(z; c)$ written in the form

$$z = c + \frac{1}{c_1}(w - \alpha) \cdots$$

Next consider an element $P(z; c)$ of the form

$$(1) \quad w = \alpha + c_\mu(z - c)^{\frac{\mu}{k}} + c_{\mu+1}(z - c)^{\frac{k+1}{k}} + \cdots$$

$$(\mu \geq 1, k \geq 1, \text{ integer, } c_\mu \neq 0)$$

where $k-1$ denotes the order of ramification. As already remarked, this element may be put in the form

$$(2) \quad z = c + t^k, \quad w = S(t) \equiv \alpha + c_\mu t^\mu + c_{\mu+1} t^{\mu+1} + \cdots$$

An easy calculation shows that there is a function

$$(3) \quad t = t(\tau) \equiv \gamma_1 \tau + \gamma_2 \tau^2 + \cdots \quad (\gamma_1 \neq 0),$$

such that $t(\tau)$ is convergent and satisfies the identity

$$(4) \quad S(t(\tau)) \equiv \alpha + \tau^\mu$$

for $|\tau| < r$, provided that r is taken to be sufficiently small. Substitute $t = t(\tau)$ into (2), then

$$(2)' \quad z = c + (\gamma_1 \tau + \gamma_2 \tau^2 + \cdots)^k, \quad w = \alpha + \tau^\mu,$$

and then invert the order of z and w and expand z into power series of τ , then

$$(5) \quad w = \alpha + \tau^\mu, \quad z = R(\tau) \equiv c + b_k \tau^k + b_{k+1} \tau^{k+1} + \cdots$$

where $b_k = \gamma^k \neq 0$, which gives a functional element $Q(w; \alpha)$ of the form:

$$(6) \quad z = c + b_k(w - \alpha)^{\frac{k}{\mu}} + b_{k+1}(w - \alpha)^{\frac{k+1}{\mu}} + \cdots$$

and it is called the inverse element of $P(z; c)$ ⁽²⁰⁾. When an element with a distinct form from (1) is given, it is only necessary to modify slightly the above process.

⁽²⁰⁾ To define the element $Q(w; \alpha)$, we have used a function $t(\tau)$ which satisfies the identity (4). In the case where $k > 1$, such a function $t(\tau)$ is not necessarily unique; there are μ such functions written as follows; $t(\tau)$, $\eta t(\tau)$, $\dots$, $\eta^{\mu-1} t(\tau)$ where $\eta = e^{\frac{2\pi i}{\mu}}$. However it is easily seen that either of them gives the same inverse element $Q(w; \alpha)$.

2. Consider now the analytic neighbourhoods U and V of $P(z; c)$, of the form (1), and the inverse element $Q(w; a)$ defined for $|z - c| < \rho$ and $|w - a| < \sigma$ respectively. It has already been stated that U and V can be transformed topologically into the circles $|z - c| < \rho$ and $|w - a| < \sigma$ by local parameters t and τ associated with $P(z; c)$ and $Q(w; a)$ respectively. It is obviously possible to find a positive number r less than $\sigma^{\frac{1}{\mu}}$, such that the function $t(\tau)$, in (3), is regular and maps the circular domain $\Delta_\tau: |\tau| < r$ on a Jordan domain Δ_t lying entirely within the circle $|t| < \rho^{\frac{1}{k}}$. Let us denote by D_τ and D_t the domains inside U and V corresponding to Δ_τ and Δ_t by means of local parameters τ and t . If r is taken to be sufficiently small, it is seen, considering (2) and (5), that D_t and D_τ consist of all simple regular elements except possible $P(z; c)$ and $Q(w; a)$ and moreover that any element P_{t_0} , corresponding to the point t_0 , of D_t is inverse to the element Q_{τ_0} , corresponding to the point τ_0 , of D_τ provided that $\tau_0 = t(\tau_0)$; the converse also holds good. We shall give a proof for this assertion. The functions

$$(5) \quad w = \alpha + \tau^\mu, \quad z = R(\tau) = c + b_k \tau^k + \dots \quad (b_k = \gamma_1^k \neq 0)$$

are regular and univalent in a sufficiently small neighbourhood (not containing $\tau = 0$) of $\tau = \tau_0$, say $|\tau - \tau_0| < \varepsilon$, for $0 < |\tau_0| < r$. By eliminating $\tau - \tau_0$ from the Taylor expansions of w and z near $\tau = \tau_0$, we can obtain two elements $P_{\tau_0}(w; z_0)$ and $Q_{\tau_0}(w; w_0)$, of which one is inverse to the other, where $z_0 = R(\tau_0) = c + b_k \tau_0^k$, $w_0 = \alpha + \tau_0^\mu$. The fact that $t = t(\tau)$ is univalent for $|\tau - \tau_0| < \varepsilon$ shows that $P_{\tau_0}(z; z_0)$ is no other than the element P_{t_0} , corresponding to t_0 , of D_t and on the other hand it is evident that $Q_{\tau_0}(w; a)$ is identical with Q_{τ_0} , corresponding to τ_0 , of D_τ . Now we shall enunciate an important lemma:

Lemma. Suppose that there exists an analytic continuation

$$(7) \quad P(z; z(\lambda)) \quad (0 \leq \lambda \leq 1)$$

along a given curve

$$C: z = z(\lambda) \quad (0 \leq \lambda \leq 1).$$

The image Λ of the curve C by this continuation may be written in the form $\Lambda: w = w(\lambda)$ ($0 \leq \lambda \leq 1$) where $w(\lambda) \equiv P(z(\lambda); z(\lambda))$ for every λ . Then the system

$$(8) \quad Q(w; w(\lambda)) \quad (0 \leq \lambda \leq 1),$$

where every $Q(w; w(\lambda))$ denotes the inverse element of $P(z; z(\lambda))$ necessarily defines an analytic continuation along the curve Λ .

Proof. Let λ_0 be an arbitrarily fixed value of λ . To fix ideas, let it be supposed that the element $P(z; z(\lambda_0))$ is of the form (1). Using the notations which have just been used, we may define two domains $D_t(\lambda_0)$ and $D_\tau(\lambda_0)$ contained entirely inside $U(\lambda_0)$ and $V(\lambda_0)$ such that any element of $D_t(\lambda_0)$ is inverse to an element of $D_\tau(\lambda_0)$, its converse being also true, where $U(\lambda_0)$ and $V(\lambda_0)$ denote analytic neighbourhoods of $P(z; z(\lambda_0))$ and $Q(w; w(\lambda_0))$ respectively. By hypothesis, it is possible to find a positive number ε such that any element $P(z; z(\lambda))$ for $|\lambda - \lambda_0| < \varepsilon$ is contained in $D_t(\lambda_0)$. As an immediate result, it is seen that all the elements $Q(w; w(\lambda))$ for $|\lambda - \lambda_0| < \varepsilon$ must be contained in $D_\tau(\lambda_0)$ and so in $V(\lambda_0)$; in other words, all elements $Q(w; w(\lambda))$ for $|\lambda - \lambda_0| < \varepsilon$ are direct continuations of the element $Q(w; w(\lambda_0))$. (Q. E. D)

Let $w = f(z)$ be a given analytic function, F be the Riemann covering surface associated with $w = f(z)$ and let $w = f(p)$ be the function considered on the surface F . Applying the lemma, we see without difficulty that the set of all the elements $Q(w; a)$, each of which is inverse to an element $P(z; c)$ of $w = f(z)$, defines completely an analytic function, say $z = \varphi(w)$. Let ϕ denote the Riemann covering surface associated with the analytic function $z = \varphi(w)$ and $z = \varphi(q)$ the function considered on the surface ϕ . If, for instance, an element $p = P(z; c)$ of $w = f(z)$ is represented by

$$(2) \quad z = c + t^k, \quad w = S(t) = a + c_\mu t^\mu + \dots \dots$$

$$(\mu \geq 1, k \geq 1 \text{ integer, } c_\mu \neq 0)$$

where $k-1$ is the order of ramification, then the inverse element $q = Q(w; a)$ will be of the form

$$(5) \quad w = a + \tau^\mu, \quad z = R(\tau) = c + b_k \tau^k + \dots \dots \quad (b_k \neq 0),$$

where $\mu-1$ is the order of ramification—in this case, we will say that the function $w = f(p)$, uniform and meromorphic on F , takes the value a μ times at the point $p \equiv P(z; c)$ and so the inverse function $z = \varphi(q)$ takes the value c k times at the point $q \equiv Q(w; a)$. Accordingly, it is easy to define univalent or multivalent analytic functions. Further, notice that the transformation $p \rightarrow q$ where $p \equiv Q(w; a)$ is inverse to $p \equiv P(z; c)$ gives a conformal mapping between F and ϕ , since by (3) the function $t = t(\tau)$ is regular and univalent near $\tau = 0$, where t and τ are local parameters associated with the points p and q respectively.

4. Definition of transcendental singularities and some theorems.

1. Let us denote

$$(1) \quad C: z = z(t) \quad (0 \leq t < 1), \quad \text{where} \quad \lim_{t \rightarrow 1} z(t) = \omega,$$

a given continuous curve in the z -plane. Suppose that a system of functional elements

$$(2) \quad P(z; z(t)), \quad (0 \leq t < 1)$$

satisfies the following conditions:

(i) for any positive number λ less than unity, the system

$$P(z; z(t)), \quad (0 \leq t \leq \lambda)$$

defines an analytic continuation but

(ii) for any element $P(z; z(1)) = \omega$, putting $z(1) = \omega$, the system

$$P(z; z(t)), \quad (0 \leq t \leq 1)$$

never defines an analytic continuation. Then system (2) $P(z; z(t))$, $(0 \leq t < 1)$, or rather the continuation (2) along C , will be said to define a transcendental singularity Ω lying above the point $z = \omega$, ω being called the *projection* (or *coordinate*) of Ω . Describe a circle $c: |z - \omega| = r$ which intersects with the curve C and denote by C_r the arc $\widehat{ab}$ where b is the last point of intersection of the curve C , counting from $z(0)$ to ω , with the circle c ; we call C_r the end-part of C lying inside $(c): |z - \omega| < r$. Selecting an arbitrary element $P(z; z(t_r))$ of the system (2), $z(t_r)$ lying on the arc C_r and continuing analytically this element along any path starting from $z = z(t_r)$ and lying within the circle $(c): |z - \omega| < r$, in so far as analytic continuation with algebraic characters is possible, it is possible to obtain a branch $w = f_r(z)$, that is, the set of all the elements obtained in this way; the set of values taken by $w = f_r(z)$ in (c) is clearly a connected domain (say $\mathcal{A}_r$), its closure $\overline{\mathcal{A}_r}$ being a continuum (closed connex set). We will call the set

$$(3) \quad S_\Omega = \bigcap_{r>0} \overline{\mathcal{A}_r}$$

the cluster set of the analytic function $w = f(z)$, having $w = f_r(z)$ as its branch, at Ω . It is evident that the set S_Ω consists of a single point or a continuum of more than one point. If the set S_Ω contains only one point, then the transcendental singularity Ω will be said to be "ordinary"

and otherwise, Ω will be said to be "essential". Regarding the Riemann surface F associated with the function $w = f(z)$, then the curve $L: P(z; z(t))$, $(0 \leq t < 1)$ on F will define an accessible boundary point Ω of F .

2. Let R be an abstract Riemann surface. If a sequence $p_\nu (\nu = 1, 2, \dots)$ of points on R has no point of accumulation (in this case R is clearly open), this sequence will be said to converge to the boundary of R . If a continuous curve $\Gamma: p(t) (0 \leq t < 1)$ on R satisfies that for any increasing sequence $t_\nu (\nu = 1, 2, \dots)$ tending to unity, the corresponding sequence $p_\nu \equiv p(t_\nu) (\nu = 1, 2, \dots)$ has no point of accumulation, then the curve Γ will be said to converge to the boundary of R . And, if we have $\lim_{t \rightarrow 1} \psi(p(t)) = \alpha$, where $w = \psi(p)$ is a given function uniform and meromorphic on R , then α will be called an asymptotic value (*Zielwert*) along the path Γ , Γ being called an asymptotic path (*Zielweg*) or a path of determination. On the contrary, if the limit $\lim_{t \rightarrow 1} \psi(p(t))$ does not exist, then the curve Γ will be said to be a path of indetermination (*Irrweg*). For instance, the curve $L: p(t) \equiv P(z; z(t)) (0 \leq t < 1)$, which defines an accessible boundary point Ω , converges to the boundary of the Riemann surface F associated with $w = f(z)$ and L is either an asymptotic path or a path of indetermination of $w = f(p)$, according as $\lim_{t \rightarrow 1} f(p(t))$ either exists or does not.

3. Some theorems on transcendental singularities will be enunciated. Suppose that an analytic continuation

$$(1) \quad P(z; z(t)), \quad (0 \leq t < 1)$$

along a given curve $C: z = z(t)$, $(0 \leq t < 1)$, $\lim_{t \rightarrow 1} z(t) = \omega$, defines a transcendental singularity Ω above the point ω . Next, suppose that a system of elements

$$(2) \quad Q(z; \zeta(t)), \quad (0 \leq t < 1),$$

where $\Lambda: z = \zeta(t)$, $(0 \leq t < 1)$, $\lim_{t \rightarrow 1} \zeta(t) = \omega$, is another curve, which satisfies that for any positive number λ less than unity the system $Q(z; \zeta(t)) (0 \leq t \leq \lambda)$ always defines an analytic continuation. In the same way as a branch $w = f_r(z)$ was defined in (c): $|z - \omega| < r$, another branch $w = \varphi_r(z)$ can be also defined inside (c), by using an element $Q(z; \zeta(t_r))$ with centre $\zeta(t_r)$ on the end-part Λ_r of the curve Λ . Finally, suppose that for arbitrary small numbers r , both branches $f_r(z)$ and $\varphi_r(z)$ are identical with each other. Then follows

Theorem 1. *Under these conditions, it is seen that the continuation (2) along the curve Λ also defines a transcendental singularity, which will*

be denoted by Ω' , with coordinate ω . In this case, Ω' will be the same as Ω .

Proof. If it were not true, by adding a suitable element $Q(z; z(1))$ to system (2), where $\zeta(1) = \omega$, the system $Q(z; \zeta(t))$ ($0 \leq t \leq 1$) would define an analytic continuation. Denote by ρ a positive number less than the radius of convergence of the element $Q(z; \zeta(1))$. Then the branch $\varphi_\rho(z)$ would consist of all direct continuations of $Q(z; \zeta(1))$ including $Q(z; \zeta(1))$ itself. Accordingly, by adding $P(z; z(1)) \equiv Q(z; z(1))$ to the system (1), the system $P(z; z(t))$ ($0 \leq t \leq 1$) would also define an analytic continuation. Thus we have reached a contradiction.

Next is shown the following

Theorem 2. Assume that an analytic continuation

$$(1) \quad P(z; z(t)) \quad (0 \leq t < 1)$$

along a given curve $C: z = z(t)$, ($0 \leq t < 1$), $\lim_{t \rightarrow 1} z(t) = \omega$, defines a transcendental singularity Ω with coordinate ω . Then it is possible to find another analytic continuation

$$(2) \quad Q(z; \zeta(t)) \quad (0 \leq t < 1)$$

along a suitable curve $A: z = \zeta(t)$, ($0 \leq t < 1$), $\lim_{t \rightarrow 1} \zeta(t) = \omega$, such that every $Q(z; \zeta(t))$ is regular and the system $Q(z; \zeta(t))$ ($0 \leq t < 1$) also defines the same transcendental singularity as Ω .

Proof. Let r_ν ($\nu = 1, 2, \dots$) be a decreasing sequence of positive numbers tending to zero. Consider the branch $f_\nu(z)$ obtained from system (3) inside the circle $(c_\nu): |z - \omega| < r_\nu$. Obviously the branch contains an infinity of regular elements, any one of which may be denoted by $P(z; z_\nu)$. Since $P(z; z_\nu)$ and $P(z; z_{\nu+1})$ are both contained in the branch $f_\nu(z)$, they may be connected analytically with each other inside the circle $(c_\nu): |z - \omega| < r_\nu$, by selecting a suitable path l_ν joining z_ν and $z_{\nu+1}$. The continuation along l_ν may contain possibly a certain number of distinct irregular elements. Consequently, by keeping fixed the end-points z_ν and $z_{\nu+1}$ of the curve l_ν and letting l_ν vary slightly, we can obtain a path A_ν inside $(c_\nu): |z - \omega| < r_\nu$ and also an analytic continuation along A_ν which includes only regular elements and connects $P(z; z_\nu)$ with $P(z; z_{\nu+1})$ analytically. It is easily seen that the continuation of the element $P(z; z_1)$ along the curve A , where $A = \sum_{\nu=1}^{\infty} A_\nu$, has the required properties.

The last theorem is related to the essential singularities.

Theorem 3. *When an analytic continuation*

$$(1) \quad P(z; z(t)), \quad (0 \leq t < 1)$$

along a given curve $C: z = z(t)$, $(0 \leq t < 1)$, $\lim_{t \rightarrow 1} z(t) = \omega$ defines an essential singularity Ω with coordinate ω , it is possible to find another continuation, defining the same singularity as Ω ,

$$(2) \quad P(z; \zeta(t)), \quad (0 \leq t < 1)$$

along a suitable path $\Lambda: z = \zeta(t)$ $(0 \leq t < 1)$, $\lim_{t \rightarrow 1} \zeta(t) = \omega$ such that $\lim_{t \rightarrow 1} P(\zeta(t); \zeta(t))$ does not exist; in this case, if system (2) be considered as a path Γ on the Riemann surface F associated with the analytic function $w = f(z)$ determined by system (1), Γ becomes a path of indetermination, of $w = f(p)$, arriving at the same accessible boundary point Ω of F .

Proof. Let r_ν and ε_ν ($\nu = 1, 2, \dots$) be two decreasing sequences tending to zero. Denote by Δ_ν the set of the values taken by the branch $w = f_\nu(z)$ obtained from system (1) inside the circle $(c_\nu): |z - \omega| < r_\nu$. By hypothesis, $S_\Omega = \bigcap_{\nu=1}^{\infty} \bar{\Delta}_\nu$ is a continuum of more than one point where $\bar{\Delta}_\nu$ is the closure of Δ_ν . Let $\alpha^{(1)}$ and $\alpha^{(2)}$ ($\alpha^{(1)} \neq \alpha^{(2)}$) be two arbitrary values belonging to S_Ω . Then, since $\alpha^{(1)}$ and $\alpha^{(2)}$ belong to $\bar{\Delta}_\nu$, it is possible to find two elements $P(z; z_\nu^{(1)})$ and $P(z; z_\nu^{(2)})$ of the branch $f_\nu(z)$ such that $|P(z_\nu^{(1)}; z_\nu^{(1)}) - \alpha^{(1)}| < \varepsilon_\nu$ and $|P(z_\nu^{(2)}; z_\nu^{(2)}) - \alpha^{(2)}| < \varepsilon_\nu$ where $z_\nu^{(1)}$ and $z_\nu^{(2)}$ are chosen to lie inside the circle $(c_\nu): |z - \omega| < r_\nu$. For any ν , $P(z; z_\nu^{(1)})$ and $P(z; z_\nu^{(2)})$ may be joined analytically by a suitable path Δ'_ν inside (c_ν) and on the other hand, as $f_\nu(z)$ is branch of $f_{\nu-1}(z)$, $P(z; z_\nu^{(2)})$ is joined to $P(z; z_{\nu-1}^{(1)})$ analytically by a suitable path Δ''_ν inside the circle $(c_{\nu-1}): |z - \omega| < r_{\nu-1}$. Consequently, if a curve $\Lambda = \sum_{\nu=2}^{\infty} \Delta_\nu$ be constructed where $\Delta_\nu = \Delta'_\nu + \Delta''_\nu$, the continuation of $P(z; z_1^{(1)})$ along Λ will have the required property. Strictly speaking, the continuation along Λ necessarily defines the same transcendental singularity as Ω and the image of the curve Λ by this continuation surely oscillates.

§ 5. A generalization of Iversen's theorem.

