27. Theorems on interval-functions and h -measures.

By

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I. The object of the present note is to see that the theorem of S. Saks and A. Zygmund,⁽¹⁾ an extension of the theorem of A. S. Besicovitch,⁽²⁾ can be extended as far as to be stated in terms of so called h -measures, which are more general than α -dimensional measures with which their theorem is concerning.

Their methods will be also used effectively with a slight modification.

II. Let us begin with some definitions and notations. Given a set E of points in Euclidean plane R^2 , let us denote the diameter and the Lebesgue measure of E , as usual, by $\delta(E)$ and $|E|$ respectively.

Let $h(t)$ be an increasing and continuous function with $h(0) = 0$ defined for $t \geq 0$ near the origin.

We shall denote, for each $\varepsilon > 0$, by $L_\varepsilon^h(E)$ the lower bound of all the sums $\sum_i h[\delta(E_i)]$ where $\{E_i\}_{i=1,2,\dots}$ is an arbitrary partition of E into a sequence of sets that have diameters less than ε and no two of which have common points. Making ε approach to 0, the number $L_\varepsilon^h(E)$ tends, in a monotone non-decreasing way, to a unique limit (finite or infinite) which is called the h -measure of E and will be denoted by $L^h(E)$.

$L^h(E)$ is known to possess the property of Carathéodory's outer measure⁽³⁾ and therefore all the Borel sets (B) are measurable in the sense of h -measure.

(1) S. Saks and A. Zygmund, On functions of rectangles and their application to the analytic functions, Ann. Scuola Norm. Sup. Pisa, Vol. 3 (1934), pp. 1-6.

(2) A. S. Besicovitch, On sufficient conditions for a function to be analytic, and on behaviour of analytic functions in the neighbourhood of non-isolated singular points, Proc. London Math. Soc. (2), Vol. 32 (1931), pp. 1-9.

(3) F. Hausdorff, Dimension und äusseres Mass, Math. Ann., Vol. 79 (1919), pp. 157-179.

If we put $h = t^\alpha$ ($\alpha > 0$) or $1/(\log t^{-1})$, then $L^h(E)$ represents the so-called α -dimensional or the logarithmic measure.

We shall consider only intervals (rectangles) and squares that are closed and have sides parallel to the coordinate-axes. Let $F(I)$ be an additive and continuous function defined for all the intervals I in R^2 or in an open set. Let Q be any square containing a point x and let us write:

$$\begin{aligned}\overline{F}(x) &= \limsup_{\delta(Q) \rightarrow 0} \frac{F(Q)}{|Q|}, & \underline{F}(x) &= \liminf_{\delta(Q) \rightarrow 0} \frac{F(Q)}{|Q|}, \\ \overline{F}_h(x) &= \limsup_{\delta(Q) \rightarrow 0} \frac{F(Q)}{h[\delta(Q)]}, & \underline{F}_h(x) &= \liminf_{\delta(Q) \rightarrow 0} \frac{F(Q)}{h[\delta(Q)]}.\end{aligned}$$

If $\overline{F}(x) = \underline{F}(x)$, then the common value is denoted, as usual, by $F'(x)$ and is called the derivative of $F(I)$ at x .

For each integer n , we shall mean by the n -th net $\mathfrak{N}_n$ on R^2 , the sequence of all the squares into which R^2 is divided by the two systems of parallel lines

$$x = k/2^n, \quad y = k'/2^n$$

where k and k' are arbitrary integers. Then the diameter of every square belonging to $\mathfrak{N}_n$ is $\sqrt{2}/2^n$ which will be denoted by $\Delta(\mathfrak{N}_n)$. Thus, we have $\Delta(\mathfrak{N}_n) = \frac{1}{2}\Delta(\mathfrak{N}_{n-1})$.

III. It is the following simple lemma that makes us possible to generalize the result of Saks and Zygmund in an analogous way. The lemma itself is a generalization of their corresponding lemma, though the method of the proof needs a delicate change.

