5. On stochastic processes (I).*

(Infinitely divisible laws of probability)

By Kiyosi ITÔ.

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The main aim of this paper is to give a new rigorous proof of the known formula of P. Lévy⁽¹⁾ on the infinitely divisible law (of probability) in making use of the scheme of stochastic differential processes introduced recently by J. L. Doob.⁽²⁾ Chapter I is devoted to preliminary explanations of concepts and notations used later on. In Chapter II, I shall prove several theorems concerning the topology of the probability distributions on the real number space; these theorems will be useful in the following chapters.

In Chapter III, I shall establish a relation between "an infinitely divisible law" and "a stochastic process." Logically to say, they constitute two different concepts; the former is a probability distribution on the real number space and the latter is a random variable which takes values in a functional space. It is the chief object of Chapter III to relate these two concepts (Cf. Theorem 3.1.). The " $\mathfrak{L}$ -process" appearing there is not essentially different from what is called a stochastic differential process with no fixed discontinuity.

In Chapter IV, I shall determine a canonical form of $\mathfrak{L}$ -processes (Cf. Theorem 4.4, and 4.5.), whence the formula of Lévy is easily deduced. The idea of this chapter is essentially due to the work of Lévy, but it may be not without interest to express it in a clearer form. In Chapter V, I shall prove two theorems concerning temporally homogeneous $\mathfrak{L}$ -processes, by which we can easily prove the uniqueness theorem of Lévy⁽³⁾ concerning his formula.

I. Preliminaries.

A probability distribution on a space Ω , as is well-known, is defined as a real-valued set-function P on Ω , defined for any set belonging to a

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(1) See P. Lévy: *Théorie de l'addition des variables aléatoires* (1937), p. 180 Th. 54.1, and see also A. Khintchine: *Déduction nouvelle d'une formule de M. Paul Lévy*. Bull. Univ. Etat Moscou, Sér. Int. Sect. A. Math. et Mechan. 1 (1937).

(2) See J. L. Doob: *Stochastic processes depending on a continuous parameter*, Th. 3.8 (Transactions of the American Mathematical Society, Vol. 42).

(3) See P. Lévy: loc. cit. (1) p. 186 Th. 55.1.

certain completely additive class of subsets of Ω and satisfying the following conditions:

$$(1) \quad P(E) \geq 0$$

$$(2) \quad P(\Omega) = 1$$

(3) for disjoint subsets $E_1, E_2, \dots$, of Ω , we have

$$P\left(\sum_1^\infty E_k\right) = \sum_1^\infty P(E_k).$$

We shall often make use of the term “a P -measurable set” instead of “a set for which P is defined.”

Let B be a certain completely additive class of subsets of a space X . When we consider the space together with B , we shall call it a *field* and denote it by (X, B) . We can consider any topological space X as a field in taking as B the least completely additive class of subsets of X containing all open sets on X . A space Ω on which a probability distribution P is defined can also be considered as a field (Ω, B_P) , B_P being the class of all P -measurable subsets. Such a field is called a *probability field* and is denoted by (Ω, P) .

One of the most important concepts in the probability theory is that of the “*random variable*”. An (X, B) -valued random variable on a probability field (Ω, P) is defined as a function $x(\omega)$ on Ω which takes values in X and such that $E(\omega; x(\omega) \in E)$ is P -measurable for any subsets E in B . The image $x(\Omega)$ of Ω by x is a subset of X . If we denote this subset by X' , we shall be able to define a completely additive class B' on X' as the class of the subsets such as $X' \cdot E$, E running over B . The random variable x may be considered also as an (X', B') -random variable. Such a consideration may be not important if X is a simple space such as the real number space or the n -dimensional vector space, but it was shown by J. L. Doob⁽⁴⁾ how important it is in case X is a functional space.

A function defined on Ω except on a subset of Ω with probability 0 is called a random variable in the larger sense, if it satisfies the above condition of P -measurability.

In order to avoid ambiguities we shall make use of different notations to represent a random variable itself and the value which the variable takes for a certain element ω in Ω ; the former will be denoted by the symbol such as “ x ” or “ $(x(\omega); \omega \in \Omega)$ ”, and the latter by the symbol such as $x(\omega)$.