1. The following beautiful theorem is proposed by F. Iversen: any asymptotic path of a transcendental meromorphic function $z = \varphi(w)$ corresponds to a transcendental singularity (say Ω) of the inverse function $w = f(z)$, its converse being also true. Two asymptotic paths of $z = \varphi(w)$ corresponding to the same transcendental singularity of $w = f(z)$ will be said to be equivalent to each other. Under the convention that equivalent

asymptotic paths give the same asymptotic value, there exists a one-one correspondence between asymptotic values of the meromorphic function $z = \varphi(w)$ and transcendental singularities of the inverse $w = f(z)$.

2. In this section it is proposed to extend this theorem to the case of a many-valued analytic function instead of a meromorphic function. Suppose that an analytic continuation (of an analytic function $w = f(z)$)

$$(1) \quad P(z; z(t)), \quad (0 \leq t < 1)$$

along a path, lying in the z -plane,

$$(2) \quad C: z = z(t), \quad (0 \leq t < 1), \quad \lim_{t \rightarrow 1} z(t) = \omega$$

defines a transcendental singularity Ω with projection ω . Then the image will be denoted by

$$(3) \quad A: w = w(t) \quad (0 \leq t < 1)$$

where

$$(4) \quad w(t) \equiv P(z(t); z(t)), \quad (0 \leq t < 1).$$

For the behaviour of $w(t)$ as t tends to unity, two cases will be distinguished: either $\lim_{t \rightarrow 1} w(t)$ exists or does not; geometrically speaking, on the Riemann surface F associated with $w = f(z)$, the curve $L: p(t) \equiv P(z; z(t))$, $(0 \leq t < 1)$, which defines an accessible boundary point Ω , is either an asymptotic path of the function $w = f(p)$ uniform and meromorphic on F or a path of indetermination. In either case, denoting by

$$(5) \quad Q(w; w(t)), \quad (0 \leq t < 1)$$

the inverse element of $P(z; z(t))$ $(0 \leq t < 1)$, we see that

$$(6) \quad Q(w(t); w(t)) \equiv z(t), \quad (0 \leq t < 1),$$

from which it follows at once that

$$(7) \quad \lim_{t \rightarrow 1} Q(w(t); w(t)) = \lim_{t \rightarrow 1} z(t) = \omega.$$

Consider now the Riemann surface ϕ associated with the inverse function $z = \varphi(w)$. On the surface ϕ the function $z = \varphi(w)$ may be considered as a uniform meromorphic function: $z = \varphi(q)$. System (5) gives obviously a path on ϕ ; that is

$$(8) \quad \Gamma: q(t) = Q(w; w(t)), \quad (0 \leq t < 1).$$

It must be proved that the curve Γ necessarily converges to the boundary of Φ . First, to treat the case where $\lim_{t \rightarrow 1} w(t)$ exists, put $\lim_{t \rightarrow 1} w(t) = w(1)$. If, by adding a suitable element $Q(w; w(1))$, the system

$$(9) \quad Q(w; w(t)) \quad (0 \leq t \leq 1)$$

could define an analytic prolongation, then, denoting the inverse element of $Q(w; w(1))$ by $P(z; z(1))$, where $z(1) = \omega$, the system of elements

$$(10) \quad P(z; z(t)) \quad (0 \leq t \leq 1)$$

also would define an analytic continuation. By hypothesis, this is impossible. Thus, in this case, we have seen that the continuation (5) along the curve Λ necessarily defines a transcendental singularity with coordinate $w(1)$; geometrically speaking, the curve $\Gamma: q(t)$ ($0 \leq t < 1$) on the surface Φ defines an accessible boundary point $\mathcal{Q}'$ above the point $w = w(1)$.

Next consider the other case where $\lim_{t \rightarrow 1} w(t)$ does not exist. To prove that the curve Γ converges to the boundary of Φ , it is sufficient to show that for any increasing sequence t_ν ($\nu = 1, 2, \dots$) tending to unity, the corresponding sequence $q(t_\nu)$ ($\nu = 1, 2, \dots$) of points, where $q(t_\nu) \equiv Q(w; w(t_\nu))$ ($\nu = 1, 2, \dots$), has no point of accumulation on Φ . Suppose that there were a sequence $q(t_\nu)$ ($\nu = 1, 2, \dots$) having the limit point q_0 on Φ . We will assume that q_0 is identical with a functional element $Q(w; w_0)$ of $z = \varphi(w)$. Denote by ρ a sufficiently small positive number less than the radius of convergence of $Q(w; w_0)$ and let V_ρ be the ρ -neighbourhood of the point q_0 , that is, the set, consisting of the element $q_0 \equiv Q(w; w_0)$ and the direct analytic continuations with centres lying in $|w - w_0| < \rho$. The hypothesis that $\lim_{t \rightarrow 1} w(t)$ does not exist, shows that there is another sequence t'_ν ($\nu = 1, 2, \dots$) tending to unity, such that all corresponding points $q(t'_\nu)$ ($\nu = 1, 2, \dots$) lie outside the neighbourhood V_ρ . It may suppose, without loss of generality, that, between t_ν and t'_ν ,

$$(11) \quad t_1 < t'_1 < t_2 < t'_2 < \dots < t_\nu < t'_\nu < \dots$$

holds and, further, that all the points $q(t_\nu)$ ($\nu = 1, 2, \dots$) lie inside V_ρ . For any ν , let $q(\bar{t}_\nu)$ be the first point of intersection, counting from $q(t_\nu)$, of the arc $\widehat{q(t_\nu)q(t'_\nu)}$ of Γ with the boundary of V_ρ and let Γ_ν be the arc $\widehat{q(t_\nu)q(\bar{t}_\nu)}$. Using the facts that (7) gives

$$(12) \quad \lim_{t \rightarrow 1} \varphi[q(t)] = \lim_{t \rightarrow 1} Q[w(t), w(t)] = \omega$$

and that the arc $\Gamma_\nu (\nu = 1, 2, \dots)$ of Γ joins $q(t_\nu)$, $q(\bar{t}_\nu)$ tending to q_0 as $\nu \rightarrow \infty$, to the point $q(\bar{t}_\nu)$ inside V_ρ , except the end-point $q(\bar{t}_\nu)$, we may prove that the function $z = \varphi(q)$ would reduce to a constant ω . To fix ideas, let $q_0 \equiv Q(w; w_0)$ be represented by

$$(13) \quad w = w_0 + \tau^k, \quad z = S(\tau),$$

where $S(\tau)$ is a Laurent's series convergent in $0 < |\tau| < \rho^{\frac{1}{k}}$, $k-1$ being the order of ramification. As is well-known, the neighbourhood V_ρ will be mapped by the local parameter $\tau(q)$ on the circle $|\tau| < \rho^{\frac{1}{k}}$; in this case, the images of $q(t_\nu)$, $q(\bar{t}_\nu)$ and Γ_ν will be denoted by τ_ν , τ_ν^* and l_ν respectively. It is evident that the sequence $\tau_\nu (\nu = 1, 2, \dots)$ of points inside $|\tau| < \rho^{\frac{1}{k}}$ tends to $\tau = 0$ as $\nu \rightarrow \infty$ and the curve l_ν lying inside the circle $|\tau| < \rho^{\frac{1}{k}}$ except the end-point τ_ν^* , joins the point τ_ν to the point τ_ν^* . Let r be an arbitrary positive number less than $\rho^{\frac{1}{k}}$ and describe a circle $c: |\tau| = r$. Then it follows, by (12), that $S(\tau)$ should take the value ω at least once on the circumference $c: |\tau| = r$. As r may be taken arbitrarily, $S(\tau)$ would reduce to a constant ω . But evidently this is a contradiction. Thus the assertion that the curve Γ converges to the boundary of Φ has been completely proved. Again by the equality (12), we see at once that Γ is an asymptotic path of $z = \varphi(q)$, ω being the asymptotic value along the path Γ . Thus we have shown the following

Theorem 4. *Let $w = f(z)$ be an analytic function and $z = \varphi(w)$ be the inverse analytic function. Denote by Φ the Riemann covering surface associated with $z = \varphi(w)$; then the function $z = \varphi(w)$ may be considered as a function $z = \varphi(q)$ uniform and meromorphic on Φ . Let it be supposed that $w = f(z)$ has at least one transcendental singularity⁽²¹⁾. Then, any transcendental singularity Ω , with coordinate $z = \omega$, of $w = f(z)$ corresponds to an asymptotic path Γ , on Φ , of $z = \varphi(q)$, ω being the asymptotic value of $z = \varphi(q)$ along the path Γ . It is obvious that the converse is also true. Let us say two asymptotic paths of $z = \varphi(q)$ corresponding to the same transcendental singularity of $w = f(z)$ to be equivalent to each other. If we consider as equivalent paths give the same asymptotic value of $z = \varphi(q)$, then there is a one-one correspondence between transcendental singularities of $w = f(z)$ and asymptotic values of $z = \varphi(q)$.*

Remark. When $z = \varphi(w)$ is a meromorphic function for $|w| < +\infty$, with an essential singularity at $w = \infty$, the inverse function $w = f(z)$

(21) In the present case, it is clear that Riemann surfaces F and Φ associated with $w = f(z)$ and $z = \varphi(w)$ respectively are both open surfaces.

always has only ordinary (transcendental) singularities. This is a well-known result proved by F. Iversen. However, from the theorem which has just been stated it follows that the same holds good for any uniform analytic function $z = \varphi(w)$ provided that the domain of existence has a totally discontinuous boundary; that is, the boundary contains no continuum (closed connex set) of more than one point. Let it further be noticed that, in theorem 4, an asymptotic path Γ (on Φ) of $z = \varphi(w)$ does not necessarily define an accessible boundary point of Φ . In fact, an example given by G. Valiron⁽²²⁾ shows that in the case where a function $z = \varphi(w)$ is meromorphic in the domain $D: |w| < 1$, D being the domain of existence of $z = \varphi(w)$, the inverse function $w = f(z)$ may have an essential singularity; in other words, it is possible that an asymptotic path of $z = \varphi(w)$ converges to the circumference $|w| = 1$ without arriving at one point on $|w| = 1$ (cf. theorem 3 in § 4).

§ 6. Another interpretation of the generalized theorem of Iversen.

1. Let $R(t)$ and $S(t)$ be two Laurent series, whose principal parts have only a finite number of terms, convergent for $0 < |t| < r$ where r is a certain positive number. Suppose further that the equalities $R(t_1) = R(t_2)$ and $S(t_1) = S(t_2)$ never hold simultaneously, provided that $|t_1| < r$, $|t_2| < r$ and $t_1 \neq t_2$. Then, the ordered pair of $z = R(t)$ and $w = S(t)$ defines a functional element

$$(1) \quad e: z = R(t), \quad w = S(t)$$

in H. Weyl's sense (Cf. his excellent book: Die Idee der Riemannschen Flächen). Two elements $e: z = R(t)$, $w = S(t)$ and $e': z = H(\tau)$, $w = K(\tau)$ are said to be identical, if and only if there exists a one-to-one and conformal mapping

$$(2) \quad \tau = \gamma_1 t + \gamma_2 t^2 + \cdots \quad (\gamma_1 \neq 0)$$

such that $H(\tau(t)) \equiv R(t)$, $K(\tau(t)) \equiv S(t)$ in a certain neighbourhood of $t = 0$. Between the element in Weyl's sense and the element $P(z; a)$ stated before, the following relation holds: any normal expression (any one of the simplest expressions) of the element $e: z = R(t)$, $w = S(t)$ gives an element $P(z; a)$, where $a = R(0)$, $P(a; a) = S(0)$. For example, if $R(t)$ is of the form $R(t) = a + c_k t^k + c_{k+1} t^{k+1} + \cdots$ ($k \geq 1$, integer), the element e may be also represented by the normal expression

⁽²²⁾ G. Valiron: Sur les singularités des fonctions holomorphes dans un cercle. C. R. t. 198, (1934), pp. 2065-2067.

$e: z = a + \tau^k$, $w = P(\tau)$, so that e gives the element $P(z; a) = P((z-a)^{\frac{1}{k}})$. Without difficulty we see that Taylor expansions of $R(t)$ and $S(t)$ in the neighbourhood of any point $t = t_0$ inside the circle $|t| < \rho$, may define a new element $e_{t_0}: z = R_{t_0}(\tau)$, $w = S_{t_0}(\tau)$, ($\tau = t - t_0$), provided that ρ is sufficiently small ($0 < \rho < r$). We call the set of all elements e_{t_0} (including e), obtained by letting t vary throughout inside $|t| < \rho$, a t -neighbourhood of the element $e: z = R(t)$, $w = S(t)$. After H. Weyl, a system of functional elements

$$(3) \quad e_\lambda: z = R_\lambda(t), \quad w = S_\lambda(t) \quad (0 \leq \lambda \leq 1)$$

will be said to define an analytically connected sequence (analytisch zusammenhängende Reihe) provided that for each value λ_0 of λ and for any t -neighbourhood V of e_{λ_0} , it is possible to find a positive number ε such that all elements e_λ for $|\lambda - \lambda_0| \leq \varepsilon$ are included in V . Notice that an analytically connected sequence $e_\lambda: z = R_\lambda(t)$, $w = S_\lambda(t)$ ($0 \leq \lambda \leq 1$) gives a spherically continuous curve $C: z = z(\lambda)$ ($0 \leq \lambda \leq 1$), where $z(\lambda) = R_\lambda(0)$, in the z -plane and an analytic continuation $P(z; z(\lambda))$ ($0 \leq \lambda \leq 1$) along the curve C , where $P(z; z(\lambda))$ denotes the corresponding elements to e_λ ; its converse also holds good. (Cf. Weyl's book mentioned above). Now the inverse element e^{-1} of a given element

$$(4) \quad e: z = R(t), \quad w = S(t)$$

is defined by the pair obtained by inverting the order of R and S , that is

$$(5) \quad e^{-1}: w = S(t), \quad z = R(t);$$

of course the elements $P(z; a)$ and $Q(w; b)$ ⁽²³⁾ corresponding to e and e^{-1} respectively are inverse.

2. Now suppose that there is given a system of elements

$$(6) \quad e_\lambda: z = R_\lambda(t), \quad w = S_\lambda(t) \quad (0 \leq \lambda < 1)$$

with the following properties:

i) for any positive number λ_0 less than unity the system

$$e_\lambda: z = R_\lambda(t), \quad w = S_\lambda(t) \quad (0 \leq \lambda \leq \lambda_0)$$

always defines an analytically connected sequence, however

ii) for any element $e_1: z = R_1(t)$, $w = S_1(t)$ the system obtained by adding e_1 to (6)

(23) a and b denote the values $R(0)$ and $S(0)$ respectively.

$$e_\lambda: z = R_\lambda(t), \quad w = S_\lambda(t) \quad (0 \leq \lambda \leq 1)$$

never defines an analytically connected sequence. In this case, for convenience, the system (6) $e_\lambda: z = R_\lambda(t), w = S_\lambda(t)$ ($0 \leq \lambda < 1$) is said to define a *transcendental singular element*. From the definition it follows directly that if system (6) defines a transcendental singular element, the corresponding system of inverse elements

$$(7) \quad e_\lambda^{-1}: w = S_\lambda(t), \quad z = R_\lambda(t) \quad (0 \leq \lambda < 1)$$

defines also a transcendental singular element. According to systems (6) and (7), there are two paths $C: z = z(\lambda)$ ($0 \leq \lambda < 1$) and $A: w = w(\lambda)$ ($0 \leq \lambda < 1$) where $z(\lambda) \equiv S_\lambda(0)$, $w(\lambda) \equiv Q_\lambda(0)$. Concerning the behaviour of $z(\lambda)$ and $w(\lambda)$ as λ tends to unity, three cases are distinguished: (i) both limits $\lim_{\lambda \rightarrow 1} z(\lambda)$ and $\lim_{\lambda \rightarrow 1} w(\lambda)$ exist, (ii) only one of these limits exists, (iii) neither limit exists.

3. Let us denote by $w = f(z)$ and $z = \varphi(w)$ the analytic functions obtained from systems (6) and (7) respectively; it is clear that $f(z)$ and $\varphi(w)$ are inverse to each other. We denote by F and Φ the Riemann covering surfaces associated with $w = f(z)$ and $z = \varphi(w)$ and by $w = f(p)$ and $z = \varphi(q)$ the functions on F and Φ . Further let us notice that to systems (6) and (7) there correspond two analytic continuations: $P(z; z(\lambda))$ ($0 \leq \lambda < 1$) along C and $Q(w; w(\lambda))$ ($0 \leq \lambda < 1$) along A ; for brevity, let $p(\lambda)$ and $q(\lambda)$ denote the points $P(z; z(\lambda))$ on F and $Q(w; w(\lambda))$ on Φ respectively. The following relation between $w = f(z)$ and $z = \varphi(w)$ will give another meaning of Iversen's theorem stated before and include theorem 4 as a special case.

Case (i) both limits $\lim_{\lambda \rightarrow 1} z(\lambda)$ and $\lim_{\lambda \rightarrow 1} w(\lambda)$ exist.

The curve $L: P(0 \leq \lambda < 1)$ defines an accessible boundary point Ω of F above the point $z = \omega$, putting $\lim_{\lambda \rightarrow 1} z(\lambda) = \omega$ and is an asymptotic path of $w = f(p)$ on F , ω' being the asymptotic value.

The curve $\Gamma: p(\lambda) (0 \leq \lambda < 1)$ defines an accessible boundary point Ω' of Φ above the point $w = \omega'$, putting $\lim_{\lambda \rightarrow 1} w(\lambda) = \omega'$ and is an asymptotic path of $z = \varphi(q)$ on Φ , ω being the asymptotic value.

Case (ii) To fix ideas, we will suppose that $\lim_{\lambda \rightarrow 1} z(\lambda)$ exists but $\lim_{\lambda \rightarrow 1} w(\lambda)$ does not.

The curve $L: p(\lambda)$ ($0 \leq \lambda < 1$) defines an accessible boundary point Ω of F above the point $z = \omega$, putting $\lim_{\lambda \rightarrow 1} z(\lambda) = \omega$ and is a path of indetermination of $w = f(p)$ on F , hence Ω is the essential singularity of the analytic function $w = f(z)$.

The curve $\Gamma: q(\lambda)$ ($0 \leq \lambda < 1$) converges to the boundary of Φ without arriving at an accessible boundary point and is an asymptotic path of $z = \varphi(q)$ on Φ , ω being the asymptotic value along Γ .

Case (iii) Neither $\lim_{\lambda \rightarrow 1} z(\lambda)$ nor $\lim_{\lambda \rightarrow 1} w(\lambda)$ exists. This is a trivial case, since, from the hypothesis, it does not necessarily follow that the curve $L: p(t)$ ($0 \leq t < 1$) on F converges to the boundary of F . Therefore let the assumption be added that it converges to the boundary of F (without arriving at an accessible boundary point of F). Then, L is a path of indetermination and consequently the curve $\Gamma: (0 \leq t < 1)$ on Φ converges to the boundary of Φ without reaching an accessible boundary point of Φ and is a path of indetermination of $z = \varphi(q)$.

Remark. Theorem 4 relates to cases (i) and (ii) only.

§ 7. Analytic functions having no transcendental singularity.

1. In this section an application of theorem 4 will be considered. We first prove

Theorem 5. Consider an analytic function $w = f(z)$ having no transcendental singularity. If we denote by $n(a)$ the number of elements $P(z; a)$, with centre a , of $w = f(z)$, where an algebraic element is counted k times if the order of ramification is equal to $k-1$, then the enumerative function $n(z)$ is a constant throughout in the z -plane (including $z = \infty$) and moreover $n(z) \equiv +\infty$ except the case where $w = f(z)$ is an algebraic function.