Lemma 1. *Given a set E with finite h -measure in R^2 and a positive integer N , there exists for each positive ε , a sequence $\{Q_n\}$ of squares belonging to the nets $\mathfrak{N}_n$ for $n \geq N$, which satisfies the following conditions:*

$$(1) \quad \sum_n h[\delta(Q_n)] \leq 16(L^h(E) + \varepsilon),$$

(2) *to each point x of E , there corresponds a positive integer $n(\geq N)$ such that all the (there are at most 4) squares of the net $\mathfrak{N}_n$ that contain the point x belong to the sequence $\{Q_n\}$.*

Proof. Let us begin with some preliminary considerations. Given a set A with positive diameter $\leq \Delta(\mathfrak{N}_1)$, let m be the smallest of all the integers n such that $\Delta(\mathfrak{N}_n) \leq \delta(A)$. Then $\Delta(\mathfrak{N}_m) \leq \delta(A) < \Delta(\mathfrak{N}_{m-1}) = 2\Delta(\mathfrak{N}_m)$, from which follows:

$$(3) \quad \frac{1}{2}\delta(A) < \Delta(\mathfrak{N}_m).$$

We shall now see that there exist at most 16 different squares of $\mathfrak{N}_m$ that have points in common with A .

Denoting by A_x the projection of the set A on the axis of x , from the obvious relation $\delta(A) \geq \delta(A_x)$ and (3), we have

$$\frac{1}{3} \delta(A_x) < \frac{1}{2\sqrt{2}} \delta(A_x) < \frac{1}{\sqrt{2}} \mathcal{A}(\mathfrak{N}_m) = \frac{1}{2^m}$$

or
$$\delta(A_x) < \frac{3}{2^m}.$$

This shows evidently that there exist at most 4 different (consecutive) intervals on the x -axis of the type $\left[\frac{i}{2^m}, \frac{i+1}{2^m}\right]$, that have points in common with A_x . As to the projection of A on the axis of y , the same as above holds by symmetry.

Thus we have proved that at most $4^2 = 16$ different squares of $\mathfrak{N}_m$ can have points in common with A .

Let us now show the validity of the present lemma. From the definition of $L^h(E)$, there exists a sequence $\{E_i\}_{i=1,2,\dots}$ of sets that covers E and such that

$$(4) \quad \sum_i h[\delta(E_i)] \leq L^h(E) + \epsilon, \quad \text{where } 0 < \delta(E_i) < 1/2^N$$

for $i = 1, 2, \dots$. Denoting by n_i , for each $i = 1, 2, \dots$, the smallest of the integers n such that $\mathcal{A}(\mathfrak{N}_n) \leq \delta(E_i)$, we have

$$(5) \quad \mathcal{A}(\mathfrak{N}_{n_i}) \leq \delta(E_i).$$

From the obvious relation $2^{-n_i} = 2^{-\frac{1}{2}} \mathcal{A}(\mathfrak{N}_{n_i}) \leq 2^{-\frac{1}{2}} \delta(E_i) < 2^{-\frac{1}{2}-N} < 2^{-N}$, we see $n_i > N$.

By the preliminary considerations, there exist at most 16 squares of $\mathfrak{N}_{n_i}$ that have points in common with E_i . Let $\{Q_n\}_{n=1,2,\dots}$ be the sequence of all the squares belonging to the nets $\mathfrak{N}_{n_1}, \mathfrak{N}_{n_2}, \dots, \mathfrak{N}_{n_i}, \dots$ and having points in common with the sets $E_1, E_2, \dots, E_i, \dots$ respectively. It is clear that the sequence $\{Q_n\}$ satisfies the condition (2). From (5) and (4), it follows that

$$\begin{aligned} \sum_n h[\delta(Q_n)] &\leq 16 \sum_i h[\mathcal{A}(\mathfrak{N}_{n_i})] \leq 16 \sum_i h[\delta(E_i)] \\ &\leq 16(L^h(E) + \epsilon), \end{aligned}$$

which gives the condition (1) and this completes the proof.