(4) See J. L. Doob: loc. cit. (3).

We shall now define the direct product of the fields (X_α, B_α) for $\alpha \in A$, A being a parameter domain. The direct product X of the space X_α for α in A , as is well-known, is the set of all correspondences such as

$$(x_\alpha; \alpha \in A),$$

where x_α is an element in X_α for any value α in A . The direct product of E_α for $\alpha \in A$, E_α being respectively any element in B_α , will be called a cylinder subset of X , if $E_\alpha \neq X_\alpha$ for a finite number of values of α . Let B denote the least completely additive class of subsets of X which contains all cylinder subsets. The field (X, B) is the required direct product.

Definition 1.1. Let x_α be any (X_α, B_α) -valued random variable on (Ω, P) , where α runs over a parameter domain A . Let (X, B) denote their product field. The (X, B) -valued random variable on (Ω, P) :

$$((x_\alpha(\omega); \alpha \in A); \omega \in \Omega)$$

is called the combination of x_α for $\alpha \in A$ and is denoted by $(x_\alpha; \alpha \in A)$. Except in the case that A is enumerable, this definition may perhaps lose its sense, if we replace here "random variable in the larger sense" instead of "random variable".

Definition 1.2. Let x be an (X, B) -valued random variable, and f a transformation from (X, B) to (X', B') such that the set $f^{-1}(E')$ may belong to B for any set E' in B' . Then $(f(x(\omega)); \omega \in \Omega)$ will be an (X', B') -valued random variable. This variable is called a B -measurable function of x and is denoted by $f(x)$.

Definition 1.3. Let x_α be any (X_α, B_α) -valued random variable on (Ω, P) where α runs over a parameter domain A . When, for any finite system of parameter-values $\alpha_1 \alpha_2 \dots \alpha_n (\in A)$ and for any subset E_{α_i} of $X_{\alpha_i} (i = 1, 2, \dots, n)$, we have

$$P((x_{\alpha_1} \in E_1)(x_{\alpha_2} \in E_2) \dots (x_{\alpha_n} \in E_n)) = P(x_{\alpha_1} \in E_1)P(x_{\alpha_2} \in E_2) \dots P(x_{\alpha_n} \in E_n)^{(5)},$$

we say that $x_\alpha (\alpha \in A)$ are independent.

Corollary 1.1. Let $f_\alpha(x_\alpha)$ be a B -measurable function of x_α where α runs over A . If $x_\alpha (\alpha \in A)$ be independent, then $f_\alpha(x_\alpha)$ will be also independent.

(5) Let $A(\omega)$ be a proposition depending on an element ω in a probability field (Ω, P) . Then we shall denote the set of the ω -elements for which $A(\omega)$ is true by A itself, if we have no fear of confusions; this notation may be consistent with the conventional use.

If two random variables x_1, x_2 are independent, we say also that x_1 is independent of x_2 .

Corollary 1.2. If $x_1, x_2, \dots, x_n$ be independent, then x_i will be also independent of the combination of $x_{i+1}, x_{i+2}, \dots, x_n$ for any suffix i , and conversely.

Definition 1.4. Let x be an (X, B) -valued random variable on (Ω, P) . $P(x^{-1}(E))$ will be a probability distribution on X . We shall call this distribution the probability law of x , and denote it by P_x ; we also say that x is subject to this law P_x .

Definition 1.5. Let x be a random variable on (Ω, P) which takes values in the n -dimensional vector space. In case $\int_{\Omega} \|x(\omega)\| P(d\omega) < \infty^{(6)}$, we shall define the mean value of x as $\int_{\Omega} x(\omega) P(d\omega)$, and denote it by $\bar{x}$ or $m(x)$.

Definition 1.6. Let P be a probability distribution on the n -dimensional vector space R^n . In case $\int_{R^n} \|\lambda\| P(d\lambda) < \infty$ we shall define the mean value of P as $\int_{R^n} \lambda P(d\lambda)$, and denote it by $m(P)$.