Proof. First it will be proved that $n(z) \equiv +\infty$, if $n(b) = +\infty$, b being a certain point in the z -plane. Without loss of generality, we may suppose that b is at a finite distance. If $n(z)$ were not a constant $+\infty$, then a point⁽²⁴⁾ (say c) would be found such that $n(c) < +\infty$. It is evident that there are $n(c)+1$ regular elements

$$(1) \quad P_1(z; a), \quad P_2(z; a), \quad \dots, \quad P_{n(c)+1}(z; a)$$

where a denotes a suitable point near $z = b$. Remembering the fact that any analytic function contains at most an enumerable number of irregular elements, we may choose a suitable circular arc Γ joining the

(24) For simplicity, c is supposed to be finite.

points a and c such that there is no irregular element on the interior of Γ . Now let us continue analytically (with algebraic characters) each of $n(c)+1$ regular elements $P_\nu(z; a)$ ($\nu = 1, 2, \dots, n(c)+1$) along the curve Γ ; each continuation is possible, by hypothesis. Denote by μ the total number of all the elements $P_i(z; a)$ ($i = 1, 2, \dots$) obtained at $z = c$ by these continuations, where an algebraic element is counted k times if the order of ramification is equal to $k-1$. It is clear that $\mu \leq n(c)$. Let ρ be the minimum of radii of convergence of the elements $P_i(z; c)$ ($i = 1, 2, \dots, \lambda$) and consider μ direct continuations of them at a point $z = c'$ on Γ inside the circle $|z - c| < \rho$. Evidently these μ direct (analytic) continuations can be obtained by continuing $n(c)+1$ regular elements $P_\nu(z; a)$ ($\nu = 1, 2, \dots, n(c)+1$) along the curve Γ from a to c' . Hence μ must be equal to $n(c)+1$, since any analytic continuation along a path with rational characters is possible uniquely, provided that the initial element and the path are fixed. Thus we have arrived at a contradiction. From the argument used above, it is seen without difficulty that if $n(z)$ is finite at least at one point, $n(z)$ necessarily becomes a finite function and moreover must be a constant. Thus it is seen that $n(z)$ is identical with a constant (finite or infinite). Next it will be proved that, if $n(z) \equiv n < +\infty$, the function $w = f(z)$ has necessarily a finite number of irregular elements. Otherwise, as $n(z) \equiv n < +\infty$, there would exist an infinite point-set E formed by centres of irregular elements of $w = f(z)$. Denote by ω a point of accumulation of E and let ρ be the minimum of radii of convergence of all the elements $P_j(z; \omega)$ lying above $z = \omega$. If we choose a point a_1 of the set E inside the circle $|z - \omega| < \rho$, there would exist n distinct regular elements with centres $z = a_1$ (that is, n direct continuations of the elements $P_j(z; \omega)$ at $z = a_1$) and moreover at least one irregular element with centre $z = a_1$ —this contradicts the assumption $n(z) \equiv n < +\infty$. Thus we have shown that if an analytic function $w = f(z)$, having no transcendental singularity satisfies $n(z) \equiv n < +\infty$, then $w = f(z)$ reduces to an algebraic function. As an alternative, if an analytic function, different from an algebraic function, has no transcendental singularity, it follows at once that $n(z) \equiv +\infty$. (Q. E. D.)

2. Next suppose that $w = f(z)$ is an analytic function which is distinct from an algebraic function and has no transcendental singularity. Let us consider the inverse analytic function $z = \varphi(w)$. It is evident that the function $z = \varphi(w)$ is not also an algebraic function, hence the Riemann surface Φ associated with $z = \varphi(w)$ must be an open surface. Considering $z = \varphi(w)$ as a function $z = \varphi(q)$ uniform and meromorphic on Φ , we may, in virtue of theorem 4, assert that $z = \varphi(q)$ has no asymptotic

tic path on Φ . Obviously we see that the converse of the above also holds good. Thus we have

Theorem 6. *Let $z = \varphi(w)$ be an analytic function different from an algebraic function and let Φ denote the Riemann (covering) surface associated with $z = \varphi(w)$. Suppose further that the function $z = \varphi(q)$ considered on Φ has no asymptotic path. Then $z = \varphi(q)$ takes every value without any exception an infinity of times on the surface Φ .*

As an immediate consequence, the following will be obtained:

Theorem 7. *Suppose that $z = \varphi(w)$ is a function uniform, meromorphic in a (schlicht) domain D , where D does not coincide with the whole w -plane containing $w = \infty$, such that $z = \varphi(w)$ has no asymptotic path inside D . Then $z = \varphi(w)$ takes every value infinitely often.*

Proof. Denote by C the boundary of D . Then it is easily seen that C must be the natural boundary of the function $z = \varphi(w)$, for, if otherwise, there would exist at least one asymptotic path of $z = \varphi(w)$ inside D . Hence, applying theorem 6 just stated above, it is seen that the assertion is true.

Remark. When $z = \varphi(w)$ is a transcendental meromorphic function in $|w| < +\infty$, this theorem is an immediate result from a theorem of Iversen, which is stated in the following way: any exceptional value of $z = \varphi(w)$ in Picard's sense is necessarily an asymptotic value. The present writer in one⁽²⁵⁾ of his previous papers obtained this theorem under the condition that D is the unit-circle: $|w| < 1$, by using a well-known theorem of Fatou. It is known that when D is either the w -plane: $|w| < +\infty$ or the unit-circle: $|w| < 1$, every function belonging to the first category of class (A) defined by K. Yosida and the present writer⁽²⁶⁾ satisfies the condition of theorem 7 and so takes every value infinitely often.

§ 8. On Iversen's property of an analytic function.

1. F. Iversen obtained in his thesis the following important theorem: Suppose that $z = \varphi(w)$ is a transcendental meromorphic function for $|w| < \infty$ and denote by (c) an arbitrary circle $|z - \omega| < r$ in the z -plane. Then, denoting by $P(z; a)$ any element, with centre inside (c) , of the inverse function $w = f(z)$, we can find a path C , joining a and ω inside

⁽²⁵⁾ K. Noshiro: Contributions to the theory of meromorphic functions in the unit-circle, Jour. Fac. Sc., Hokkaido Imperial University, Ser. I, vol. 7 (1939), pp. 149-157. See especially theorem 5, p. 154.

⁽²⁶⁾ K. Yosida: On a class of meromorphic functions, Proc. Phys.-Math. Soc. Japan, 3. ser., vol. 16 (1934), pp. 227-235. And K. Noshiro: Contributions to the theory of meromorphic functions, loc. cit.

(c), along which the analytic continuation of $P(z; a)$ is possible except at the end-point $z = \omega$. The object of this section is to extend this theorem to the case of a general analytic function $z = \varphi(w)$. We shall start with some lemmas and some definitions. Let $w = f(z)$ and $z = \varphi(w)$ be analytic functions which are inverse to each other. Denote by F and Φ the Riemann surfaces associated with $w = f(z)$ and $z = \varphi(w)$ and denote by $w = f(p)$ and $z = \varphi(q)$ the functions considered on the Riemann surfaces F and Φ respectively. It has already been stated that the correspondence $p \rightarrow q$, where $q \equiv Q(w; a)$ is inverse to $p \equiv P(z; c)$, gives a one-one and conformal mapping between F and Φ . Let D be an arbitrary domain in the z -plane and let $P(z; a)$ be an element of $w = f(z)$ where a lies inside D . Continuing $P(z; a)$ inside D with algebraic characters, in so far as the continuation is possible, we may define a branch $w = f_D(z)$ within D . Denote by $n(c)$ the number of the elements, of $f_D(z)$, with a common centre $z = c$, each ramified element being counted in the same way as before. Then applying the argument (with a slight modification) used for the proof of theorem 5, we obtain at once

Theorem 8. *If the branch $w = f_D(z)$ has no transcendental singularity with the projection inside D , the enumerative function $n(z)$ of $w = f_D(z)$ must reduce to a finite or infinite constant throughout inside D .*

Now let us denote by F_D the domain defined on F by the branch $w = f_D(z)$. Obviously F_D is a covering surface of the basic domain D . Denote by Δ the domain which is the image on Φ of F by means of the transformation $q = f(p)$. According as Δ is either compact or non-compact, the domain will be called an island (*Insel*) or a peninsula (*Zunge*). From the above theorem there follows at once

Theorem 9. *Suppose that the domain Δ is an island on Φ . Then the branch $w = f_D(z)$ cannot have any transcendental singularity and so, by theorem 8. the enumerative function $n(z)$ must reduce to a constant (say n), in this case n being necessarily finite. In other words, inside Δ the function $z = \varphi(q)$ takes every value belonging to D exactly n times.*

Proof. In either case if branch $w = f_D(z)$ has at least one transcendental singularity or if the enumerative function $n(z)$ becomes infinite at one point at least inside D , then the domain Δ would be necessarily non-compact. This contradicts the hypothesis.

2. Here some definitions will be introduced :

(1) (W)-property. Let the Riemann surface Φ be open. If there exists a sequence $q_\nu (\nu = 1, 2, \dots)$ tending to the boundary of Φ such that $\lim_{\nu \rightarrow \infty} \varphi(q_\nu) = \beta$, then β will be called a cluster value of $w = \varphi(q)$ at the boundary of Φ . If the set of all the cluster values of $z = \varphi(q)$ at

the boundary of ϕ covers the whole z -plane, it may be said that the function $z = \varphi(w)$ has (W) -property or Weierstrass' property. It is well-known that a transcendental meromorphic function has (W) -property (Weierstrass' theorem).

(2) (L) -property. Suppose first that the set of values taken by $z = \varphi(w)$ is everywhere dense in the z -plane. Let Δ be any domain (on ϕ) which corresponds to a covering surface F_D , by means of the transformation $q = f(p)$, where D denotes an arbitrary circle $(c) : |z - \omega| < r$ ⁽²⁷⁾ in the present case. Suppose next that, whatever the domain Δ may be chosen to be, there should exist a sequence $q_\nu (\nu = 1, 2, \dots)$ of points inside Δ tending to the boundary of ϕ such that $\lim_{\nu \rightarrow \infty} \varphi(q_\nu) = \omega$, provided that $z = \varphi(q)$ does not take the value ω anywhere inside Δ . Then, it is said that $z = \varphi(w)$ has (L) -property or Lindelöf's property. It is well-known that a transcendental meromorphic function $z = \varphi(w)$ has (L) -property as a direct result of Lindelöf's principle of maximum modulus.

(3) (I) -property. Suppose that the point-set obtained by projecting the covering surface F associated with $w = f(z)$ upon the z -plane is everywhere dense in the z -plane. Further suppose that for any circle $(c) : |z - \omega| < r$ (or $|z| > r$, if $\omega = \infty$) and for any possible element $P(z; a)$ of $w = f(z)$ whose centre a lies inside (c) , we can find a suitable path joining $z = a$ and $z = \omega$ inside (c) along which the element $P(z; a)$ can be continued except possibly at $z = \omega$. Then the function $w = f(z)$ will be said to have (I) -property or Iversen's property. Iversen's theorem states that the inverse function of a transcendental meromorphic function has this property. It is easily proved that (I) -property is equivalent to such a property that given an arbitrary curve C , whose end-points are denoted by a and b , and an arbitrary strip S containing C entirely in its interior, we can find a path, connecting a and b inside S , along which the continuation of any element $P(z; a)$ (supposed to exist) is possible except perhaps at $z = b$.

3. Using these definitions, we can prove the following

Theorem 10. *Suppose that an analytic function $z = \varphi(w)$ has both (W) -property and (L) -property. Then the inverse analytic function $w = f(z)$ necessarily has (I) -property.*

Proof. Consider any circular disc $(c_0) : |z - \omega| < r$. By (W) -property, there exists at least one element of $w = f(z)$ having the centre inside (c_0) . Let $P(z; a_0)$ be such an element. Then, it is clear that this element can be continued by a suitable path across the circumference $|z - \omega| = r$. Let $f_0(z)$ be the branch determined by the element $P(z; a_0)$ inside (c_0)

⁽²⁷⁾ When $\omega = \infty$, (c) is considered to mean $|z| > r$ instead of $|z - \omega| < r$.

and let F_0 be the corresponding covering surface of (c_0) . Then two cases will be considered; either the function $z = \varphi(q)$ takes the value ω at least once or $z = \varphi(q)$ does not take ω anywhere inside Δ_0 where Δ_0 denotes the image on ϕ mapped by the transformation $q = f(p)$ of F_0 . Notice that in either case the function $z = \varphi(q)$ takes a value lying on the circumference $c_0: |z - \omega| = r$ at any boundary point of Δ_0 relative to ϕ and on the contrary $z = \varphi(q)$ takes a value lying inside (c_0) at each point inside Δ_0 . Put $p_0 \equiv p(z; a)$ and $q_0 = f(p_0)$. In the former case when there exists an ω -point (say q^*) within Δ_0 , q_0 and q_0^* may be connected by a suitable path Γ lying entirely inside Δ_0 . Then the image on F of the curve Γ by the inverse mapping $p = \varphi(q)$ becomes a path L connecting p_0 and p^* inside F_0 where $p^* = \varphi(q^*)$. It is evident that as the projection of p^* is equal to $z = \omega$, the projection of L upon the z -plane is a curve C connecting a_0 and ω inside (c_0) . From this it follows at once that the element $P(z; a_0)$ can be continued along the path C including the point $z = \omega$.

Consider next the latter case where the function $z = \varphi(q)$ does not take ω anywhere inside Δ_0 . By (L)-property, it is possible to find a sequence $q_v^{(0)}$ ($v = 1, 2, \dots$) of points inside Δ_0 tending to the boundary of ϕ such that $\lim_{v \rightarrow \infty} \varphi(q_v^{(0)}) = \omega$. Accordingly, inside Δ_0 it is possible to select a point q_1 at which the function $z = \varphi(q)$ takes a value a_1 lying inside $(c_1): |z - \omega| < \frac{r}{2}$. Denote by Γ_1 a path joining q_0 and q_1 inside Δ_0 . Then the image (on F) of the curve Γ_1 by the inverse mapping $p = \varphi(q)$ becomes L_1 connecting p_0 and p_1 inside F_0 where $p_1 = \varphi(q_1)$; it is evident that the projection of p_1 is equal to $z = a_1$. Therefore, on denoting by C_1 the projection of L_1 upon the z -plane, the element $P(z; a_0)$ can be continued along the path C_1 to obtain an element $p_1 \equiv P(z; a_1)$ at the end-point $z = a_1$. Now let $f_1(z)$ be the branch determined by the element $P(z; a_1)$ inside $(c_1): |z - \omega| < \frac{r}{2}$ and let F_1 be the corresponding covering surface of (c_1) ; obviously F_1 is included in F_0 . Consequently $z = \varphi(q)$ does not take ω anywhere inside Δ_1 , Δ_1 denoting the image on ϕ mapped by the transformation $q = f(p)$ of F_1 . Again, by (L)-property, there is a sequence $q_v^{(1)}$ ($v = 1, 2, \dots$) of points inside Δ_1 tending to the boundary of ϕ such that $\lim_{v \rightarrow \infty} \varphi(q_v^{(1)}) = \omega$. By the same reasoning as before, there exists a path C_2 , inside $(c_1): |z - \omega| < \frac{r}{2}$, which joins the point a_1 to a point a_2 lying inside $(c_2): |z - \omega| < \frac{r}{2^2}$, such that a continuation of $P(z; a_1)$ along C_2 is possible and an element $P(z; a_2)$ can be obtained. Repeating the same arguments and using the analogous notations, for

any natural number ν , we have a path C_ν with the following properties:

(i) C_ν connects two points a_ν and $a_{\nu+1}$ inside $(c_\nu): |z-\omega| < \frac{r}{2^\nu}$ where a_ν lies inside $(c_\nu): |z-\omega| < \frac{r}{2^\nu}$,

(ii) analytic continuation of $P(z; a_\nu)$ is possible along C_ν and an element $P(z; a_{\nu+1})$ is obtained.

It is evident that the curve $C = \sum_{\nu=1}^{\infty} C_\nu$ is a path, inside (c_0) , converging to $z = \omega$ such that the continuation of $P(z; a_0)$ along C is possible except at the end-point $z = \omega$ and it defines a transcendental singularity with the projection $z = \omega$. (Q. E. D.)

4. In the above theorem the (W) -property or (L) -property of an analytic function played an important part. There are two important problems to find

(A) the necessary and sufficient condition that a given analytic function should have (W) -property and

(B) the necessary and sufficient condition that a given analytic function should have (L) -property.

It seems to the present writer very difficult to give a complete answer for any of them. Here, applying Nevanlinna's theory of harmonic measure⁽²⁸⁾, it is proposed next to give a sufficient condition that a uniform analytic function $z = \varphi(w)$ should have both (W) -property and (L) -property. For this purpose, use is made of the following two lemmas:

Lemma 1. (An extension of Weierstrass' theorem) *Let $z = \varphi(w)$ be uniform and meromorphic in D except a closed set E , lying completely inside D , of transcendental singularities. Let w_0 be any point of E . Then, if E is of absolute harmonic measure zero, the set $S_{w_0}^{(D-E)}$ covers the whole z -plane, where $S_{w_0}^{(D-E)}$ denotes the set of all values such that $\lim_{\nu \rightarrow \infty} \varphi(w_\nu) = a$, $w_\nu (\nu = 1, 2, \dots)$ being a sequence of points tending to w_0 inside the domain $D-E$.*

Lemma 2. (An extension of Lindelöf's principle) *Let D be an arbitrary domain and let E be a closed set of absolute harmonic measure zero, consisting of boundary points. Suppose that $z = \varphi(w)$ is a uniform and regular function in D which satisfies $|\varphi(w)| \leq M$ inside D and $\lim_{w \rightarrow w'} |\varphi(w)| \leq m (m \leq M)$ for any boundary point w' not belonging to E . Then we have the relation $|\varphi(w)| \leq m$ throughout in D .*

(28) R. Nevanlinna: *Eindeutige analytische Funktionen*, 1936. Cf. pp. 106-153.

Both lemmas may be obtained readily from Nevanlinna's theorem⁽²⁹⁾ stated in the form of harmonic functions. For instance, lemma 1 will be considered here. Obviously it is sufficient to show that $z = \varphi(w)$ is not bounded near w_0 . We can draw a circle $K: |w - w_0| = \rho$ lying entirely inside D , for E is of linear measure zero. Consider the real part $u(w)$ of $\varphi(w)$, which is harmonic for $|w - w_0| \leq \rho$ inside D , and define, by Poisson's integral, another function $v(w)$ harmonic throughout in the circular domain $|w - w_0| < \rho$ such that $v(w) = u(w)$ on any point of K . If $\varphi(w)$ were bounded for $|w - w_0| < \rho$ inside D , then we see that using Nevanlinna's result, there would exist the relation $v(w) = u(w)$ throughout for $|w - w_0| < \rho$ inside D . Consequently, for $|w - w_0| < \rho$ inside D $\varphi(w)$ would be identical with the analytic function defined by $v(w)$ regular for any point in the circle $|w - w_0| < \rho$, while w_0 is a transcendental singularity of $\varphi(w)$.

Now we shall prove

Theorem 11. *Let $z = \varphi(w)$ be a uniform analytic function and let D denote the domain of existence. If the boundary E of D ⁽³⁰⁾ is of absolute harmonic measure zero, then $z = \varphi(w)$ has both (W)-property and (L)-property and so the inverse analytic function $w = f(z)$ has (I)-property, by theorem 10.*

Proof. By lemma 1, it is clear that $z = \varphi(w)$ has (W)-property. Hence, it is only necessary to prove that $z = \varphi(w)$ has (L)-property. Draw an arbitrary $c: |z - \omega| = r$ and denote by $w = f_r(z)$ the branch determined by an element $P(z; a)$, with centre a lying in $(c): |z - \omega| < r$ ⁽³¹⁾, of the inverse function $w = f(z)$ inside (c) . Suppose that the function does not take the value ω anywhere inside Δ_r where Δ_r denotes the set of all values taken by the branch $w = f_r(z)$; it is obvious that Δ_r is a peninsula, whose boundary consists of a finite or an enumerable number of analytic curves Γ_r lying inside D , and a subset E_r of E . Obviously E is of absolute harmonic measure zero and the function $\varphi(w) - \omega$ has the following properties: $\varphi(w) - \omega$ is regular in the closed domain $\overline{\Delta_r}$ (that is, the closed cover of Δ_r), excluding at all points belonging to E_r , not vanishing within Δ_r and $|\varphi(w) - \omega| = r$ on the contours Γ_r . Accordingly, we see that the function $\psi(w) = [\varphi(w) - \omega]^{-1}$, regular in Δ_r , is not bounded there. In other words, we can find such a sequence $w_\nu (\nu = 1, 2, \dots)$ inside Δ tending to a point belonging to E_r such that $\varphi(w_\nu) \rightarrow \omega$. Because, if $\psi(w)$ were bounded in Δ_r , then, applying

⁽²⁹⁾ R. Nevanlinna: loc. cit., page 132 and 134.