Lemma 2. *If an additive and continuous function $F(I)$ fulfils the following conditions:*

(6) $F_h(x) \geq 0$ at every point x of the interval I_0 ,

(7) $F(x) \geq 0$ everywhere in I_0 except, perhaps, at the points belonging to a set E expressible as the sum of an enumerable infinity of sets of finite h -measure, then it follows

$$F(I_0) \geq 0.$$

Proof. Since $F(I)$ is continuous, we may assume, without loss of generality, that I_0 is a square belonging to a net whose order will be denoted by N_0 .

Let $G(I) = F(I) + \varepsilon|I|$, ε being an arbitrarily given positive number. By (7), we may write $E = \sum_i E_i$, where $L^h(E_i) < \infty$ ($i = 1, 2, \dots$).

We shall denote by $R_{i,n}$ the set of all the points x in I_0 such that

$$(8) \quad G(Q) > -\varepsilon \cdot \frac{1}{2^i(L^h(E_i) + 1)} \cdot h[\delta(Q)]$$

for every square Q containing x and belonging to one of the nets $\mathfrak{N}_k$ for $k \geq n$. F and h being continuous, the sets $R_{i,n}$ are evidently measurable (B).

From the condition (6), we have $I_0 = \sum_{n=1}^{\infty} R_{i,n}$ ($i = 1, 2, \dots$) and, by the measurability of $R_{i,n}$,

$$(9) \quad L^h(E_i) = \sum_{n=1}^{\infty} L^h(E_{i,n}),$$

where $E_{i,n} = E_i(R_{i,n} - R_{i,n-1})$ for $n > 1$ and $E_{i,1} = E_i \cdot R_{i,1}$ ($i = 1, 2, \dots$). According to the preceding lemma, we see that there exists for each pair of positive integers i and n a sequence $\{Q_{i,n}^{(j)}\}_{j=1,2,\dots}$ of squares in I_0 which belong to the nets $\mathfrak{N}_k$ for $k \geq n$, and satisfy the following conditions;

$$(10) \quad \sum_j h[\delta(Q_{i,n}^{(j)})] \leq 16(L^h(E_{i,n}) + 2^{-n}),$$

(11) to each point $x \in E_{i,n}$ there corresponds an integer $k \geq N_0$ such that each square of the net $\mathfrak{N}_k$, containing x , belongs to the sequence $\{Q_{i,n}^{(j)}\}_{j=1,2,\dots}$,

(12) each square $Q_{i,n}^{(j)}$ has points in common with $E_{i,n}$ and therefore by (8) fulfils the inequality $G(Q_{i,n}^{(j)}) > -\varepsilon \cdot 2^{-i}(L^h(E_i) + 1)^{-1} \cdot h[\delta(Q_{i,n}^{(j)})]$.

We shall now show that:

(13) I_0 can be represented as a sum of a finite number of squares Q each of which either satisfies the inequality $G(Q) > 0$, or coincides with one of the squares $Q_{i,n}^{(j)}$.

Supposing the contrary, by the well known method of successive subdivisions starting with I_0 , we could determine a descending sequence $\{Q_p\}$ of squares satisfying $\delta(Q_p) \rightarrow 0$ as $p \rightarrow \infty$ and belonging to the nets $\mathcal{R}_k (k \geq N_0)$ such that each Q_p neither coincided with any of $Q_{i,n}^{(j)}$ nor satisfied the inequality $G(Q_p) > 0$.

Denoting by x_0 the common point of all these squares, then either $x_0 \in E$ or $x_0 \in I - E$.

In the first case, by (11) the sequence $\{Q_p\}$ must contain squares $Q_{i,n}^{(j)}$, which is impossible, while, in the second case, by the condition (7) $G(x_0) > 0$ and therefore $G(Q_p) > 0$ for every sufficiently large p , which is also impossible. Thus we have proved (13).