Corollary 1.3. $\bar{x} = m(P_x)$

Definition 1.7. Let x be a real-valued random variable on (Ω, P) . The standard deviation of x is defined as $\sqrt{(x - \bar{x})^2}$ if this value exists and is finite. It will be denoted by $\sigma(x)$. The concentration function of x with regard to a positive number l is defined as $\text{l.u.b. } P(\lambda \leq x \leq \lambda + l)$ and denoted by $Q(x, l)$. The characteristic function of x is defined as e^{izx} and is denoted by $\varphi_x(z)$.

Definition 1.8. Let P be a probability distribution on the real number space R . The standard deviation of P is defined as $\sqrt{\int_{-\infty}^{+\infty} (\lambda - m(P))^2 P(d\lambda)}$ if this value exists and is finite, and is denoted by $\sigma(P)$. The concentration function of P with regard to l is defined as $\text{l.u.b. } P([\lambda, \lambda + l])^{(7)}$ and is denoted by $Q(P, l)$.

The characteristic function of P is defined as $\int_{-\infty}^{+\infty} e^{iz\lambda} P(d\lambda)$, and is denoted by $\varphi_P(z)$.

Corollary 1.4. $\sigma(x) = \sigma(P_x)$, $Q(x, l) = Q(P_x, l)$, $\varphi_x(z) = \varphi_{P_x}(z)$.

(6) $\|\quad\|$ means the euclidian metric.

(7) By $[ab]$ we shall denote the interval closed on the left and open on the right. $[ab]$, (ab) or (ab) may also be understood similarly.

Definition 1.9. Let x be an (X, B) -valued random variable on $(\mathcal{Q}, P)$ and y on $(\mathcal{Q}', P')$. If $P_x(E) = P_y(E)$ for any set E in B , we say that x and y are *equivalent in law* as (X, B) -valued random variables; we shall write " $x \sim y$ " or simply " $x \sim y$ ".
(x, B)

Definition 1.10. If F, G be probability distributions on the real number space, the following set-function H will be also a probability distribution:

$$H(E) = \int_{-\infty}^{+\infty} F(E(+)(-\lambda))G(d\lambda)^{(8)}.$$

This distribution is called the convolution of F and G , and is denoted by $F * G$.

Corollary 1.5. If x, y be independent real-valued random variables, then we have

$$P_{x+y} = P_x * P_y$$

Definition 1.11. Let $(\mathcal{Q}_\alpha, P_\alpha)$ be a probability field separable in P_α where α runs over A . Let $\mathcal{Q}$ denote the product space of $\mathcal{Q}_\alpha$. We can constitute⁽⁹⁾ a probability distribution P on $\mathcal{Q}$ so that

$$P(E) = \prod_{\alpha \in A} P_\alpha(E_\alpha),$$

E_α being identified with $\mathcal{Q}_\alpha$ except for a finite number of values of α and E being the product space of E_α for $\alpha \in A$. Then $(\mathcal{Q}, P)$ is called the *direct product* of $(\mathcal{Q}_\alpha, P_\alpha)$ for $\alpha \in A$. The *representation* in the direct product $(\mathcal{Q}, P)$ of a random variable x_β on $(\mathcal{Q}_\beta, P_\beta)$ will be defined as the random variable x'_β on $(\mathcal{Q}, P)$ satisfying $x'_\beta(\omega) = x_\beta(\omega_\beta)$ for $\omega = (\omega_\alpha; \alpha \in A)$.

Corollary 1.6. Let x_α be a random variable on $(\mathcal{Q}_\alpha, P_\alpha)$ where α runs over A . Let x'_α denote the representation of x_α on the direct product of $(\mathcal{Q}_\alpha, P_\alpha)$ for $\alpha \in A$. Then $x'_\beta (\alpha \in A)$ are independent.

II. On probability distributions on the real number space.

In this chapter we shall treat probability distributions on the real number space and call them distributions simply. Let F be any distribution. Then $F([-\infty, \lambda])$ will have the following properties, if we consider it as a function of λ :

⁽⁸⁾ $E(+)\alpha$ means the set $E(x+\alpha; x \in E)$.

⁽⁹⁾ See J. L. Doob: Stochastic processes with an integral-valued parameter, Th. 1.1 (Transactions of the American Mathematical Society, Vol. 44), and E. Hopf: Ergodentheorie, p. 4, Mass und Integral in Produkträumen (Ergebnisse der Mathematik, Vol. 5, No. 2). The word "separable" used here is explained at the end of this paper.