⁽³⁰⁾ In other words, E is the natural boundary.

⁽³¹⁾ When ω is the infinity, we take (c) as $|z| > \rho$.

Nevanlinna's extension of Lindelöf's principle, we would have $|\psi(w)| \leq \frac{1}{r}$ in Δ_r , that is, $|\varphi(w) - \omega| \leq r$ inside Δ_r . This is a contradiction.

(Q. E. D.)

Remark. The present writer has already used both lemmas stated above to prove a theorem⁽³²⁾ analogous to theorem 11.

5. In this next paragraph we shall give some sufficient conditions under which an analytic function should have (W)-property. On the other hand, we cannot give a useful condition under which a multiform analytic function $z = \varphi(w)$ should have (L)-property. In the proof of theorem 11, it is seen easily that (L)-property is related intimately to the maximum modulus principle of an analytic function. Concerning the maximum modulus principle for multiform analytic functions only a few results have been obtained by G. Hössjer and R. Nevanlinna⁽³³⁾. Next, suggested by Hössjer's paper, a sufficient condition will be proposed under which an analytic function $z = \varphi(w)$ should have (L)-property. Suppose first that *the set of values taken by $z = \varphi(w)$ is everywhere dense in the z -plane*. Consider any possible peninsula Δ corresponding to the covering surface F_D (stated before) where D is a circular disc $(c): |z - \omega| < r$ or $|z| > r$. Suppose secondly that *the projection of any peninsula Δ on the w -plane has at least one exterior point, if r is sufficiently small or sufficiently large according as (c) denotes either $|z - \omega| < r$ or $|z| > r$* . It is clear that Δ has at least one boundary point relative to the Riemann surface ϕ associated with $z = \varphi(w)$. Denote by Γ the relative boundary of Δ ; Γ consists of a finite or an enumerable number of analytic curves on ϕ . Obviously Δ is not necessarily simply connected. Hence, let us consider the universal covering surface Δ^* of the basic domain Δ . Then Δ^* is a simply connected Riemann covering surface of hyperbolic type, by the second assumption. By a well-known theorem on conformal mapping, Δ^* may be mapped conformally upon the unit-circle $(K): |\zeta| < 1$ in a one-one manner. We denote by $w = \psi(\zeta)$ such a mapping function. By the second assumption, $w = \psi(\zeta)$ is a function, meromorphic in $|\zeta| < 1$, of bounded type. Accordingly, for any θ not belonging to an exceptional set of measure 0, $w = \psi(\zeta)$ tends to a finite limit $\psi(e^{i\theta})$ (Fatou's theorem). We suppose finally that *the image on Δ^* transformed by the transformation $q^* = \psi(\zeta)$ of any radius with the end-point $e^{i\theta}$, θ not belonging to E , has a projection L*

(32) K. Noshiro: On the theory of the cluster sets of an analytic function, Journ. Fac. Sc., Hokkaido Imp. Univ., Ser. I, vol. 6, 1938, pp. 217-231. See p. 226, theorem 3.

(33) G. Hössjer: Über funktionentheoretische Nullmengen. und das Maximumprinzip bei mehrdeutigen analytischen Functionen, 8, Congr. scand. Stockholm, 1934.

R. Nevanlinna: Über die Lösbarkeit des Dirichletschen Problems für eine Riemannsche Fläche. Göttingen Nachr., Bd. 1, (1939).

(on Δ) which converges to a point on the relative boundary L . Under these conditions, we can assert that $z = \varphi(q)$ has (L) -property. Clearly the function $\varphi(q) - \omega$ is regular inside Δ and at all points of Γ and has such properties that $|\varphi(q) - \omega| < r$ inside Δ and $|\varphi(q) - \omega| = r$ on Γ . Next it will be shown that under the assumption that $\varphi(q) - \omega$ does not vanish inside Δ , the regular function $H(q) \equiv \frac{1}{\varphi(q) - \omega}$ cannot be bounded there. If $H(q)$ were bounded in Δ , considering $H(q)$ as a function on Δ^* which will be denoted by $H^*(q^*)$, and putting $G(\zeta) \equiv H^*[\psi(\zeta)]$, a bounded regular function $G(\zeta)$ inside $|\zeta| < 1$ would be obtained such that for any θ not belonging to E , $G(\zeta)$ tends to a value with the modulus $\frac{1}{r}$ as ζ tends to $e^{i\theta}$ along the radius. Consequently $|G(\zeta)| \leq \frac{1}{r}$ would follow throughout in $|\zeta| < 1$; in other words, the inequality $|\varphi(q) - \omega| \geq r$ would hold for all points inside Δ ⁽³⁴⁾. Thus we arrived at contradiction.

§ 9. On the behaviour of an analytic function in the neighbourhood of a transcendental singularity.

1. Let $w = f(z)$ be an analytic function and F be the Riemann surface, associated with $w = f(z)$, on which the function will be considered as a uniform meromorphic function $w = f(p)$. Let Ω be a transcendental singularity with coordinate ω , of $w = f(z)$ defined by the continuation $P(z; z(t))$ ($0 \leq t < 1$) along a path $C: z = z(t)$ ($0 \leq t < 1$), $\lim_{t \rightarrow 1} z(t) = \omega$. It has already been remarked that the system $P(z; z(t))$ ($0 \leq t < 1$) can be also chosen to contain regular elements only and moreover such that $\lim_{t \rightarrow 1} P(z(t); z(t))$ does not exist in the case where Ω is an essential singularity⁽³⁵⁾. We denote by $f_r(z)$ the branch determined by an element $P(z; z(t_r))$, $z(t_r)$ lying on the end-part C_r of C inside $|z - \omega| < r$ and call the domain F_r , on F , defined by the branch $f_r(z)$ an r -neighbourhood of an accessible boundary point Ω of F (Cf. § 4). Now we shall introduce some new definitions. If a sequence p_ν ($\nu = 1, 2, \dots$) of points on F has such a property that for any positive number r , all points excluding a finite number of ν , lie within the neighbourhood F_r , then the sequence will be said to converge to the boundary point Ω , as it clearly converges to the boundary of F ⁽³⁶⁾.

⁽³⁴⁾ When $\omega = \infty$, it is necessary to modify the argument only slightly.

⁽³⁵⁾ Cf. theorems 2 and 3 in § 4.

⁽³⁶⁾ Because, if the sequence $p_\nu \equiv P(z; z_\nu)$ ($\nu = 1, 2, \dots$) had a point of accumulation on F , say $p_0 \equiv P(z; z_0)$, z_0 would be equal to ω , as $z_\nu \rightarrow \omega$. Denote by ρ_0 the radius of convergence of $P(z; z_0)$. Then the neighbourhood F_r of Ω , where $r < \rho_0$, would have at least one common point with the ρ_0 -neighbourhood of p_0 . This contradicts that $P(z; z(t))$ ($0 \leq t < 1$) defines a transcendental singularity Ω with coordinate ω .

Let us associate the following point-sets with the transcendental singularity Ω .

(1) The cluster set S_Ω . Denote by A_r the set of values taken by $w = f_r(z)$. Then this set is defined by the formula $S_\Omega \equiv \bigcap_{r>0} \bar{A}_r$, where $\bar{A}_r$ is the closure of A_r . In other words, S_Ω is the set of values α such that $\lim_{\nu \rightarrow \infty} f(p_\nu) = \alpha$ where $p_\nu (\nu = 1, 2, \dots)$ is a sequence of points converging to the boundary point Ω ; S_Ω consists of a single point or is a continuum more than one point.

(2) The range of values R_Ω . This is the set defined by the formula $R_\Omega \equiv \bigcap_{r>0} A_r$. In other words, R_Ω is a set of values α such that $f(p_\nu) = \alpha$ ($\nu = 1, 2, \dots$) where $p_\nu (\nu = 1, 2, \dots)$ is a sequence of points on F converging to the boundary point Ω ⁽³⁷⁾.

(3) The convergence set Γ_Ω . This is the set of all asymptotic values of $w = f(p)$ at the accessible boundary point Ω .

It is clear that both R_Ω and Γ_Ω are included in S_Ω . Without difficulty, we see that if $S_\Omega \equiv \bigcap_{r>0} \bar{A}_r$ has an interior point (say α), then $R_\Omega \equiv \bigcap_{r>0} A_r$ cannot be empty. To prove this, let $r_\nu (\nu = 1, 2, \dots)$ be a decreasing sequence of positive numbers tending to zero. Denote by K a circular disc: $|w - \alpha| < \rho$ such that the closure $\bar{K}$ of K is included entirely in S_Ω . As α belongs to $\bar{A}_{r_1}$, we may find a point α_1 in the common part $K \cdot A_{r_1}$ of K and A_{r_1} ; hence, it is possible to find a circular disc $K_1: |w - \alpha_1| < \rho_1$ such that the closure $\bar{K}_1$ of K_1 lies entirely in the product $K \cdot A_{r_1}$. Next, as α_1 belongs to $\bar{A}_{r_2}$, we may find similarly a closed circular disc $\bar{K}_2$ contained in the product $K_1 \cdot A_{r_2}$. Repeating the same process, we can find a sequence of $\bar{K}_\nu (\nu = 1, 2, \dots)$ of closed circular discs such that $\bar{K}_1 \supset \bar{K}_2 \supset \dots \supset \bar{K}_\nu \supset \dots$ and $\bar{K}_\nu \supset A_{r_\nu} (\nu = 1, 2, \dots)$. Consequently it follows $\bigcap_{\nu=1}^{\infty} \bar{K}_\nu \subset \bigcap_{\nu=1}^{\infty} A_{r_\nu} = R_\Omega$; it is evident that $\bigcap_{\nu=1}^{\infty} \bar{K}_\nu$ is not empty.

(37) Supposing that α is a value belonging to R_Ω , let r_ν be a decreasing sequence of positive numbers tending to zero. The present assertion is true when there is a sequence $P_\nu(z; a_\nu) (\nu = 1, 2, \dots)$ of elements such that $0 < |a_\nu - \omega| < r_\nu$ and $P(a_\nu; a_\nu) = \alpha$ where $P(z; a_\nu)$ denotes an element of the branch $f_{r_\nu}(z)$. Consider the other case in which from a certain ν (say ν_0) onwards, any element of $f_{r_\nu}(z)$ which takes the value α at its centre must have its centre at the point $z = \omega$. For each $\nu \geq \nu_0$, let us select such an element $P_\nu(z; \omega)$ of the branch $f_\nu(z) \equiv f_{r_\nu}(z)$. First denote by ρ_0 the radius of convergence of $P_{\nu_0}(z; \omega)$ and select a number ν_1 greater than ν_0 such that $r_{\nu_1} < \rho_0$. Then it is clear that $P_{\nu_1}(z; \omega)$ is distinct from $P_{\nu_0}(z; \omega)$. Next, denoting by ρ_1 the radius of convergence of $P_{\nu_1}(z; \omega)$ and selecting a number ν_2 greater than ν_1 such that $r_{\nu_2} < \rho_1$; we see that $P_{\nu_2}(z; \omega)$ is different from $P_{\nu_1}(z; \omega)$. Repeating the same arguments, we shall finally obtain an infinity of distinct elements $P_{\nu_n}(z; \omega)$ ($n = 1, 2, \dots$), each $P_{\nu_n}(z; \omega)$ being an element of $f_{\nu_n}(z)$.

2. Now some applications of normal family to the study of the behaviours of an analytic function near its transcendental singularities will be considered. First the following theorem will be proved:

Theorem 12. *Let $w = f(z)$ be an analytic function distinct from an algebraic function. Suppose that the curve*

$$(1) \quad L: p(t) \equiv P(z: z(t)) \quad (0 \leq t < 1)$$

is an asymptotic path of $w = f(p)$ on the Riemann surface F associated with $w = f(z)$, its asymptotic value being equal to a , where every $P(z; z(t))$ ($0 \leq t < 1$) is a regular element with radius $\rho(t)$ of convergence⁽³⁸⁾. Suppose further that the family

$$(2) \quad f_t(\zeta) \equiv P[z(t) + \rho(t)\zeta; z(t)] \quad (0 \leq t < 1)$$

of functions, regular in the unit-circle $|\zeta| < 1$, is normal in Montel's sense. Then, if we denote by

$$(3) \quad U(p(t); \theta), \quad (0 < \theta < 1, \theta \text{ fixed})$$

the $\theta\rho(t)$ -neighbourhood of the point $p(t)$, we have $\lim_{\nu \rightarrow \infty} f(p_\nu) = a$ for any sequence p_ν ($\nu = 1, 2, \dots$) of points inside Δ converging to the boundary of F , where Δ is the domain which consists of the points belonging to any $U(p(t); \theta)$ ($0 \leq t < 1$); that is, Δ is defined by the expression

$$\Delta = \sum_{0 \leq t < 1} U(p(t); \theta).$$

Proof. We denote by $C: z = z(t)$ ($0 \leq t < 1$) the projection of the asymptotic path L . Let θ be an arbitrary positive number less than unity. For each t_0 ($0 \leq t_0 < 1$), let K_{t_0} be a circle with centre $z(t_0)$ and radius $\theta\rho(t_0)$. As C is the projection of the asymptotic path L of $w = f(p)$ it is seen that the part $C_{t_0}: z = z(t)$ ($t_0 \leq t < 1$) of the curve C necessarily intersects with the circle $K_{t_0}: |z - z(t_0)| = \theta\rho(t_0)$; we will denote by $z(t'_0)$ the first point of intersection of C_{t_0} counting from $z(t_0)$ and by ζ'_0 the image of $z(t'_0)$ transformed by $\zeta = \frac{1}{\rho(t_0)}(z - z(t_0))$.

Now consider any increasing sequence t_ν ($\nu = 1, 2, \dots$) such that $0 < t_\nu < 1$, $t_\nu \rightarrow 1$ and put $z_\nu = z(t_\nu)$ ($\nu = 1, 2, \dots$). By hypothesis, from from the corresponding sequence

$$f_{t_\nu}(\zeta) \equiv P[z_\nu + \rho(t_\nu)\zeta; z_\nu] \quad (\nu = 1, 2, \dots),$$

⁽³⁸⁾ Suppose $\rho(t)$ to be finite for $0 \leq t < 1$. Then it is clear that $\rho(t)$ is a continuous function for $0 \leq t < 1$.

each member $f_{t_\nu}(\zeta)$ (which is denoted by $\psi_\nu(\zeta)$ for simplicity) is regular in $|\zeta| < 1$, we may choose a subsequence $\psi_{\nu_n}(\zeta)$ ($n = 1, 2, \dots$) converging uniformly in the interior of the unit-circle $|\zeta| < 1$, the limiting function being denoted by $f_0(\zeta)$ ⁽³⁹⁾. It is obvious that the sequence

$$\psi_{\nu_n}(\zeta'_{\nu_n}) = P[z_{\nu_n} + \rho(t_{\nu_n})\zeta'_{\nu_n}; z_{\nu_n}] \quad (n = 1, 2, \dots)$$

where

$$\zeta'_{\nu_n} = \frac{1}{\rho(t_{\nu_n})}(z(t'_{\nu_n}) - z(t_{\nu_n})) \quad (n = 1, 2, \dots),$$

tends to the limit α as $n \rightarrow +\infty$. Since every ζ'_{ν_n} lies on the circumference $|\zeta| = \theta$, the limiting function $f_0(\zeta)$ must have at least one α -point on $|\zeta| = \theta$. On the other hand, as θ can be taken arbitrarily, we see that the function $f_0(\zeta)$ reduces to a constant α . Hence it follows without difficulty, that the initial sequence $f_{t_\nu}(\zeta)$ ($\nu = 1, 2, \dots$) converges to the constant limit α inside the unit-circle $|\zeta| < 1$. Here let us remark that the above argument is independent of the choice of the initial sequence $f_{t_\nu}(\zeta)$ ($\nu = 1, 2, \dots$). Thus, it is concluded that for any positive number ε , we may find a positive number $\delta = \delta(\varepsilon)$ such that $|f_t(\zeta) - \alpha| < \varepsilon$ in the closed domain $|\zeta| \leq \theta$, θ being arbitrarily fixed, provided that $t \geq 1 - \delta$. Now, let us consider the just have been obtained conclusion on the Riemann surface F associated with $w = f(z)$. Then, it is easily seen that we have $\lim_{\nu \rightarrow \infty} f(p_\nu) = \alpha$ for any p_ν ($\nu = 1, 2, \dots$) of points inside Δ converging to the boundary of F , where Δ is the domain which consists of all points belonging to any $U(p(t); \theta)$ ($0 \leq t < 1$), $U(p(t); \theta)$ denoting the $\theta\rho(t)$ -neighbourhood of the point $p(t)$. Indeed, to prove this, it is sufficient to show that for any positive number λ less than unity, every point p_ν , from a certain ν onwards, cannot lie within $\Delta_\lambda^* = \sum_{0 \leq t \leq \lambda} U(p(t); \theta)$. If otherwise, we can find a subsequence p_{ν_n} ($n = 1, 2, \dots$) of points lying in Δ_λ^* . With each p_{ν_n} associate a neighbourhood $U(p(t_n^*); \theta)$, where $0 \leq t_n^* \leq \lambda$, containing the point p_{ν_n} in its interior. As the sequence t_n^* ($n = 1, 2, \dots$) has at least one point of accumulation in the closed interval $0 \leq t \leq \lambda$, we may without loss of generality suppose that the sequence t_n^* ($n = 1, 2, \dots$) converges to a limit t_0^* ($0 \leq t_0^* \leq \lambda$). Then, from $z(t_n^*) \rightarrow z(t_0^*)$ follows $\theta\rho(t_n^*) \rightarrow \theta\rho(t_0^*)$ as $n \rightarrow +\infty$. Accordingly, for all sufficiently large n , every $U(p(t_n^*); \theta)$ is contained in the ρ^* -neighbourhood of $p(t_0^*)$ where $\rho^* = \frac{1}{2}(1 + \theta)\rho(t_0^*)$, and so all $p_{\nu_n}^*$ (excluding a finite

(39) "To converge uniformly in the interior of D " means "to converge uniformly in any closed domain inside D ".

number of n) are contained in the ρ^* -neighbourhood of $p(t_0^*)$; p_n^* must have at least one point of accumulation on F . (Q. E. D.)

Remark. Theorem 12 can be regarded as an extension of a well-known theorem of Lindelöf. In the particular case where $w = f(z)$ is a function meromorphic in $|z| < R \leq +\infty$, analogous results have been obtained by F. Ullrich, T. Shimizu and the present writer⁽⁴⁰⁾.

Let us notice that in the particular case of theorem 12 such that the projection $z(t)$ ($0 \leq t < 1$) of the curve L in (1) satisfies the relation $\lim_{t \rightarrow 1} z(t) = \omega$, ($\omega \neq \infty$), the curve L defines an accessible boundary point Ω , of F , with coordinate $z = \omega$ and it follows $\lim_{t \rightarrow 1} \rho(t) = 0$. Then, from theorem 12, we easily obtain

Theorem 13. *Let $w = f(z)$ be an analytic function and F be the Riemann surface associated with $w = f(z)$. Suppose that the curve $L: p(t) \equiv P(z; z(t))$ ($0 \leq t < 1$) defines an accessible boundary point Ω with coordinate $z = \omega$ ($\omega \neq \infty$) and is an asymptotic path of $w = f(p)$ on F , the asymptotic value being equal to α , where every $P(z; z(t))$ ($0 \leq t < 1$) is a regular element with radius $\rho(t)$ of convergence. Denote by $U(p(t); \theta)$ ($0 < \theta < 1$, θ fixed) the $\theta\rho(t)$ -neighbourhood of the point $p(t)$, by Δ the domain which consists of all the points belonging to any $U(p(t); \theta)$ ($0 \leq t < 1$), that is $\Delta \equiv \sum_{0 \leq t < 1} U(p(t); \theta)$ and by $S_\Omega^{(\Delta)}$ the set of all values β such that $\lim_{\nu \rightarrow \infty} f(p_\nu) = \beta$ where p_ν ($\nu = 1, 2, \dots$) is a sequence of points, converging to Ω , inside Δ . Under these conditions, if $S_\Omega^{(\Delta)}$ contains at least one value distinct from α , then the function $w = f(p)$ takes every finite value, with one possible exception, an infinity of times in any domain Δ' defined in the same way as Δ , by using θ' instead of θ where $0 < \theta < \theta' < 1$. Consequently, the range of values R_Ω becomes the domain obtained by excluding possibly one point from $|w| < +\infty$, so that S_Ω becomes the whole w -plane including $w = \infty$; in other words, $w = f(z)$ has (W)-property.*

Proof. It should first be remarked that a sequence p_ν ($\nu = 1, 2, \dots$) inside Δ converging to the boundary of F necessarily converges to the accessible boundary point Ω . For any r -neighbourhood F_r of Ω , we can find a positive number λ less than unity, such that the domain $\Delta_\lambda \equiv \sum_{\lambda \leq t < 1} U(p(t); \theta)$ may be included in F_r , since $\lim_{t \rightarrow 1} z(t) = \omega$ and $\lim_{t \rightarrow 1} \rho(t) = 0$.