Now, on the one hand, if I_0 is a sum of a finite number of non-overlapping squares each of which coincides with one of $Q_{i,n}^{(j)}$, then by (12), (10) and (9), we have

$$\begin{aligned} G(I_0) &\geq -\varepsilon \cdot \sum_{i,n,j} \frac{h[\delta(Q_{i,n}^{(j)})]}{2^i(L^h(E_i)+1)} \geq -\varepsilon \sum_i \frac{1}{2^i(L^h(E_i)+1)} \cdot \sum_n 16 \left(L^h(E_{i,n}) + \frac{1}{2^n} \right) \\ &= -16\varepsilon. \end{aligned}$$

On the other hand, if I_0 is a sum of a finite number of non overlapping squares for each of which the function G takes a positive value, then $G(I_0) > 0$ and *a fortiori* $G(I_0) \geq 16\varepsilon$. Thus, from (13) we have always $G(I_0) \geq 16\varepsilon$. Hence by the definition of $G(I)$, $F(I_0) \geq -(16 + |I_0|)\varepsilon$, and, since ε is given arbitrarily, we have $F(I_0) \geq 0$, which proves our lemma.

We shall restrict ourselves hereafter to the case that the function $h(t)$ should fulfil the following condition

$$(14) \quad \lim_{t \rightarrow +0} \frac{t^2}{h(t)} = 0.$$

From this condition we see easily that $L^h(E) < \infty$ implies $|E| = 0$, since, as is known, 2-dimensional measure 0 is identical with Lebesgue plane measure 0⁽⁴⁾ and therefore the exceptional set E will be of Lebesgue plane measure 0.

Theorem 1. Suppose that an additive and continuous function $F(I)$ satisfies the following conditions everywhere in I_0 ,

$$\underline{F}_h(x) \geq 0,$$

(15) $-\infty \neq \underline{F}(x) \geq g(x)$, except perhaps, at the points belonging to a set expressible as the sum of an enumerable infinity of sets of finite h -measure, where $g(x)$ is a summable function over I_0 .

(4) F. Hausdorff, Loc. cit.

Then, for any interval $I < I_0$, we have

$$F(I) \geq \int_I g(x) dx.$$

Proof. From the summability of $g(x)$, there exists, for each $\varepsilon > 0$, an additive and continuous function $V(I)$, a minor function of $g(x)^{(5)}$, such that for every $x \in I_0$,

$$(16) \quad +\infty \neq \bar{V}(x) \leq g(x) \quad \text{and} \quad 0 \leq \int_I g(x) dx - V(I) \leq \varepsilon \quad \text{for every } I < I_0, \quad \text{where } \varepsilon \text{ is an arbitrarily given positive number.}$$

From the first of (16), we have $\bar{V}(x) < \infty$ and consequently by (14), $\bar{V}_h(x) \leq 0$. Therefore, putting $G(I) = F(I) - V(I)$, we have $\underline{G}_h(x) \geq \underline{F}_h(x) - \bar{V}_h(x) \geq 0$ everywhere in I_0 , and moreover $\underline{G}(x) \geq \underline{F}(x) - \bar{V}(x) \geq \underline{F}(x) - g(x) \geq 0$ at every point where (15) is satisfied.

Since $G(I)$ is seen to have satisfied all the assumptions of the preceding lemma, we have $G(I) \geq 0$ or $F(I) \geq V(I)$ which gives, by the second part of (16), $F(I) \geq \int_I g(x) dx - \varepsilon$ and this completes the proof, as ε is arbitrary.

Remark 1. If the conditions $\underline{F}_h(x) \geq 0$ and (15) are replaced by the conditions $\bar{F}_h(x) \leq 0$ and $+\infty \neq \bar{F}(x) \leq g(x)$ respectively in the preceding theorem, then by symmetry we have for every $I < I_0$, $F(I) \leq \int_I g(x) dx$.