- (1) monotone increasing⁽¹⁰⁾
 (2) if $\lambda \rightarrow \infty$, then it will tend to 1, and if $\lambda \rightarrow -\infty$, then it will tend to 0,
 (3) continuous on the right hand.

We shall denote this function by the same notation F . As Lévy⁽¹¹⁾ has remarked, it is convenient to consider that, at each discontinuity point λ of F , $F(\lambda)$ may represent any values between $F(\lambda-0)$ and $F(\lambda+0)$. Then the curve $y = F(x)$ proves to be a continuous curve. By the distance ρ between two distributions F, G we shall understand the least upper bound of the distance between the points at which any straight line perpendicular to the bisector of the first quadrant in the (xy) -plane intersects the curves $y = F(x)$ and $y = G(x)$ respectively.

$q(F)$, as defined in the following expression, will be called *the mean concentration function of F* :

$$(4) \quad q(F) = \int_0^\infty Q(F, l)e^{-l}dl,$$

$Q(F, l)$ being the concentration function of F with regard to l .

Let F_1 and F_2 be distributions. If there exists a real number λ such that $F_1(E(+)\lambda) = F_2(E)$ for any B -measurable set E , we say that F_1 is equivalent to F_2 and write $F_1 \sim F_2$. This relation is evidently reflexive, symmetric, and transitive. We classify all distributions by means of this equivalence relation, and represent classes by small letters $f, g, \dots$, and especially by e the *unit class* containing the unit distribution E which attributes to 0 the probability 1. *The concentration function or the mean concentration function of a class* is defined as that of some distribution belonging to the class, and *the convolution of two classes* is defined as the class which contains the convolution of distributions chosen from each of the classes; these definitions are clearly independent of the choice of distributions from the classes in question.

If there exists a class h such that $f = g * h$, we shall write, by definition, $f \leq g$. This relation satisfies evidently the following conditions:

$$(5a) \quad f \leq f, \quad (5b) \quad f \leq g \text{ and } g \leq h \text{ imply } f \leq g,$$

$$(5c) \quad f \leq g \text{ and } g \leq f \text{ imply } f = g.$$

The set of all classes forms thus a *partially ordered set with regard to this order relation*.

⁽¹⁰⁾ In this paper we shall make use of the term "monotone increasing" in the sense of "monotone non-decreasing".

⁽¹¹⁾ See P. Lévy: loc. cit. (1) p. 28, foot-note (1).

The distance ρ between two classes is defined as the minimum of the distance between two distributions respectively chosen from the classes. The distance ρ satisfies clearly the following conditions;

$$\begin{aligned} (6a) \quad \rho(f, g) &\geq 0 & (6b) \quad \rho(f, g) &= \rho(g, f) & (6c) \quad \rho(f, g) + \rho(g, h) &\geq \rho(f, h) \\ (6d) \quad \rho(f, f) &= 0 & (6e) \quad \rho(f, g) &= 0 \text{ implies } f = g. \end{aligned}$$

It is also easy to see that the set of all classes forms a complete set with regard to this distance.

We shall prove a theorem concerning the relation of this order and this distance ρ .

Theorem 2.1. Any set $\mathfrak{K}$ of classes lowerly bounded with regard to the above order relation by a class f is totally bounded with regard to the above distance ρ . (We shall adopt the term "order-bounded" or " ρ -bounded" to signify "bounded with regard to the above order" or "to the above distance ρ ".)

Proof. It is sufficient to prove that any sequence $\{f_k\}$ in $\mathfrak{K}$ may have a ρ -convergent subsequence. We will choose from each class of $f_1, f_2, \dots$, and f the distributions whose medians are equal to 0, and denote them by $F_1, F_2, \dots$, and F respectively, where the median of F means one of the values x such that $F(x) = \frac{1}{2}$. By the principle of decreasing⁽¹²⁾ of the concentration function we have

$$Q(F_k, l) \geq Q(F, l).$$

The median of $F_1, F_2, \dots$, and F being equal to 0, we can deduce from this inequality that, as far as l is so large that $Q(F, l) > \frac{1}{2}$, we have

$$1 - F_k(l) + F_k(-l) \leq 1 - Q(F_k, l) \leq 1 - Q(F, l).$$

$1 - Q(F, l)$ tending to 0 with $\frac{1}{l}$, a theorem of Lévy⁽¹³⁾ informs us that the sequence $\{F_k\}$ has a ρ -convergent subsequence $\{G_k\}$. Let $g_1, g_2, \dots$ denote respectively the classes containing $G_1, G_2, \dots$. Then g_k is a ρ -convergent subsequence of $\{f_k\}$.