(40) E. Ullrich: Über eine Anwendung des Verzerrungssatzes auf meromorphe Funktionen, Journ. f. Math., Bd. 166 (1932), Satz 8, p. 232.

T. Shimizu: On the paths of determination and indetermination of integral functions, Proc. Phys.-Math. Soc. Japan, 3. ser. vol. 12 (1930), p. 127.

K. Noshiro: Contributions to the theory of meromorphic functions in the unit-circle, Journ. Facul. Sc., Hokkaido Imp. Univ., ser. I. vol. 7 (1939), pp. 149-157. See theorem 3, p. 153.

On the other hand, it has already been shown that except a finite number of ν , all p_ν lie in the domain Δ_λ (See the last part of the proof of theorem 12). Next remark that, given any point $p_0 \equiv P(z; z_0)$ on F , we can find a positive number less than 1 such that the domain $\Delta'_\lambda \equiv \sum_{\lambda \leq t < 1} U(p(t); \theta')$ never contains $p_0 \equiv P(z; z_0)$.

Now, let us pass to the proof for the theorem. If there are two finite exceptional values (say β_1 and β_2) in Picard's sense inside Δ' , we shall find such a number λ ($0 < \lambda < 1$) that $w = f(p)$ has neither β_1 -point nor β_2 -point in the domain Δ'_λ according to the second remark above. Therefore the family $f_t(\zeta) \equiv P[z(t) + \rho(t)\zeta; z(1)]$ ($\lambda \leq t < 1$) of functions, regular in the unit-circle $|\zeta| < 1$, is normal for $|\zeta| < \theta'$. Consequently, applying theorem 12, it is seen that $\lim_{\nu \rightarrow \infty} f(p_\nu) = \alpha$ for any sequence p_ν ($\nu = 1, 2, \dots$) of points inside Δ converging to the boundary of F . Since such a sequence p_ν ($\nu = 1, 2, \dots$) converges to the accessible boundary point Ω , according to the first remark, it is seen that $S_\Omega^{(\Delta)}$ necessarily consists of only one point α . As its alternative, if $S_\Omega^{(\Delta)}$ contains at least one value different from α , $w = f(p)$ takes every finite value with possible one exception inside Δ' . Let γ be any non-exceptional value in Picard's sense inside Δ' and let p_ν ($\nu = 1, 2, \dots$) be all distinct γ -points within Δ' . It is clear that p_ν ($\nu = 1, 2, \dots$) converges to the accessible boundary point Ω , according to the first remark. Thus it was proved that the range of values R_Ω becomes the domain obtained by excluding possibly one point from $|w| < +\infty$ and so S_Ω becomes the whole w -plane including $w = \infty$. (Q. E. D.)

Next is proposed the following

Theorem 14. *Let $w = f(z)$ be an analytic function. Suppose first that the curve $L: p(t) \equiv P(z; z(t))$ ($0 \leq t < 1$), $\lim_{t \rightarrow 1} z(t) = \omega$ ($\omega \neq \infty$) defines an accessible boundary point Ω of F with the projection ω , where every element $P(z; z(t))$ ($0 \leq t < 1$) is regular and its radius of convergence is equal to $\rho(t)$. Next, Suppose that the family $f_t(\zeta) \equiv P[z(t) + \rho(t)\zeta; z(t)]$ ($0 \leq t < 1$) of functions, regular in the unit-circle $|\zeta| < 1$ is normal. Suppose finally that for a certain increasing sequence t_ν ($\nu = 1, 2, \dots$) tending to unity, the corresponding sequence $f_{t_\nu}(\zeta)$ ($\nu = 1, 2, \dots$) converges to a non-constant limiting function $f_0(\zeta)$ inside $|\zeta| < 1$, the set of all values taken by $f_0(\zeta)$ being denoted by D_0 . Under these conditions, it is seen that the range of values R_Ω covers the domain D_0 .*

Proof. With each value α lying in D , a natural number $\nu(\alpha)$ may be associated such that if $\nu \geq \nu(\alpha)$, every $f_{t_\nu}(\zeta)$ takes α at least once within a fixed circular domain: $|\zeta| < \theta(\alpha)$, ($0 < \theta(\alpha) < 1$), which contains

at least one α -point of the limiting function $f_0(\zeta)$ in its interior. Consequently $w = f(p)$ takes every value α , belonging to D , an infinity of times within the domain Δ_α defined by the expression: $\Delta_\alpha \equiv \sum_{0 \leq t < 1} U(p(t); \theta(\alpha))^{(41)}$. It will be clear that, if all α -points in Δ_α be denoted by $p_1(\alpha)$, $p_2(\alpha)$, ..., the sequence $p_\nu(\alpha)$ ($\nu = 1, 2, \dots$) converges to the accessible boundary point Ω of $F^{(42)}$. Hence it has been proved that R_Ω includes the domain D_0 .

As an immediate consequence of theorem 14, it is possible to obtain

Theorem 15. Suppose that the curve $L: p(t) \equiv P(z; z(t))$ ($0 \leq t < 1$), $\lim_{t \rightarrow 1} z(t) = \omega$ ($\omega \neq \infty$) on the surface F associated with an analytic function $w = f(z)$ defines an accessible boundary point Ω of F with the projection ω , where every element $P(z; z(t))$ ($0 \leq t < 1$) is regular and its radius of convergence is equal to $\rho(t)$. Next suppose that the range of values R_Ω is empty. Finally suppose that there is another curve, of F , $L^*: p^*(t) \equiv P^*(z; z^*(t))$ ($0 \leq t < 1$), $\lim_{t \rightarrow 1} z^*(t) = \omega$ such that

(i) for every t ($0 \leq t < 1$), $|z(t) - z^*(t)| < \theta \rho(t)$,

where θ is a fixed positive number less than unity and

(ii) for every t ($0 \leq t < 1$), $P^*(z; z^*(t))$ is a direct analytic continuation of $P(z; z(t))$ (it is evident that L^* defines the same accessible boundary point Ω as L). Under these conditions, the cluster set $S_\Omega^{(L)}$ along the curve L of $w = f(p)$ is identical with the cluster set $S_\Omega^{(L^*)}$ along the curve L^* of $w = f(p)$ where $S_\Omega^{(L)}$ denotes the set of all values α such that $f(p(t_\nu)) \rightarrow \alpha$ as $t_\nu \rightarrow 1$ and $S_\Omega^{(L^*)}$ is defined in a similar way.

Proof. Let γ and γ' ($\gamma \neq \gamma'$) any two complex numbers. Then we can find a positive number λ less than unity such that there exists neither γ -point nor γ' -point of $w = f(p)$ inside $\Delta'_\lambda \equiv \sum_{\lambda \leq t < 1} U(p(t); \theta')$, θ' being a fixed positive number between θ and 1. Consequently the family $f_t(\zeta) \equiv P[z(t) + \rho(t)\zeta; z(t)]$ ($\lambda \leq t < 1$) of functions, regular in $|\zeta| < 1$, is normal for $|\zeta| < \theta$. Consider any increasing sequence t_ν ($\nu = 1, 2, \dots$) converging to unity such that both limit $\lim_{\nu \rightarrow \infty} f(p(t_\nu))$ and $\lim_{\nu \rightarrow \infty} f(p^*(t_\nu))$ exist and put $\lim_{\nu \rightarrow \infty} f(p(t_\nu)) = \alpha$ and $\lim_{\nu \rightarrow \infty} f(p^*(t_\nu)) = \beta$. For the proof, it will be sufficient to show that α is identical with β .

Without any loss of generality it may be supposed that the corresponding sequence $f_{t_\nu}(\zeta) \equiv P[z(t_\nu) + \rho(t_\nu)\zeta; z(t_\nu)]$ ($\nu = 1, 2, \dots$) converges uniformly to a limiting function, say $f_0(\zeta)$, in the interior of the circle $|\zeta| < \theta'$. It is obvious that

⁽⁴¹⁾ Cf. the second remark in the proof of theorem 13.

⁽⁴²⁾ Cf. the first remark in the proof of theorem 13.

$$f_0(0) = \lim_{\nu \rightarrow \infty} f_{t_\nu}(0) = \lim_{\nu \rightarrow \infty} P[z(t_\nu); z(t_\nu)] = \lim_{\nu \rightarrow \infty} f(p(t_\nu)) = a.$$

As R_Ω is empty, by hypothesis, it follows, by theorem 14, that $f_0(\zeta)$ must reduce to a constant which is no other than $f_0(0) = a$. Thus, the sequence $f_{t_\nu}(\zeta)$ ($\nu = 1, 2, \dots$) converges uniformly to a constant limit a in the domain $|\zeta| < \theta^*$ where θ^* is arbitrarily fixed between θ and θ' . Now, for each natural number ν , let ζ_ν^* be the image of $z^*(t_\nu)$ by the transformation $\zeta = \rho(t_\nu)^{-1}(z - z(t_\nu))$, then all ζ_ν^* ($\nu = 1, 2, \dots$) will lie in the circle $|\zeta| < \theta$, by (i), and so there will exist at least one point of accumulation of the sequence ζ_ν^* ($\nu = 1, 2, \dots$) in the closed domain $|\zeta| \leq \theta$. It may be supposed, without loss of generality, that ζ_ν^* converges to a limiting point ζ_0^* inside $|\zeta| \leq \theta^*$. We see that, as uniform convergence gives continuous convergence in Caratheodory's sense,

$$\lim_{\nu \rightarrow \infty} f_{t_\nu}(\zeta_0^*) = \lim_{\nu \rightarrow \infty} f_{t_\nu}(\zeta_\nu^*)$$

from which follow the equalities

$$\begin{aligned} f_0(\zeta_0^*) &= \lim_{\nu \rightarrow \infty} f_{t_\nu}(\zeta_\nu^*) = \lim_{\nu \rightarrow \infty} P(z^*(t_\nu); z(t_\nu)) \\ &= \lim_{\nu \rightarrow \infty} P^*(z^*(t_\nu); z^*(t_\nu)), \quad \text{by (ii),} \\ &= \lim_{\nu \rightarrow \infty} f(p^*(t_\nu)) = \beta. \end{aligned}$$

As $f_0(\zeta) \equiv a$ inside $|\zeta| < 1$, it is seen that $a = \beta$. (Q. E. D.)

Chapter 2. On the Theory of the Cluster sets of Analytic Functions.

§ 1. Definitions of the cluster sets of a uniform analytic function.

1. At the beginning, we will state some definitions of the cluster sets of uniform analytic functions. Suppose that $f(z)$ is a uniform function, meromorphic in a connected domain D . Let z_0 be a point on the boundary C of D . We associate with z_0 the following sets of values:

(1) The cluster set $S_{z_0}^{(D)}$. This is the set of all values a such that $\lim_{\nu \rightarrow \infty} f(z_\nu) = a$ where z_ν ($\nu = 1, 2, \dots$) is a sequence of points tending to z_0 inside D . In other words, $S_{z_0}^{(D)}$ is identical with the product of all $\overline{\mathfrak{D}_r}$ where $\overline{\mathfrak{D}_r}$ is the closure of the set $\mathfrak{D}_r$ of values taken by $w = f(z)$ inside the common part of (c_r) and D , the circular disc: $|z - z_0| < r$ being denoted by (c_r) . Hence, it is obvious that $S_{z_0}^{(D)}$ is a closed (non-

empty) set but is not necessarily a continuum (closed connex set,) as is shown by an easy example.

(2) The cluster set $S_{z_0; L}^{(D)}$. In the particular case where z_0 is an accessible boundary point, there exists at least one path (Jordan arc), say L , lying inside D except the end-point z_0 . Describe a circle $c: |z - z_0| = r$ and denote by L_r the end-part $\widehat{z_r z_0}$ of L , where z_r is the first point of intersection of L with $c: |z - z_0| = r$ counting from z_0 . The common part of D and $(c): |z - z_0| < r$ is an open set D_r which consists of a finite or an enumerable number of connected domains. Denote by A_r the connected component containing the arc L_r excluding the end-points z_r and z_0 and by $\mathfrak{D}_r^*$ the set of values taken by $w = f(z)$ inside A_r . It is clear that $\mathfrak{D}_r^*$ is a connected domain. Accordingly the set $S_{z_0; L}^{(D)} = \bigcap_{r>0} \mathfrak{D}_r^*$ consists of only one point or a continuum of more than one point, as the closure $\overline{\mathfrak{D}_r^*}$ being a continuum, is decreasing with r . Let L' be another path inside D except the end-point z_0 such that both end-parts L'_r (defined in the same manner as L_r) and L_r can be connected by an arc lying entirely inside the open set D_r for all sufficiently small r (> 0). Then, it may be said that two paths L and L' are equivalent to each other and it may be considered that they define the identical accessible point. It is seen without difficulty that $S_{z_0; L}^{(D)}$ and $S_{z_0; L'}^{(D)}$ are identical with each other provided that L and L' are equivalent to each other⁽⁴³⁾.

(3) The cluster set $S_{z_0}^{(C)}$. This is the product $\prod M_r$, where M_r denotes the closure of the sum $\sum_{0 < |z' - z_0| < r} S_{z'}^{(D)}$, z' varying on C . This set is also closed and becomes empty, if and only if z_0 is an isolated boundary point.

(4) The range of values $R_{z_0}^{(D)}$. This is the product of all value-sets of $w = f(z)$ for the common part of D and (c_r) . Hence, a value α belongs to $R_{z_0}^{(D)}$, if and only if, $f(z)$ takes α infinitely often near z_0 inside D . It is clear that $R_{z_0}^{(D)}$ is a G_δ set.

(5) The convergence set $\Gamma_{z_0}^{(D)}$. Suppose that z_0 is an accessible boundary point. $\Gamma_{z_0}^{(D)}$ is the set of all values α such that $f(z) \rightarrow \alpha$ as $z \rightarrow z_0$ along a path inside D except the end-point z_0 . In other words, $\Gamma_{z_0}^{(D)}$ is the set of all asymptotic values at $z = z_0$ of $w = f(z)$. For convenience, it is taken that $\Gamma_{z_0}^{(D)} = 0$ when z_0 is non-accessible.

It is obvious that $S_{z_0}^{(D)}$ includes $S_{z_0; L}^{(D)}$, $S_{z_0}^{(C)}$, $R_{z_0}^{(D)}$ and $\Gamma_{z_0}^{(D)}$ separately.

(43) In the particular case, where C is a Jordan closed curve, $S_{z_0; L}^{(D)}$ is identical with $S_{z_0}^{(D)}$.

2. The following remark will be used later: If z_0 is non-isolated and accessible, $S_{z_0; L}^{(D)}$ intersects with $S_{z_0}^{(C)}$. Since z_0 is a non-isolated boundary point, we may find a sequence $z'_\nu (\nu = 1, 2, \dots)$ of boundary-points distinct from z_0 such that $z'_\nu \rightarrow z_0$ as $\nu \rightarrow \infty$. It is clear that if we round the circle $c_\nu: |z - z_0| = r_\nu$, putting $r_\nu = |z'_\nu - z_0|$, starting from $z = z_\nu$ where z_ν denotes the first point of intersection of the path L with the circle c_ν counting from z_0 , we meet at the boundary. Consequently, it is seen that for any natural number ν , $\bar{D}_{r_\nu}^*$ and $M_{r_\nu} = \sum_{0 < |z' - z_0| < r_\nu} S_{z'}^{(D)}$ have at least one point in common, whence it follows at once $S_{z_0; L}^{(D)} \cdot S_{z_0}^{(C)} \neq 0$.

Let us recall some well-known theorems in the particular case where z_0 is an isolated boundary point: Riemann-Weierstrass' theorem show that $S_{z_0}^{(D)}$ either contains only one point or covers the whole w -plane including $w = \infty$ and Picard's theorem states that $R_{z_0}^{(D)}$ is the domain obtained by excluding at most two points from the whole w -plane with $w = \infty$. Moreover, Iversen has proved that if a is an exceptional value in Picard's sense near z_0 , a is an asymptotic value—in other words, if a does, not belong to $R_{z_0}^{(D)}$, then it does belong to $\Gamma_{z_0}^{(D)}$. It is very important to study the properties of the sets $S_{z_0}^{(D)}$, $S_{z_0; L}^{(D)}$, $S_{z_0}^{(C)}$, $R_{z_0}^{(D)}$, $\Gamma_{z_0}^{(D)}$ separately and the relations among them when D is an arbitrary domain with a non-isolated boundary point z_0 .

2. Three equivalent forms of Beurling-Kunugi's theorems.

1. In this section three distinct form of Beurling-Kunugi's theorems will be considered.

Theorems 1. (First form). *Let D be an arbitrary connected domain and let z_0 be a non-isolated boundary point. If $w = f(z)$ is a uniform function, holomorphic and bounded in the neighbourhood of z_0 inside D , then it follows*

$$(1) \quad \lim_{z \rightarrow z_0} |f(z)| = \lim_{z' \rightarrow z_0} (\lim_{\substack{z \rightarrow z' \\ z' \neq z_0}} |f(z)|),$$

denoting any boundary point by z' ; since the left-hand side and the right-hand side of (1) denote the radii $r(D)$ and $r(C)$ of the smallest closed circular domains, with centres at the origin $w = 0$, which contain $S_{z_0}^{(D)}$ and $S_{z_0}^{(C)}$ respectively, (1) may be put in the form

$$(1') \quad r(D) = r(C).$$

Now let $\rho(E, a)$ denote the spherical distance of a point-set E from a point a on the w -plane, by introducing the Riemann sphere, which will

be denoted by Σ , with diameter 1, touching the w -plane at the origin $w = 0$. Then the above theorem is equivalent to

Theorem 2. (Second form.) *Let D be an arbitrary connected domain its boundary being denoted by C , and let z_0 be a non-isolated boundary point and suppose that $w = f(z)$ is uniform and meromorphic in D . Under these conditions, if α is not contained in $S_{z_0}^{(D)}$, it follows that*

$$\rho(S_{z_0}^{(D)}, \alpha) = \rho(S_{z_0}^{(C)}, \alpha).$$

Proof. Theorem 2 contains theorem 1. From the assumption that $w = f(z)$ is bounded in the neighbourhood of z_0 inside D , it follows that $w = \infty$ cannot lie in $S_{z_0}^{(D)}$. On the other hand, the equality $r(D) = r(C)$ is no other than $\rho(S_{z_0}^{(D)}, \infty) = \rho(S_{z_0}^{(C)}, \infty)$. Next it will be proved that theorem 2 follows from theorem 1. Suppose that α does not lie in $S_{z_0}^{(D)}$ and then consider the linear transformation $w^* = \frac{1 + \bar{\alpha}w}{w - \alpha}$ such that $w = \alpha$ corresponds to $w^* = \infty$. It is clear that $w^* = f^*(z) \equiv \frac{1 + \bar{\alpha}f(z)}{f(z) - \alpha}$ satisfies the hypothesis of theorem 1 and so there exists $r^*(D) = r^*(C)$, that is $\rho(S_{z_0}^{*(D)}, \infty) = \rho(S_{z_0}^{*(C)}, \infty)$, where $S_{z_0}^{*(D)}$ and $S_{z_0}^{*(C)}$ denote the cluster sets of $w^* = f^*(z)$. Accordingly it is seen that $\rho(S_{z_0}^{(D)}, \alpha) = \rho(S_{z_0}^{(C)}, \alpha)$ as the transformation $w^* = \frac{1 + \bar{\alpha}w}{w - \alpha}$ corresponds to a rotation of the Riemann sphere Σ . (Q. E. D.)