Remark 2. From $F(I) \geq \int_I g(x) dx$ for every $I < I_0$, we have $F(I) \geq -\int_{I_0} |g(x)| dx$ for any interval $I < I_0$, which shows that the lower variation of F on I_0 is finite and therefore F is of bounded variation on I_0 . Then by the well known theorem of Lebesgue, a finite derivative $F'(x)$ exists almost everywhere in I_0 . By these remarks, we obtain at once the following:

Theorem 2. Given an increasing and continuous function $h(t)$ with $h(0) = 0$ and the property (14), let us suppose that $F(I)$ is an additive and continuous function satisfying the following conditions:

- (i) $\bar{F}_h(x) = F_h(x) = 0$ for every x in I_0 ,
- (ii) either $\bar{F}(x)$ or $\underline{F}(x)$ is summable over $I_0^{(6)}$,

⁽⁵⁾ S. Saks, *Theory of the Integral*, Warsaw and Lwów, 1937, pp. 190-191, esp. (3.2).

⁽⁶⁾ Saks and Zygmund assumed that both $\bar{F}(x)$ and $\underline{F}(x)$ were summable, but, as we have seen from Remark 2, the summability of one follows from that of the other. Even without Remark 2, that fact is found to be true by the delicate and deep treatments of A. J. Ward, of which one will find enough in Saks' *Theory of the Integral*, pp. 133-139, esp. (11.15) Theorem.

(iii) $\overline{F}(x)$ and $\underline{F}(x)$ are finite everywhere in I_0 , except at most at the points belonging to a set expressible as a sum of an enumerable infinity of sets with finite h -measure. Then we have for any interval $I \subset I_0$

$$F(I) = \int_I F'(x) dx.$$

IV. Let us now apply this result to the theory of functions of a complex variable. Suppose that $f(z)$ is a complex valued continuous function of a complex variable z in a domain G and that it satisfies at every point z of G

$$(17) \quad |f(z+\Delta z) - f(z)| = O\{h(|\Delta z|)\} \quad \text{as } \Delta z \rightarrow 0.$$

Denoting the boundary of an interval I by (I) , the complex integral taken along (I) in the positive sense:

$$\int_{(I)} f(z) dz = S(I) + iT(I)$$

may be considered as composed of the two additive and continuous functions S and T , which satisfy the condition $\overline{S}_h(\zeta) = \underline{S}_h(\zeta) = \overline{T}_h(\zeta) = \underline{T}_h(\zeta) = 0$, since, by (17), we have $f(z) = f(\zeta) + h(|z - \zeta|) \cdot K$ where $|K| \leq M$ in a neighbourhood of ζ , and therefore

$$\frac{1}{h[\delta(Q)]} \left| \int_{(Q)} f(z) dz \right| \leq \frac{M}{h[\delta(Q)]} \int_{(Q)} h(|z - \zeta|) |dz| \leq M \cdot \frac{4\delta(Q)}{\sqrt{2}} \rightarrow 0.$$

Moreover, we see easily that $\overline{S}(\zeta)$, $\underline{S}(\zeta)$, $\overline{T}(\zeta)$ and $\underline{T}(\zeta)$ are finite at all the points ζ where $\lim_{z \rightarrow \zeta} |(f(z) - f(\zeta))/(z - \zeta)| < \infty$ is satisfied. It is also clear that if $f'(z)$ exists and is finite, then the relation $S'(z) = T'(z) = 0$ holds true. Hence, by Theorem 2 and Morera's theorem, we obtain the following:

Theorem 3. Suppose that a continuous function $f(z)$ of a complex variable z defined in a domain G satisfies the following conditions:

(i) for some $h(t)$ with (14), the condition (17) is fulfilled everywhere,
 (ii) $\lim_{z \rightarrow \zeta} |(f(z) - f(\zeta))/(z - \zeta)| < \infty$ holds everywhere in G except at most at points belonging to a set expressible as the sum of an enumerable infinity of sets of finite h -measure,

(iii) $f'(z)$ exists and is finite almost everywhere in G .

Then $f(z)$ is regular in G .

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