We shall introduce now another distance into the set of classes; we define namely:

$$(7) \quad \rho_1(f, g) = |q(f) - q(g)|$$

⁽¹²⁾ See P. Lévy: loc. cit. (1) p. 90.

⁽¹³⁾ See P. Lévy: loc. cit. (1) p. 63.

This distance ρ_1 satisfies the above conditions (6a)—(6d) but not necessarily (6e).

Theorem 2.2. *The distance ρ_1 satisfies the condition (6e) on a linearly ordered set of classes.*

Proof. It is sufficient to show that $f \leq g$ and $q(f) = q(g)$ imply $f = g$. As $f \leq g$, we have $f = g * h$ where h is a certain class. If we choose $G \in g$ and $H \in h$, then we have $G * H \in f$. As the principle of decreasing of the concentration function informs us $Q(G * H, l) \leq Q(G, l)$, we have $Q(f, l) \leq Q(g, l)$. But we have by the assumption $q(f) = q(g)$, namely,

$$(8) \quad \int_0^\infty Q(f, l) e^{-l} dl = \int_0^\infty Q(g, l) e^{-l} dl.$$

As $Q(f, l)$ and $Q(g, l)$ are monotone functions continuous on the right hand, we have $Q(f, l) = Q(g, l)$ i.e. $Q(G * H, l) = Q(G, l)$. By a theorem of Lévy⁽¹⁴⁾ we can deduce, from this identity, $H \in e$, namely $f = g$.

Theorem 2.3. *The distances ρ and ρ_1 induce equivalent topologies on any set $\mathfrak{R}$ of classes of distributions which is linearly ordered and lowerly order-bounded.*

The proof of this theorem consists of two parts. In the first part we shall show that there exists a positive number $\delta(\mathfrak{E})$ for any positive number $\mathfrak{E}$ such that $\rho(f, g) < \delta(\mathfrak{E})$ may imply $\rho_1(f, g) < \mathfrak{E}$, and in the second part we shall show conversely that there exists a positive number $\delta(\mathfrak{E})$ for any positive number $\mathfrak{E}$ such that $\rho_1(f, g) < \delta(\mathfrak{E})$ may imply $\rho(f, g) < \mathfrak{E}$.

The first part: Let F_0, G_0 denote distributions in f and g respectively such that $\rho(F_0, G_0) = \rho(f, g)$. On the curve $y = F_0(x)$ there exist now two points $(x_1 y_1)$ $(x_2 y_2)$ such that

$$(9) \quad x_2 - x_1 = l$$

$$(10) \quad y_2 - y_1 = Q(F_0, l).$$

Draw two straight lines perpendicular to the bisector of the first quadrant through $(x_1 y_1)$ and $(x_2 y_2)$; and let $(x'_1 y'_1)$ and $(x'_2 y'_2)$ denote respectively the points of intersection with the curve $y = G_0(x)$. Let δ denote $\rho(F_0, G_0)$. Then we have

$$|x_1 - x'_1| \leq \delta, \quad |y_1 - y'_1| \leq \delta, \quad |x_2 - x'_2| \leq \delta, \quad \text{and} \quad |y_2 - y'_2| \leq \delta,$$

and so

⁽¹⁴⁾ See P. Lévy: loc. cit. (1) p. 19, Th. 29,1.

$$x'_2 - x'_1 \leq x_2 - x_1 + 2\delta = l + 2\delta$$

$$y'_2 - y'_1 \geq y_2 - y_1 - 2\delta = Q(F_0, l) - 2\delta,$$

from which we deduce

$$Q(G_0, l + 2\delta) \geq Q(F_0, l) - 2\delta$$

i.e.