It will next be shown that theorem 2 is also equivalent to

Theorem 3. (Third form.) *Let D be an arbitrary connected domain its boundary being denoted by C , and let z_0 be a non-isolated boundary point and suppose that $w = f(z)$ is uniform and meromorphic in D . Under these conditions, it follows that*

$$B(S_{z_0}^{(D)}) \subset B(S_{z_0}^{(C)})$$

where $B(S_{z_0}^{(D)})$ and $B(S_{z_0}^{(C)})$ denote the boundaries of $S_{z_0}^{(D)}$ and $S_{z_0}^{(C)}$ respectively.

If the relation $S_{z_0}^{(C)} \subset S_{z_0}^{(D)}$ be remembered, then the equivalence between theorem 2 and 3 is a direct consequence of the following two lemmas.

Lemma 1. *Let M and N ($M \supset N$) be two closed sets, in the w -plane, such that, for any exterior point P of M , they always satisfy $\rho(M, P) = \rho(N, P)$. Then it follows that $B(M) \subset B(N)$.*

Proof. Denote by Q any point in $B(M)$, then, since M is closed, there is a sequence P_ν ($\nu = 1, 2, \dots$) of exterior points of M (and so N)

such that $P_\nu \rightarrow Q$. Clearly it follows that $\rho(M, Q) = \lim_{\nu \rightarrow \infty} \rho(M, P_\nu)$, $\rho(N, Q) = \lim_{\nu \rightarrow \infty} \rho(N, P_\nu)$ and, by hypothesis, $\rho(M, P_\nu) = \rho(N, P_\nu)$ for any ν . Consequently we have $\rho(N, Q) = \rho(M, Q) = 0$, whence follows the above assertion.

Lemma 2. *From the assumption that $B(M) < B(N)$ and $M > N$, it follows that, for any exterior point P of M , $\rho(M, P) = \rho(N, P)$.*

Proof. By hypothesis $\rho(M, P) \leq \rho(N, P)$ and $\rho(B(M), P) \geq \rho(B(N), P)$, while $\rho(B(M), P) = \rho(M, P)$, $\rho(B(N), P) = \rho(N, P)$, whence follows at once $\rho(M, P) = \rho(N, P)$ for any exterior point P of M .

Thus it has been shown that these three theorems, any of them being called Beurling-Kunugui's theorem, are equivalent to one other. Consequently it may be asserted that to prove these three theorems, it is only necessary to give a direct proof for any one of them.

2. The following remark may be useful to some readers. In the particular case where D is a Jordan domain, W. Seidel⁽⁴⁴⁾ has proved theorem 1 by using Lindelöf's principle of maximum modulus and J. L. Doob⁽⁴⁵⁾ has shown the equivalence between theorem 1 and 3. Under the assumption that z_0 is regular concerning the problem of Dirichlet, A. Beurling⁽⁴⁶⁾ has proved theorem 1, by using harmonic majoration, and the present writer proved the equivalence among theorems 1, 2 and 3⁽⁴⁷⁾. Roughly speaking's Beurling's condition of regularity on z_0 clearly means that any neighbourhood of z_0 contains a great deal of boundary point, whereas theorem 1 holds good even when there is a comparatively small quantity of boundary point in a certain neighbourhood, say $|z - z_0| < r$, of z_0 ; for instance it is so when the common part of C and $|z - z_0| < r$ is enumerable or of absolute harmonic measure zero, because $w = f(z)$ becomes meromorphic at $z = z_0$. On these grounds, the present writer proposed the problem, whether Beurling's inequality $B(S_{z_0}^{(D)}) < B(S_{z_0}^{(C)})$ holds good without imposing any condition of regularity on z_0 ⁽⁴⁸⁾. Recently K. Kunugui⁽⁴⁹⁾ has given an affirmative answer for this problem; he has shown that any condition of regularity on z_0 is unnecessary and given a direct elementary proof for theorem 1. In the next section § 3 some lemmas will be proposed and then in § 4 another proof for theorem 1 will be given.

(44) W. Seidel: loc. cit.

(45) J. L. Doob; loc. cit.

(46) A. Beurling: loc. cit.

(47) K. Noshiro: loc. cit.

(48) K. Noshiro: loc. cit. See remark 2, p. 230.

(49) K. Kunugui: loc. cit.

§ 3. Two lemmas used for the proof of Beurling-Kunugui's theorems.

1. We first prove the following Irie's form of Beurling's theorem :

Lemma 1. (S. Irie) *Let D be an arbitrary connected domain such that the boundary C contains a simple arc $\gamma: z_0 z_1$ joining two points z_0 and z_1 . Under the restriction imposed on z_0 , theorem 1 holds⁽⁵⁰⁾.*

Proof. Let Δ_z denote the z -plane cut along the arc γ . Then, clearly Δ_z includes the domain D . Transform Δ_z into a domain, say Δ_ξ , by the function $\xi = \frac{z-z_0}{z-z_1}$, then Δ_ξ is the ξ -plane cut along a simple curve γ_ξ , γ_ξ being the image of γ ; γ_ξ runs from $\xi=0$ to $\xi=\infty$. Next map Δ_ξ conformally upon a domain, say Δ_η , by any branch of $\eta = \sqrt{\xi}$, then Δ_η is a domain enclosed by a simple closed curve γ_η , γ_η being the image of γ_ξ and passing through $\eta=0$ and $\eta=\infty$. Finally, applying Carathéodory's theorem on conformal mapping, Δ_η can be mapped conformally on the normal domain $\Delta_{z^*}: R(z^*) > 0$ such that $\eta=0$ and $\eta=\infty$ correspond to $z^*=0$ and $z^*=\infty$; it is clear that the function $z^*(\eta)$ is bi-uniform and (spherically) bi-continuous in the closed domain $\bar{\Delta}_\eta$, where $\bar{\Delta}_\eta$ denotes the closure of Δ_η . Composing all the transformations stated above, we may define a transformation $z^*(z)$ mapping Δ_z conformally upon $\Delta_{z^*}: R(z^*) > 0$ such that z_0 corresponds to the origin $z_0^*=0$. Let D^* be the image of D transformed by $z^*(z)$. Then D^* is included in $R(z^*) > 0$ and the imaginary axis γ_{z^*} is included in the boundary C^* of D^* . Putting $f^*(z^*) \equiv f(z(z^*))$, it is seen that $S_{z_0}^{(D)} = S_{z_0}^{(D^*)}$ and $S_{z_0}^{(C)} = S_{z_0}^{(C^*)}$ where $S_{z_0}^{(D^*)}$ and $S_{z_0}^{(C^*)}$ denote the cluster sets of $w = f(z^*)$ at $z^*=0$.

Consequently, it is only necessary to prove the lemma under the particular hypothesis that D lies inside the half-plane: $R(z) > 0$ and the boundary point z_0 is identical with the origin $z=0$. Put

$$\overline{\lim}_{z \rightarrow 0} |f(z)| = M, \quad \overline{\lim}_{z' \rightarrow 0} (\overline{\lim}_{\substack{z \rightarrow z' \\ z' \neq 0}} |f(z)|) = m,$$

where z' denotes any boundary point, and let ϵ be an arbitrary positive number. We may select a positive number r , such that

$$|f(z)| < M + \epsilon \quad \text{inside } D \text{ for } 0 < |z| \leq r$$

and

$$\lim_{\substack{z \rightarrow z' \\ z' \neq 0}} |f(z)| < m + \epsilon \quad \text{for } 0 < |z'| \leq r$$

(50) S. Irie: loc. cit. He has given another elementary proof for this lemma.

simultaneously, and then we keep r fixed. Put $M + \varepsilon = \bar{M}$ and $m + \varepsilon = \bar{m}$ for simplicity. Further we may find a positive number p , such that on the semi-circle $\Gamma: |z| = r, R(z) > 0$ there holds

$$\left| \frac{1}{1+pz} \right| \leq \frac{1}{p|z|-1} = \frac{1}{pr-1} < \frac{\bar{m}}{\bar{M}},$$

and then keep p fixed. Now let us consider a function $\phi(z) = \frac{f(z)}{1+pz}$ inside the common part D_r of D and $|z| < r$. Clearly D_r is not necessarily connex and consists of a finite or an enumerable number of connected domains (components). As $|1+pz| > 1$ for $R(z) > 0$, we get

$$|\phi(z)| \leq |f(z)| < \bar{M}$$

for all points inside D_r , whence follows

$$\lim_{\substack{z \rightarrow z' \\ z \in D_r}} |\phi(z)| \leq \lim_{\substack{z \rightarrow z' \\ z \in D_r}} |f(z)| < \bar{m} \quad \text{for } 0 < |z'| \leq r$$

and the selection of p gives

$$|\phi(z)| = \left| \frac{f(z)}{1+pz} \right| < \frac{\bar{m}}{\bar{M}} \cdot \bar{M} = \bar{m}$$

on the common part Γ of the circumference $|z - z_0| = r$ and the domain D ; clearly Γ consists of a finite or an enumerable number of cross-cuts. Applying a classical theorem of Lindelöf to each component of D_r , we see that $|\phi(z)| < \bar{m}$ throughout inside D_r , namely $|f(z)| < \bar{m}|1+pz|$ throughout, inside D_r . Consequently, letting z tend to the origin, $\lim_{z \rightarrow 0} |f(z)| \leq \bar{m}$, that is, $M \leq \bar{m} = m + \varepsilon$ and then, letting ε tend to zero, we have $M \leq m$. (Q. E. D.)

Remark. For the argument used above in the proof, we owe to S. Irie, and to M. Tsuji⁽⁵¹⁾.

2. Next we shall state the other

Lemma 2. (An extension of Iversen's theorem) Let $w = f(z)$ be uniform and meromorphic in an arbitrary domain D whose boundary is denoted by C . Suppose that a is a value belonging to $S_{z_0}^{(D)} - S_{z_0}^{(C)}$ but not belonging to $R_{z_0}^{(D)}$, where z_0 is a boundary point of D . Then z_0 becomes necessarily accessible and a is an asymptotic value of $f(z)$ at z_0 ; that is, a belongs to $\Gamma_{z_0}^{(D)}$.

(51) S. Irie: loc. cit. M. Tsuji: Theory of functions, 1934, cf. p. 69.

Proof. Since a precise proof for this lemma has been published in another of the author's papers⁽⁵²⁾, here only an outline of the proof will be stated. It is needed only to consider the case where z_0 is non-isolated (then, it is clear that $S_{z_0}^{(C)} \neq 0$) and α is finite. Since α does not belong to $S_{z_0}^{(C)}$, by its definition, there exists a positive number r_1 such that α does not belong to $\sum_{0 < |z' - z_0| \leq r_1} S_{z'}^{(D)}$ where z' varies on C . Consequently α has a positive distance ρ_1 from $\sum_{0 < |z' - z_0| \leq r_1} S_{z'}^{(D)}$. By the assumption that α does not belong to $R_{z_0}^{(D)}$, we may also find another positive number r_2 such that $f(z) - \alpha$ never vanishes for $|z - z_0| \leq r_2$ inside D . Put $r = \min\left(\frac{r_1}{2}, \frac{r_2}{2}\right)$. Denote by K and k the circular domain $|z - z_0| < r$ and its circumference $|z - z_0| = r$ respectively. Then k is either contained entirely in D or the common part Γ of k and D consists of a finite or an enumerable number of cross-cuts. Consequently the common part D_r of K and D is a single connected domain in the former case and contains finite or an enumerable infinity of connected domains D_i ($i = 0, 1, 2, \dots$) in the latter. We easily see that in either case the modulus of $f(z) - \alpha$ has a positive minimum ρ_2 on the set Γ . Now put $\rho = \min\left(\frac{\rho_1}{2}, \frac{\rho_2}{2}\right)$. Thus we have obtained two positive numbers r and ρ such that the function $f(z) - \alpha$ does not vanish for $|z - z_0| \leq r$ inside D and further $\sum_{0 < |z' - z_0| \leq r} S_{z'}^{(D)}$ and $\sum_{z \in \Gamma} f(z)$ both lie outside the circle $(c): |w - \alpha| < \rho$. Remember that α is a cluster value of $f(z)$ at z_0 . Then there is a sequence z_ν ($\nu = 1, 2, \dots$) of points inside D such that $z_\nu \rightarrow z_0$ and $f(z_\nu) \rightarrow \alpha$. Accordingly we may find a point ζ_0 in D_r , whose image $w_0 = f(\zeta_0)$ lies within (c) ; suppose ζ_0 to be contained in the component D_0 of D_r and denote by $e_{\zeta_0} \equiv P(w; w_0)$ the element inverse to the expansion of $w = f(z)$ near $z = \zeta_0$. Continuing analytically the element e_{ζ_0} inside $(c_0): |w - \alpha| < |w_0 - \alpha|$ (with algebraic characters), we obtain a branch $z = \varphi_{\Delta_0}(w)$ of the inverse function of $w = f(z)$; the set of values taken by $z = \varphi_{\Delta_0}(w)$ is a connected domain Δ_0 , ζ_0 being its boundary point. Without difficulty we see that the domain Δ_0 is included in the component D_0 , that Δ_0 has exactly one common boundary point z_0 with D_0 and that the boundary of Δ_0 consists of at most an enumerable number of analytic curves γ_0 and $z = z_0$. Since $f(z) - \alpha$ is regular in the closed domain Δ_1 except at z_0 , satisfying the relation $0 < |f(z) - \alpha| < |w_0 - \alpha|$ inside Δ_0 and $|f(z) - \alpha| = |w_0 - \alpha|$ on γ_0 except $z = z_0$, we can assert, after Iversen, that there exists a simple continuous curve L contained in Δ_0 , except the end-point z_0 , along which $f(z)$ tends to ω .

(52) K. Noshiro: loc. cit. theorem 1. p. 221.

§ 4. A Proof of Beurling-Kunugui's theorem.

In this section we will give a proof for theorem 1 which is stated as follows: *Let D be an arbitrary connected domain, whose boundary is denoted by C , and let z_0 be a non-isolated boundary point. If $w = f(z)$ is a uniform function meromorphic and bounded in a certain neighbourhood of z_0 inside D , then $r(D) = r(C)$, where $r(D)$ and $r(C)$ denote the radii of the smallest closed circular domains, with centres at $w = 0$, which contain $S_{z_0}^{(D)}$ and $S_{z_0}^{(C)}$ respectively.*

It is clear that $r(D) \geq r(C)$. Accordingly, we shall show that if $r(D) > r(C)$, there would exist a contradiction. Denote by α an arbitrary point which belongs to $S_{z_0}^{(D)}$, on the circumference $|w| = r(D)$. Then α either belongs to $R_{z_0}^{(D)}$ or does not belong to $R_{z_0}^{(D)}$.

(I) First consider the case in which α belongs to $R_{z_0}^{(D)}$. Since the set of all α -points inside D is at most enumerable, we can find a positive number r such that there exists neither a multiple point⁽⁵³⁾ nor α -point of $w = f(z)$ on Γ , the common part of the circumference $|z - z_0| = r$ and the domain D being denoted by Γ , and such that $f(z)$ is bounded inside D_r where D_r denotes the common part of the domain D and the circular disc $|z - z_0| < r$. It is seen that Γ is either the complete circle $|z - z_0| = r$ or that it consists of at most an enumerable number of cross-cuts and hence that the open set D_r is not necessarily connected and may consist of a finite or an enumerable number of connected domains (components). Further notice that any boundary point of D_r is either a point z' belonging to C or a point on Γ . By the hypothesis, there are an infinity of α -points of $f(z)$ inside D_r . Let $z_1, z_2, \dots, z_\nu, \dots$ denote all the α -points inside D_r and $e_{z_1}, e_{z_2}, \dots, e_{z_\nu}, \dots$ all the corresponding inverse elements with centres at $w = \alpha$ ⁽⁵⁴⁾. It is well-known that any analytic function may contain at most an enumerable infinity of ramified elements. Consequently a ray (half straight line) L may be drawn, running from $w = \alpha$ to $w = \infty$ and lying outside the circle $|w| = r(D)$ with a single exception $w = \alpha$, such that there exists no centre of ramified element of the inverse function $z = \varphi(w)$ on L excluding the end-points $w = \alpha$ and $w = \infty$. Let us associate with every ν one analytic continuation, say $z = \varphi_\nu(w)$, selected arbitrarily, of the element e_{z_ν} along the ray L , though there are k_ν distinct analytic continuations, by the choice of the ray L , provided that e_{z_ν} is a ramified element whose order of ramifica-

(53) When the derivative $f'(z)$ vanishes at $z = b$, $z = b$ will be called a multiple point.

(54) More precisely, for each ν , e_{z_ν} denotes the element inverse to the expansion of $f(z)$ in the neighbourhood of $z = z_\nu$. For a certain ν , e_{z_ν} may be a ramified element.

tion equals to $k_\nu - 1$. As $w = f(z)$ is bounded inside the open set D_r and for any ν the continuation $z = \varphi_\nu(w)$ has always rational characters except possibly e_{z_ν} , $z = \varphi_\nu(w)$ necessarily defines a critical point ω_ν , lying at a finite distance from the origin $w = 0$, such that either ω_ν is a transcendental singularity of $z = \varphi_\nu(w)$ or the image, by $z = \varphi_\nu(w)$, of the segment $\Delta_\nu: \overline{\alpha\omega_\nu}$ is a path (simple curve) which starts from $z = z_\nu$ and determines an accessible boundary point of the open set D_r . Denote by Δ_ν^0 the segment $\overline{\alpha\omega_\nu}$ without the end-point ω_ν . Consequently, in either case the image of the segment Δ_ν^0 by $z = \varphi_\nu(w)$ is an asymptotic path (say l_ν) of $w = f(z)$ in the open set D_r , ω_ν being the asymptotic value. Denote by E_ν the cluster set of $z = \varphi_\nu(w)$ along Δ_ν^0 at $w = \omega_\nu$ ⁽⁵⁵⁾. Then, E_ν is either one point or a continuum of more than one point. It is easily seen that in either case the set E_ν for any ν , cannot contain the boundary point z_0 . For, otherwise, there would exist a sequence $z_1^{(\nu)}, z_2^{(\nu)}, \dots, z_k^{(\nu)}, \dots$ of points on the path l_ν tending to z_0 such that $\lim_{k \rightarrow \infty} f(z_k^{(\nu)}) = \omega_\nu$, as ω_ν is the asymptotic value of $w = f(z)$ along l_ν ; in other words, ω_ν would belong to the set $S_{z_0}^{(D)}$. This is a contradiction, as ω_ν lies outside of the closed domain $|w| \leq r(D)$ which includes $S_{z_0}^{(D)}$.

(i) Consider the case in which there exists at least one ν such that E_ν is a continuum of more than one point. Then ω_ν is an essential singularity defined by $z = \varphi_\nu(w)$ and E_ν never contains any interior point of D , because, if it did, $w = f(z)$ would reduce to a constant ω_ν . Accordingly, the set E_ν consists of all boundary-points z' belonging to C excluding z_0 . Whence it follows at once that ω_ν must belong to the set

$\sum_{0 < |z' - z_0| \leq r} S_{z'}^{(D)}$ and so the common part

$$(1) \quad L \cdot \sum_{0 < |z' - z_0| \leq r} S_{z'}^{(D)} \neq 0.$$

(ii) Next consider the other case in which for any ν the set E_ν contains only one point. Suppose that there is at least one ν such that l_ν defines an accessible boundary point z' (distinct from z_0), lying on C , of D_r . Hence the same argument as in (i) shows that ω_ν must belong to the set $\sum_{0 < |z' - z_0| \leq r} S_{z'}^{(D)}$ and so, in this case, the relation (1) also holds good. It remains to consider the case where for any ν the path l_ν defines an accessible boundary point ζ_ν on the set Γ , Γ denoting the common part of D and $|z - z_0| = r$. The selection of the ray L shows that if $\nu \neq \nu'$, any two paths l_ν and $l_{\nu'}$ cannot intersect with each other

(55) E_ν is the set of all values z such that $\lim_{i \rightarrow \infty} \varphi_\nu(w_i) = z$, where w_i ($i = 1, 2, \dots$) is a sequence of points on Δ_ν^0 tending to ω_ν .

for $|z - z_0| \leq r$ and so $\zeta_\nu \neq \zeta_\nu$ particularly. Thus we obtain an enumerable set of all distinct points ζ_ν such that $f(\zeta_\nu) = \omega_\nu$ ($\nu = 1, 2, \dots$) has no point of accumulation in the set Γ . If there were a point of accumulation (say ζ_0) on Γ , then an infinity of l_ν would have common points with a sufficiently small circular disc $\Delta: |z - \zeta_0| < \epsilon$ selected in such a way that $w = f(z)$ maps Δ conformally upon a convex domain⁽⁵⁶⁾. Evidently, this is a contradiction. Consequently, any point of accumulation (say z') of the sequence ζ_ν ($\nu = 1, 2, \dots$) must be a boundary point, of D_r , belonging to C and lying on $|z - z_0| = r$; whence it follows that the relation (1) also holds good, because, if a subsequence ζ_{ν_n} ($n = 1, 2, \dots$) be selected such that $\zeta_{\nu_n} \rightarrow z'$ and $\omega_{\nu_n} \rightarrow \omega'$, where $\omega_{\nu_n} = f(\zeta_{\nu_n})$ ($n = 1, 2, \dots$), then ω' belongs to $\sum_{0 < |z' - z_0| \leq r} S_{z'}^{(D)}$. Thus it is shown that in either case of (i) and (ii) we have

$$L \cdot \sum_{0 < |z' - z_0| \leq r} S_{z'}^{(D)} \neq 0.$$

Remembering that r may be taken arbitrarily small, it is seen that $L \cdot S_{z_0}^{(C)} \neq 0$. On the other hand, as the ray L lies outside the circle $|w| = r(D)$ except the end-point a , it is concluded that $L \cdot S_{z_0}^{(C)}$ contains only one point a . Thus it has been completely proved that a belongs to $S_{z_0}^{(C)}$.

II. Next consider the case where a does not belong to $R_{z_0}^{(D)}$. It will next be shown that the hypothesis $r(C) < r(D)$ leads to a contradiction. In this case, applying lemma 2 in § 3, it is seen that there is a simple arc L inside D , except the end-point z_0 , along which a is an asymptotic value of $w = f(z)$ at $z = z_0$. Let D^* be the new domain, obtained by cutting the domain D along the arc L , whose boundary is denoted by C^* . Clearly $C^* = C + L$. Now let us consider the function $w = f(z)$ in the domain D^* . Without difficulty, it is seen that

$$S_{z_0}^{(C^*)} = S_{z_0}^{(C)} + (a) \quad \text{and} \quad S_{z_0}^{(D^*)} = S_{z_0}^{(D)}.$$

Applying lemma 1 to the domain D^* , it follows

$$(2) \quad B(S_{z_0}^{(D^*)}) \subset B(S_{z_0}^{(C^*)}).$$

Since the cluster set $S_{z_0}^{(D)} L$, defined in § 1, is a continuum (closed connex set), contained in $S_{z_0}^{(D)} = S_{z_0}^{(D^*)}$ and containing a and having at least one common point with $S_{z_0}^{(C)}$, there will exist at least one point (say β) of $S_{z_0}^{(D)} L$ (and so of $S_{z_0}^{(D)}$) on a circle $|w| = \rho$ where $r(C) < \rho < r(D)$ and ρ is

(56) The selection of such a domain Δ is possible as $f'(z)$ never vanishes on Γ .

fixed. Let γ be an arbitrary point outside the circle $|w| = r(D)$; obviously γ never belongs to $S_{z_0}^{(D)} = S_{z_0}^{(D*)}$. We may find a path Δ , joining β and γ and having no common point with $S_{z_0}^{(C*)} = S_{z_0}^{(C)} + (\alpha)$. Then, on the path Δ there will exist at least one point which belongs to $B(S_{z_0}^{(D*)}) = B(S_{z_0}^{(D)})$ but does not belong to $S_{z_0}^{(C*)}$. This contradicts the relation $B(S_{z_0}^{(D*)}) \subset B(S_{z_0}^{(C*)})$. (Q. E. D.)

§ 5. Application of Beurling-Kunugui's theorems.

1. Let $w = f(z)$ be a uniform and meromorphic function in an arbitrary domain D , with a non-isolated boundary point z_0 , whose boundary is denoted by C . Then, theorem 3 shows that $B(S_{z_0}^{(D)}) \subset B(S_{z_0}^{(C)})$. Consequently, $G = S_{z_0}^{(D)} - S_{z_0}^{(C)}$ is an open set which may be empty. The complementary set Ω of $S_{z_0}^{(C)}$ with respect to the w -plane is an open set and so decomposed into a finite or an enumerable number of connected components Δ_i ($i = 1, 2, \dots$). On the components of the set Δ_i , we can assert

Theorem 4. *Let $w = f(z)$ be a uniform and meromorphic function in an arbitrary domain D , with a non-isolated boundary point z_0 , whose boundary is denoted by C . Denote by Δ any one of the components Δ_i ($1, 2, \dots$) of the complementary set Ω of $S_{z_0}^{(C)}$ with respect to the w -plane. Then either Δ is entirely included in $S_{z_0}^{(D)}$ or Δ never contains any point lying in $S_{z_0}^{(D)}$.*

Proof. Suppose that Δ contains a point α belonging to $S_{z_0}^{(D)}$. If in Δ there were a point β , not belonging to $S_{z_0}^{(D)}$, then a boundary point of $S_{z_0}^{(D)}$, (of $S_{z_0}^{(C)}$ by theorem 3), would be found on a continuous curve L , joining two points α and β inside Δ , since Δ is a connected domain. Thus, a contradiction is arrived at.

Remark. This theorem is an improved form of a result which has been obtained in another place⁽⁵⁷⁾.

It is easy to show⁽⁵⁸⁾ that if $S_{z_0}^{(D)}$ has an interior point α , in any neighbourhood of α we can find a value β taken by $w = f(z)$ an infinity of times near z_0 .

Using this fact, we have at once

Theorem 5. *Let $w = f(z)$ be uniform and meromorphic in a domain D and let z_0 be a non-isolated boundary point. Then, the set $R_{z_0}^{(D)}$ is everywhere dense in the open set $G = S_{z_0}^{(C)} - S_{z_0}^{(D)}$, provided that G is not empty.*

⁽⁵⁷⁾ K. Noshiro: loc. cit. p. 228.

⁽⁵⁸⁾ For the proof of this fact, cf. K. Noshiro: loc. cit. p. 229 or § 9, Chapter 1, in this paper, as a proof for an entirely similar result is found there.

As an alternative,

Theorem 6. *Let $w = f(z)$ be uniform and meromorphic in a domain D and let z_0 be a non-isolated boundary point. Then $S_{z_0}^{(D)} = S_{z_0}^{(C)}$, provided that $f(z)$ does not take any value an infinity of times in a certain neighbourhood of z_0 .*

Remark. Theorem 5 and theorem 6 are improved forms of results which have been obtained in another place⁽⁵⁹⁾ where we have proved theorem 6 under the assumption that $w = f(z)$ is at most p -valent near z_0 .

As a corollary of theorem 4, we may obtain a result of G. Hössjer⁽⁶⁰⁾. Suppose that $w = f(z)$ is regular and bounded in a domain D enclosed by a simple closed curve C . Divide C by two points z_0 and z_1 into two simple arcs C_1 and C_2 and suppose further that $w = f(z)$ is continuous at any point in the closed domain $\bar{D}$, excluding $z = z_0$. Denote by $S_{z_0}^{(C_1)}$ and $S_{z_0}^{(C_2)}$ the cluster sets of $w = f(z)$ at $z = z_0$ along C_1 and C_2 respectively (it is evident that any one of $S_{z_0}^{(C_1)}$ and $S_{z_0}^{(C_2)}$ consists of one point or is a continuum of more than one point). Now let E_1 and E_2 be two bounded closed Jordan domains which contain $S_{z_0}^{(C_1)}$ and $S_{z_0}^{(C_2)}$ respectively. Then, G. Hössjer's result states that the sets E_1 and E_2 have at least one point in common and moreover that the component $\mathcal{A}_\infty$, containing $w = \infty$, of the complementary set $\mathcal{Q}$ of $E = E_1 + E_2$ with respect to the w -plane never contains any point lying in $S_{z_0}^{(D)}$. The proof is easy. If E_1 and E_2 had no point in common, the complementary set $\mathcal{Q}$ would be a connected domain containing $w = \infty$ in its interior since E_1 and E_2 are bounded. However, as, in this case, $S_{z_0}^{(D)}$ is a continuum containing $S_{z_0}^{(C)} = S_{z_0}^{(C_1)} + S_{z_0}^{(C_2)}$, $\mathcal{Q}$ would contain at least one point belonging to $S_{z_0}^{(D)}$. Hence, by theorem 4, $\mathcal{Q}$ would be included in $S_{z_0}^{(D)}$. This contradicts the hypothesis that $f(z)$ is bounded in D . Consequently, we see that $E = E_1 + E_2$ is a continuum. Denoting by E^* the set obtained by adding every bounded component $\mathcal{A}$ of $\mathcal{Q}$ to $E = E_1 + E_2$, theorem 4 shows that $S_{z_0}^{(D)}$ will be included in E^* .

2. To the study of transcendental singularities of univalent multi-form analytic functions, it is possible to apply the theory of the cluster sets. Consider first the case where $w = f(z)$ is a meromorphic function, in $|z| < \infty$ with an essential singularity at $z = \infty$, the inverse function being denoted by $z = \varphi(w)$. We suppose that an analytic continuation of $z = \varphi(w)$, $P(w; w(t))$ ($0 \leq t < 1$), along a path $\mathcal{A}: w = w(t)$ ($0 \leq t < 1$), $\lim_{t \rightarrow 1} w(t) = \omega$, defines a transcendental singularity $\mathcal{Q}$ with coordinate ω .

(59) K. Noshiro: loc. cit. See theorem 7, p. 230 and theorem 2, p. 223.

(60) G. Hössjer: Bemerkung über einen Satz von E. Lindelöf. Fysiogr. Sällsk. Lunds. Förh. 6, 23-29, 1937.

Now, describe any circle $c: |w-\omega| = r$ (to fix ideas, we will suppose ω to be finite) and denote by $z = \varphi_r(w)$ the branch defined by an element $P(w; w(t_r))$ with centre $w(t_r)$ on the end-part A_r of the path A inside $(c): |w-\omega| < r$ (Cf. § 4, Chapter 1). Denoting by Δ the set of all the values taken by $z = \varphi_r(w)$, we may find in the interior of Δ at least one asymptotic path L , of $w = f(z)$, which defines an accessible boundary point $z_0 = \infty$ of Δ and along which ω becomes an asymptotic value of $w = f(z)$ at $z_0 = \infty$; for instance, one may take as L the image of the end-part A_r of the path A by the continuation $P(w; w(t))$ ($0 \leq t < 1$) of $z = \varphi(w)$. Evidently $w = f(z)$ is regular at any point in the closed domain $\bar{\Delta}$ (the closure of Δ), excluding the end-point z_0 , and satisfies $|f(z)-\omega| < r$ inside Δ and $|f(z)-\omega| = r$ on C excluding $z_0 = \infty$, where C denotes the boundary of Δ . It is easy to see that the point at infinity $z_0 = \infty$ is a non-isolated boundary point of Δ , for if it were not so, we would find a domain of the form: $\rho < |z| < +\infty$ contained entirely in Δ , and so in the neighbourhood of $z_0 = \infty$, $w = f(z)$ would not take any value lying outside $(c): |w-\omega| < r$. This contradicts a classical theorem of Weierstrass on an isolated essential singularity. Accordingly, theorem 4 can be applied to the regular function $w = f(z)$ inside Δ . Since $S_{z_0}^{(C)}$ contains only points on the circumference $c: |w-\omega| = r$ and the centre ω belongs to $S_{z_0}^{(\Delta)}$, by theorem 4, it is seen that the circular disc $(c): |w-\omega| < r$ must be contained in the set $S_{z_0}^{(\Delta)}$, so that the cluster sets $S_{z_0}^{(\Delta)}$ and $S_{z_0}^{(C)}$ coincide with the closed domain $|w-\omega| \leq r$ and its circumference $|w-\omega| = r$ respectively. Consequently it is concluded that $R_{z_0}^{(\Delta)}$ is everywhere dense in the circle $(c): |w-\omega| < r$. In other words, denoting by $n(a)$ the number of the elements, of the branch $z = \varphi_r(w)$, with a common centre a , each ramified element being counted the same number of times as the multiplicity of ramification, we can assert that the enumerative function $n(w)$ of $z = \varphi_r(w)$ becomes infinite at every point on a set, say E , which is everywhere dense in the circle $(c): |w-\omega| < r$.

Thus we have obtained the following important

Theorem 7. *Suppose that $w = f(z)$ is a function, meromorphic in $|z| < +\infty$, with an essential singularity at $z = \infty$. Denote by $z = \varphi(w)$ the inverse analytic function and by Φ the Riemann covering surface associated with $z = \varphi(w)$. Suppose moreover that $z = \varphi(w)$ has at least one transcendental singularity. We denote by Φ_r the r -neighbourhood of any accessible point Ω (not necessary to be isolated) with the projection ω , where r denotes an arbitrary positive number. Then, the set E of all the projections above any of which there are an infinity of points of the r -neighbourhood Φ_r , is everywhere dense in the basic domain $(c): |z-\omega| < r$.*

Remark 1. Theorem 7 is not new in the particular case where Ω is an isolated accessible boundary point of Φ , as is known from Iversen's investigations (Cf. his thesis, loc. cit.). It is remarkable that in the present theorem it is unnecessary for Ω to be isolated and that this theorem gives an answer for a problem proposed by A. Speiser⁽⁶¹⁾. Theorem 7 may have been obtained already but the present writer has not yet found the proof for it anywhere.

Remark 2. It may also be remarked that it is possible to prove the above theorem by using only Irie's form of Beurling's theorem. Let the same notations be preserved as in the proof which has been stated above. It is well-known that concerning the boundary C of the domain $\mathcal{A}$, two cases are distinguished: either C consists of only bounded analytic contours γ_i ($i = 1, 2, \dots$) except $z_0 = \infty$ or C contains at least one analytic curve reaching $z_0 = \infty$. If for any sufficiently small r , the boundary C always consists of bounded analytic curves γ_i ($i = 1, 2, \dots$), except $z_0 = \infty$, the number of which becomes necessarily infinite, by Weierstrass' theorem, then it is evident that the inverse function $z = \varphi(w)$ admits no transcendental singularity other than Ω . Therefore, the enumerative function $n(w)$ of the branch $\varphi_r(w)$ becomes identically a constant $+\infty$ in the domain: $0 < |w - \omega| < r$ (Cf. theorem 8, chapter 1 above), since the image of any closed curve γ_i by $w = f(z)$ covers entirely the circumference $c: |w - \omega| = r$ completely. In the other case when C contains at least one analytic curve reaching $z_0 = \infty$, theorem 4 can be obtained immediately, by applying Irie's form of Beurling's theorem, and so it is easily seen that theorem 7 holds good, as is shown in the proof above.

Next we shall extend theorem 7 in the following way:

Theorem 8. *Let $w = f(z)$ be uniform and meromorphic in an arbitrary domain D , the boundary being denoted by C , and let z_0 be a non-isolated boundary point. Assume that inside D there exists an asymptotic path L (of $w = f(z)$) converging to the boundary point z_0 , such that $w = f(z)$ converges to a value ω belonging to the open set $G = S_{z_0}^{(D)} - S_{z_0}^{(C)}$ along the path L ; for instance, this assumption may be satisfied provided that ω is a value belonging to the set $G = S_{z_0}^{(D)} - S_{z_0}^{(C)}$ but not belonging to $R_{z_0}^{(D)}$ ⁽⁶²⁾. Under these conditions, it may be asserted that the analytic continuation of the inverse analytic function $z = \varphi(w)$, which corresponds to the asymptotic path L of $w = f(z)$, defines a transcendental singularity, say Ω , above the point ω (by theorem 4, chapter 1) and moreover that the enumerative function $n(z)$ of the branch $z = \varphi_r(w)$, corresponding to the*

⁽⁶¹⁾ A. Speiser: *Commentarii Mathematici Helvetici*, vol. 1 (1929), pp 289-312; see particularly p. 290.

⁽⁶²⁾ Cf. lemma 2, § 3.

r -neighbourhood Φ_r of the accessible boundary point Ω of the Riemann surface Φ associated with $z = \varphi(w)$, becomes infinite at every point of a set which is everywhere dense inside the circle: $|w - \omega| < r$, provided that r is sufficiently small.

Proof. By the hypothesis, we can find a positive number ρ , which may be taken to be arbitrarily small, such that the point ω has a positive distance, say r_1 , from the set $\sum_{0 < |z' - z_0| \leq \rho} S_{z'}^{(D)}$ where z' varies on the boundary C , and moreover such that $f(z) - \omega$ never vanishes on the common part Γ of the domain D and the circumference $|z - z_0| = \rho$. Denote by r_2 the positive lower bound of the modulus of the function $f(z) - \omega$ on Γ and put $2r_0 = \min(r_1, r_2)$. Now let us consider the branch $z = \varphi_r(w)$, stated in the theorem, of the inverse function $z = \varphi(w)$, where r denotes an arbitrary positive number less than r_0 , the set of values taken by the branch being denoted by Δ_r . The argument often used shows that the connected domain Δ_r is included inside the open set D_ρ which denotes the common part of the domain D and the circular disc $|z - z_0| < \rho$, and that the boundary C_r of Δ_r has only one common point z_0 with the boundary of D_ρ ; hence, any boundary point of Δ_r distinct from z_0 lies inside D_ρ . Here, we can apply the same argument as in the proof of theorem 7 to the function $w = f(z)$ which is regular in the closed domain $\bar{\Delta}_r$ except at the boundary point z_0 , since $f(z)$ satisfies that $|f(z) - \omega| < r$ inside Δ_r and $|f(z) - \omega| = r$ on the boundary C_r excluding z_0 and moreover ω belongs to the cluster set $S_{z_0}^{(\Delta_r)}$. Consequently, using theorem 4, it is seen that $S_{z_0}^{(\Delta_r)}$ coincides with the closed circular disc $|w - \omega| \leq r$. Whence follows at once the above assertion. (Q. E. D.)

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The present writer wishes to add the following section to the above paper.

§ 6. On the behaviour of a meromorphic function in a peninsula.

1. Suppose that $w = f(z)$ is a meromorphic function for $|z| < +\infty$ with an essential singularity at $z = \infty$. Let Ω be any transcendental singularity with coordinate ω of the inverse function $z = \varphi(w)$ supposed to have at least one transcendental singularity. If we denote by Δ_ρ the

set of all the values taken by the branch $z = \varphi_\rho(w)$ corresponding to the ρ -neighbourhood Φ_ρ (on Φ) of the accessible boundary point Ω where Φ denotes the Riemann covering surface associated with the inverse function $z = \varphi(w)$, then we have already seen that the range of values $R_{z_0}^{(\Delta_\rho)}$, letting $z_0 = \infty$, of $w = f(z)$ becomes everywhere dense in the circle (c) ; $|w - \omega| < \rho$ for all sufficiently small ρ (Theorem 7)(¹). The object of this section is to investigate the behaviour of $w = f(z)$ more precisely as z tends to $z_0 = \infty$ inside Δ_ρ . It has often been stated that the boundary C_ρ of the peninsula Δ_ρ consists of at most an enumerable number of analytic contours, bounded or not, and the point at infinity $z_0 = \infty$ and that $f(z)$ is regular in the closed domain $\bar{\Delta}_\rho$ excluding at $z_0 = \infty$ and satisfies the relation: $|f(z) - \omega| < \rho$ inside Δ_ρ and $|f(z) - \omega| = \rho$ on C_ρ , excluding $z_0 = \infty$. In the present case, we notice that there exists an asymptotic path say Γ lying inside Δ_ρ and reaching $z_0 = \infty$, along which $w = f(z)$ tends to ω as z tends to $z_0 = \infty$. Now, let r_0 be a fixed positive number such that the circle $K_{r_0}: |z| = r_0$ contains the end-point ζ_0 , at a finite distance, of Γ and at least one boundary point of Δ_ρ in its interior. For $r \geq r_0$, let Θ_r denote the common part of $K_r: |z| = r$ and the domains Δ_ρ ; it is easily seen that Θ_r consists of a finite number of cross-cuts(²), except the case in which K_r is entirely contained in Δ_ρ . In either case, let $A(r)$ denote the area of the Riemannian image of the open set G_r by the function $w = f(z)$ where G_r means the common part of Δ_ρ and the interior of K_r . Here it will be proved that $\lim_{r \rightarrow +\infty} A(r) = +\infty$. In the case where for all sufficiently small ρ , the boundary C_ρ , excluding the point $z_0 = \infty$, consists of only bounded analytic contours, the assertion is evident, because $z = \varphi(w)$ has no transcendental singularity other than Ω . Therefore, let us consider the case in which C_ρ contains at least one analytic curve running from $z = \infty$ to $z = \infty$. Denote by Γ_r the part of the asymptotic path Γ which runs from the last point of intersection $\zeta(r)$ of Γ with K_r , counting from ζ_0 , to the infinity $z_0 = \infty$. Clearly there is a positive number $r_1 (> r_0)$ with the following properties:

- i) the image of the curve Γ_{r_1} by $w = f(z)$ lies entirely within the concentric circle $(c_1): |w - \omega| < \frac{\rho}{2}$,
- ii) for any r greater than r_1 , Θ_r contains a cross-cut, say $\gamma(r)$, such

(1) In the case when the projection ω of the transcendental singularity is infinite, the statement must be modified in the familiar way.

(2) If Θ_r contains an infinity of cross-cuts, the set formed by all the end-points of these cross-cuts has at least one point of accumulation (say ζ^*) on $K_r: |z| = r$. On the other hand, since $w = f(z)$ is regular at $z = \zeta^*$ and satisfies that $|f(\zeta^*) - \omega| = \rho$, we can find a small neighbourhood of ζ^* containing only a finite number of arcs whose images by means of $w = f(z)$ are circular arcs on $c: |w - \omega| = \rho$.

that $\gamma(r)$ contains the point $\zeta(r)$ in its interior and the end-points $\zeta'(r)$, $\zeta''(r)$ of $\gamma(r)$ lie on C_ρ . Notice that the image of the circular arc $\gamma(r)$ is a curve which starts from a point on c : $|w-\omega|=\rho$, passing through a point lying on (c_1) : $|w-\omega|<\frac{\rho}{2}$ and then reaches a point on c : $|w-\omega|=\rho$, whence it follows that the image of $\gamma(r)$ is of length at least equal to ρ . Now, denote by $r\theta(r)$ and $L(r)$ the total length of θ_r and that of the image of θ_r by $w=f(z)$ respectively. Then, it is evident that

$$L(r) = \int_{\theta_r} |f'(z)| r d\theta \quad (z = re^{i\theta})$$

Applying Schwarz's inequality

$$[L(r)]^2 \leq \int_{\theta_r} r d\theta \cdot \int_{\theta_r} |f'(z)|^2 r d\theta = r\theta(r) \cdot \int_{\theta_r} |f'(z)|^2 r d\theta,$$

from which follows

$$\frac{[L(r)]^2}{r\theta(r)} \leq \int_{\theta_r} |f'(z)|^2 r d\theta,$$

hence

$$(1) \quad \int_{r_1}^r \frac{[L(r)]^2}{r\theta(r)} dr \leq \int_{r_1}^r \int_{\theta_r} |f'(z)|^2 r dr d\theta = A(r) - A(r_1).$$

As $r\theta(r) \leq 2\pi r$ and $L(r) \geq \rho$ for $r_1 \leq r < +\infty$, we have

$$\int_{r_1}^r \frac{[L(r)]^2}{r\theta(r)} dr \geq \frac{\rho^2}{2\pi} \int_{r_1}^r d \log r = \frac{\rho^2}{2\pi} \log \frac{r}{r_1},$$

whence it follows that $\lim_{r \rightarrow +\infty} A(r) = +\infty$.

2. Hereafter we will impose another somewhat large restriction upon the peninsula $\mathcal{A}_\rho$. Let us assume that $\mathcal{A}_\rho$ is a *simply connected domain*—for instance, if $w=f(z)$ is a transcendental integral function, any peninsula $\mathcal{A}_\rho$, which corresponds to a transcendental singularity Ω with the projection ω at a finite distance from the origin $w=0$, is necessarily simply connected.⁽³⁾ Now, with the aid of the brilliant investiga-

⁽³⁾ It is clear that if Ω is a logarithmic singularity, the corresponding peninsula $\mathcal{A}_\rho$ is simply connected, for $\mathcal{A}_\rho$ is enclosed by one simple closed analytic curve passing through $z_0 = \infty$. However, this is a trivial case for the present purpose.

tion of Ahlfors⁽⁴⁾ on covering surfaces, we shall obtain a result on the behaviour of $w = f(z)$, as z tends to $z_0 = \infty$, inside the simply connected peninsula Δ_p . If $r \geq r_0$, the open set G_r consists of a certain number of simply connected components which we will denote by

$$\Delta^{(1)}(r), \Delta^{(2)}(r), \dots, \Delta^{(i)}(r), \dots, \Delta^{(m)}(r),$$

where $m = m(r)$ ($m \geq 1$) depends on r . The boundary of each component $\Delta^{(i)}(r)$ ($i = 1, 2, \dots, m$) consists of at least one analytic arc $C_i^{(\lambda)}$ ($\lambda = 1, 2, \dots, k; k \geq 1$) included in the boundary C_p and at least one circular arc $\gamma_i^{(\mu)}$ ($\mu = 1, 2, \dots, h; h \geq 1$) lying on the circle $K_r: |z| = r$. It is clear that $w = f(z)$ transforms any $C_i^{(\lambda)}$ into a curve lying on $c: |w - \omega| = \rho$ and any $\gamma_i^{(\mu)}$ into a curve lying inside $(c): |w - \omega| < \rho$ with the end-points on c . Next denote by $\phi^{(i)}(r)$ the Riemannian image of $\Delta^{(i)}(r)$ transformed by $w = f(z)$ in a one-one manner, where $i = 1, 2, \dots, m$.

Denote by F any one of $\phi^{(i)}(r)$ ($i = 1, 2, \dots, m$). Then, F is a covering surface of the basic surface $F_0: |w - \omega| < \rho$ (F_0 is nothing but the circle $(c): |w - \omega| < \rho$). Let $D_1, D_2, \dots, D_q$ ($q \geq 1$) be q closed circular discs such that each lies entirely inside F_0 and no two of them have any point in common and let $\bar{F}_0$ be the domain obtained by excluding all the discs $D_1, D_2, \dots, D_q$ from $F_0: |w - \omega| < \rho$. For simplicity, let D denote any one of $D_1, D_2, \dots, D_q$. The part of F which lies above D consists of a certain number of connected pieces which are of two types. Those which are unbounded are referred to as islands, whereas the remaining pieces will be called peninsulas. For each island D^i , the negative sign of its Euler characteristic $\rho(D^i)$ of D^i will be called the simple multiplicity of D^i ; the simple multiplicity of D^i is equal to $+1$ in the case where D^i is simply connected and is less than or equal to zero in the other cases. Let $p(D)$ denote the sum of the simple multiplicities of the islands D^i of F above the disc D . Here let Ahlfors' method be applied with a slight modification (Cf. R. Nevanlinna: loc. cit., pp. 333-335). After Ahlfors, we first exclude all the peninsulas above $D_1, D_2, \dots, D_q$ from F and, then, we shall have a certain number, say $N(F')$, of simply connected surfaces F' . Next, from every F' exclude all islands D^i above $D_1, D_2, \dots, D_q$, then a certain number of covering surface $\bar{F}$ of the basic surface $\bar{F}_0$ will be obtained. As any F' decomposed into the islands D^i and the remaining domains $\bar{F}$ by loop-cuts (Rückkehrschnitte) only, we see that

(4) L. Ahlfors: Zur Theorie der Überlagerungsflächen. Acta Math., Bd. 65, 1935, or R. Nevanlinna: Eindeutige analytischen Funktionen, 1936, pp. 312-345.; See, particularly pp. 333-335.

$$(2) \quad \sum \rho(F') = \sum \rho(\bar{F}) + \sum \rho(D^i)$$

where $\rho(F')$ and $\rho(\bar{F})$ denote their Euler characteristics of F' and $\bar{F}$ respectively. Denoting by $N_1(\bar{F})$ the number of the simply connected $\bar{F}$ and using evident equalities: $N(F') = -\sum \rho(F')$ and $\sum \rho^+(\bar{F}) - N_1(\bar{F}) = \sum \rho(\bar{F})$, we may write (2) in the form

$$(3) \quad \sum_{j=1}^q p(D_j) = \sum \rho^+(\bar{F}) + N'(F'),$$

where $N'(F') = N(F') - N_1(\bar{F})$ denotes the number of F' any of which contains at least one island D^i . Apply Ahlfors' principal theorem⁽⁵⁾ to each covering surface $\bar{F}$ of the basic surface $\bar{F}_0$. Then we have at once, considering that $\rho(\bar{F}_0)$ is equal to $q-1$ in the present case,

$$(4) \quad \sum \rho^+(\bar{F}) \geq (q-1)S(\bar{F}_0) - h' \sum L(\bar{F}),$$

where $S(\bar{F}_0)$ denotes the average number of sheets of all $\bar{F}$; that is, $S(\bar{F}_0)$ denotes the ratio between the sum of areas of all $\bar{F}$ and the area of $\bar{F}_0$, h' being a constant dependent only upon $D_1, D_2, \dots, D_q$, and $L(\bar{F})$ the length of the boundary of $\bar{F}$ relative to $\bar{F}_0$. Let L be the length of the boundary of F relative to F_0 . Then it is evident that $\sum L(\bar{F}) \leq L(F) = L$. Accordingly, (4) is put in the form: $\sum \rho^+(\bar{F}) \geq (q-1)S(\bar{F}_0) - h'L$. Next, Ahlfors' first covering theorem gives $|S(\bar{F}_0) - S(F_0)| \leq h''L$, where $S(F_0)$ denotes the average number of sheets of F ; that is, $S(F_0)$ denotes the ratio between the area of F and that of F_0 , a constant h'' depending only upon $D_1, D_2, \dots, D_q$. Therefore, we have $\sum \rho^+(F) \geq (q-1)S(F_0) - hL$ and so, from (3), the following important formula

$$(5) \quad \sum_{j=1}^q p(D_j) \geq (q-1)S(F_0) - hL,$$

where h is a constant depending only upon $D_1, D_2, \dots, D_q$. Denote by $n_1(D)$ the number of orders of the branch-points of the islands D^i above D , then Hurwitz's relation shows that $p(D) = n(D) - n_1(D)$, where $n(D)$ denotes the average number of the sheets of all islands D^i above D . Thus we obtain

$$(6) \quad \sum_{j=1}^q n(D_j) - \sum_{j=1}^q n_1(D_j) \geq (q-1)S - hL.$$

Of course, this is a modified form of one of Ahlfors' formulas and is analogous to the second fundamental theorem of R. Nevanlinna. Now let us remark that F denotes any one of $\phi^{(i)}(r)$ ($i = 1, 2, \dots, m$). Then, writing the formula (6) for each i , we get at once

(5) R. Nevalinna: Loc. cit., p. 323.

$$\sum_{j=1}^q n^{(i)}(D_j) - \sum_{j=1}^q n_1^{(i)}(D_j) \geq (q-1)S^{(i)} - h \cdot L^{(i)} \quad (i = 1, 2, \dots, m),$$

whence it follows,

$$(7) \quad \sum_{i=1}^m \sum_{j=1}^q n^{(i)}(D_j) - \sum_{i=1}^m \sum_{j=1}^q n_1^{(i)}(D_j) \geq (q-1) \sum_{i=1}^m S^{(i)} - h \sum_{i=1}^m L^{(i)}$$

where h is a constant only depending on $D_1, D_2, \dots, D_q$. Here let us introduce some new notations: we put

$$\begin{aligned} (i) \quad & \sum_{i=1}^{m(r)} n^{(i)}(D_j) = n(r, D_j, \mathcal{A}_\rho), \\ (ii) \quad & \sum_{i=1}^{m(r)} n_1^{(i)}(D_j) = n_1(r, D_j, \mathcal{A}_\rho), \\ (iii) \quad & \sum_{i=1}^{m(r)} S^{(i)} = S(r, \mathcal{A}_\rho), \\ (iv) \quad & \sum_{i=1}^{m(r)} L^{(i)} = L(r, \mathcal{A}_\rho). \end{aligned}$$

If we denote by $\Phi^*(r)$ the Riemannian image of the open set G_r transformed by $w = f(z)$, then $\Phi^*(r)$ consists of covering surfaces $\Phi^{(i)}(r)$ of (c) : $|w - \omega| < \rho$, where $i = 1, 2, \dots, m(r)$, $m(r) \geq 1$. The meanings of the above notations will be enunciated as follows: $n(r, D_j, \mathcal{A}_\rho)$ denotes the average number of sheets of the islands of $\Phi^*(r)$ above D_j ; $n_1(r, D_j, \mathcal{A}_\rho)$ the total number of orders of the branch-points of the islands of $\Phi^*(r)$ above D_j ; $S(r, \mathcal{A}_\rho)$ the average number of sheets of $\Phi^*(r)$ relative to (c) : $|w - \omega| < \rho$; $L(r, \mathcal{A}_\rho)$ the length of the boundary of $\Phi^*(r)$ relative to (c) : $|w - \omega| < \rho$. Consequently

$$S(r, \mathcal{A}_\rho) = \frac{A(r)}{\pi \rho^2} \iint_{G_r} |f'(z)|^2 r dr d\theta,$$

$$L(r, \mathcal{A}_\rho) = L(r) = \int_{\partial_r} |f'(z)| r d\theta.$$

By using four notations, the formula (7) takes the form:

$$\sum_{j=1}^q n(r, D_j, \mathcal{A}_\rho) - \sum_{j=1}^q n_1(r, D_j, \mathcal{A}_\rho) \geq (q-1)S(r, \mathcal{A}_\rho) - hL(r, \mathcal{A}_\rho)$$

or

$$(8) \quad \sum_{j=1}^q \left[S(r, \mathcal{A}_\rho) - n(r, D_j, \mathcal{A}_\rho) \right] + \sum_{j=1}^q n_1(r, D_j, \mathcal{A}_\rho) \leq S(r, \mathcal{A}_\rho) + hL(r, \mathcal{A}_\rho)$$

where h is a constant depending only upon $D_1, D_2, \dots, D_q$ ($q \geq 1$). Remember that it has already been shown that $\lim_{r \rightarrow +\infty} S(r, \mathcal{A}_\rho) = +\infty$. Let

the defect and the ramification index of D_j relative to A_ρ be defined in the following way:

$$\delta(D_j, A_\rho) = \lim_{r \rightarrow +\infty} \left[1 - \frac{n(r, D_j, A_\rho)}{S(r, A_\rho)} \right] = 1 - \overline{\lim}_{r \rightarrow +\infty} \frac{n(r, D_j, A_\rho)}{S(r, A_\rho)} \quad (^6)$$

$$\vartheta(D_j, A_\rho) = \lim_{r \rightarrow +\infty} \frac{n_1(r, D_j, A_\rho)}{S(r, A_\rho)}.$$

Now, consider the ratio between $L(r, A_\rho) = L(r)$ and $S(r, A_\rho) = \frac{A(r)}{\pi\rho^2}$. From inequality (1) follows

$$\frac{dr}{r\theta(r)} \leq \frac{dA(r)}{[L(r)]^2},$$

hence, on denoting by E_r the set of all r such that

$$L(r) \geq A(r)^{\frac{1}{2}+\varepsilon}, \quad (\varepsilon > 0)$$

it is seen that

$$\frac{1}{2\pi} \int_{E_r} d \log r \leq \int_{E_r} \frac{dr}{r\theta(r)} \leq \int_{E_r} \frac{dA(r)}{[A(r)^{\frac{1}{2}+\varepsilon}]^2} \leq \int \frac{dt}{t^{1+2\varepsilon}} < +\infty,$$

whence it follows that $L(r) < A(r)^{\frac{1}{2}+\varepsilon}$ for all r not belonging to a set E_r where $\int_{E_r} d \log r < +\infty$. Thus we see that there is a sequence r_ν ($\nu = 1, 2, \dots$) tending to infinity such that

$$\lim_{\nu \rightarrow \infty} \frac{L(r_\nu, A_\rho)}{S(r_\nu, A_\rho)} = 0,$$

whence follows at once, by (8),

$$(I) \quad \sum_{j=1}^q \delta(D_j, A_\rho) + \sum_{j=1}^q \vartheta(D_j, A_\rho) \leq 1.$$

There is another important consequence from the formula (5). Let D be a closed circular disc lying inside F_0 : $|w - \omega| < \rho$. It may be said that the covering surface ϕ_ρ ramifies at least μ -ply above D , provided that every simply connected island of ϕ_ρ above D has at least μ sheets. It is evident that $S(D) \geq n(D) \geq \mu p(D)$, where $S(D)$ denotes the average number of the part of F which lies above D , because only simply connected islands of F above D give positive values to the quantity $p(D)$. From (5), is deduced the inequality

(⁶) It is easily seen that $0 \leq \delta(D_j, A_\rho) \leq 1$ ($j = 1, 2, \dots, q$), using the first covering theorem of Ahlfors. Cf. R. Nevalinna: Loc. cit., p. 332 formula (A).

$$\sum_{j=1}^q \frac{1}{\mu_j} S(D_j) \geq \sum_{j=1}^q p(D_j) \geq (q-1)S(F_0) - hL$$

on the other hand,

$$S(D_j) \leq S(F_0) + h_j L \quad (j = 1, 2, \dots, q)$$

by the first covering theorem of Ahlfors, whence follows

$$(9) \quad \sum_{j=1}^q \left(1 - \frac{1}{\mu_j}\right) S(F_0) \leq S(F_0) + h_0 L$$

where a constant h_0 depends only upon $D_1, D_2, \dots, D_q$. Taking as F every $\phi^{(i)}(r)$ ($i = 1, 2, \dots, m = m(r)$), we obtain

$$\sum_{j=1}^q \left(1 - \frac{1}{\mu_j}\right) S^{(i)} \leq S^{(i)} + h_0 \cdot L^{(i)} \quad (i = 1, 2, \dots, m(r)),$$

hence,

$$\sum_{j=1}^q \left(1 - \frac{1}{\mu_j}\right) S(r, \Delta_\rho) \leq S(r, \Delta_\rho) + h_0 \cdot L(r, \Delta_\rho)$$

where h_0 denotes a constant depending upon $D_1, D_2, \dots, D_q$. Consequently, the following important formula is obtained

$$(II) \quad \sum_{j=1}^q \left(1 - \frac{1}{\mu_j}\right) \leq 1$$

These results may be summarized as follows:

Theorem 9. *Let $w = f(z)$ be a meromorphic function for $|z| < +\infty$ with an essential singularity at $z = \infty$ and suppose that the inverse function $z = \varphi(w)$ has a transcendental singularity, say Ω , with the projection ω . Denote by Δ_ρ the set of values taken by the branch $z = \varphi_\rho(w)$ of $z = \varphi(w)$ which will define the ρ -neighbourhood of an accessible boundary point Ω of the Riemann covering surface Φ associated with $z = \varphi(w)$. We suppose, in addition, that the domain Δ_ρ is simply connected. Then follows the two inequalities:*

$$(I) \quad \sum_{j=1}^q \delta(D_j, \Delta_\rho) + \sum_{j=1}^q \vartheta(D_j, \Delta_\rho) \leq 1$$

where D_j ($j = 1, 2, \dots, q; q \geq 1$) are closed circular discs lying inside (c): $|w - \omega| < \rho$, such that no two of them have any point in common, and δ and ϑ denote the defects and the ramification indices, and

$$(II) \quad \sum_{j=1}^q \left(1 - \frac{1}{\mu_j}\right) \leq 1,$$

provided that the ρ -neighbourhood Φ_ρ ramifies at least μ_j -ply above D_j ($j = 1, 2, \dots, q$). Moreover the inequality (II) holds good, if we put $\mu_j = +\infty$ in the case where there exists no simply connected island of Φ_ρ above D_j for any j .

As a consequence of the inequality (I), the enumerative function $n(w)$ of the branch $z = \varphi_\rho(w)$ becomes infinite at every point with one possible exception inside (c): $|w - \omega| < \rho$.

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