$$(11) \quad \int_0^\infty Q(G_0, l + 2\delta) e^{-l} dl \geq q(f) - 2\delta.$$

But

$$\int_0^\infty Q(G_0, l + 2\delta) e^{-l} dl = e^{2\delta} \int_{2\delta}^\infty Q(G_0, l) e^{-l} dl \leq e^{2\delta} q(g) \leq (1 + 4\delta) q(g)$$

holds for $\delta < \frac{1}{4}$. Therefore we have, as $q(g) \leq 1$, for $\delta < \frac{1}{4}$,

$$(12) \quad \int_0^\infty Q(G_0, l + 2\delta) e^{-l} dl \leq q(g) + 4\delta.$$

By (11) and (12) we have $q(g) + 4\delta \geq q(f) - 2\delta$ or $q(g) \geq q(f) - 6\delta > q(f) - \varepsilon$ for $\delta < \min\left(\frac{\varepsilon}{6}, \frac{1}{4}\right)$. Similarly we have $q(f) > q(g) - \varepsilon$, and so we have finally, for $\rho(f, g) < \min\left(\frac{\varepsilon}{6}, \frac{1}{4}\right)$, $\rho_1(f, g) = |q(f) - q(g)| < \varepsilon$.

For the second part of the proof, it is sufficient to prove that the existence of a positive number ε_0 and three sequences $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ in $\mathfrak{R}$, such that $f_n = g_n * h_n$, $\rho_1(f_n, g_n) \leq \frac{1}{n}$ and $\rho(f_n, g_n) \geq \varepsilon_0$ for every natural number n , may be reduced to a contradiction. By Theorem 2.1 we can see that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\rho\text{-}\lim_{k \rightarrow \infty} f_{n_k} = f, \quad \rho\text{-}\lim_{k \rightarrow \infty} g_{n_k} = g, \quad \text{and} \quad \rho\text{-}\lim_{k \rightarrow \infty} h_{n_k} = h.$$

(The limit distributions f , g , and h may not exist in $\mathfrak{R}$, but it does not matter.) By the assumption we have $f_{n_k} = g_{n_k} * h_{n_k}$ and so

$$(13) \quad f = g * h$$

By the first part of the proof we can see that $\rho_1(f_{n_k}, g_{n_k}) \leq \frac{1}{n_k}$ implies

$$(14) \quad \rho_1(f, g) = 0.$$

From (13) and (14) follows $f = g$ by Theorem 2.2, in contradiction with the inequality $\rho(f, g) \geq \varepsilon_0$ which follows from $\rho(f_{n_k}, g_{n_k}) \geq \varepsilon_0$.

Corollary 2.1. If $f_1 \geq f_2 \geq f_3 \geq \dots \geq f$, then there exists $\rho\text{-}\lim_{n \rightarrow \infty} f_n$.

Proof. We have $q(f_1) \geq q(f_2) \geq \dots \geq q(f)$ by the hypothesis, and so $q(f_n)$ is a convergent sequence. $\{f_n\}$ is therefore a ρ_1 -fundamental sequence. As the set $\mathfrak{R}$ of f_n ($n = 1, 2, \dots$) is linearly ordered and lower-order bounded, $\{f_n\}$ is also a ρ -fundamental sequence by Theorem 2.2. By the ρ -completeness of the set of all classes of distributions we obtain the existence of $\rho\text{-}\lim_{n \rightarrow \infty} f_n$.

Corollary 2.2. If $f_1 \leq f_2 \leq \dots$, then there exists $\rho\text{-}\lim_{n \rightarrow \infty} f_n$.

Corollary 2.3. Let $\{f_t\}$ be a set of classes depending on a continuous parameter t such that $t < s$ may imply $f_t \geq f_s$. Then the set of the ρ -discontinuity t -points of f_t forms an enumerable or finite set.

Proof. By Theorem 2.3 the set of all ρ -discontinuity t -points of f_t and the set of all ρ_1 -discontinuity t -points of f_t are coincident. But the latter forms, as is well-known, an enumerable or finite set, since $q(f_t)$ is a monotone function of t . So it is also the case with the former.

Theorem 2.4. Let $\mathfrak{R}$ be a totally ρ -bounded set. Then there exists $\delta(\mathfrak{E})$ for any positive number $\mathfrak{E}$ such that $\rho(h, e) < \delta(\mathfrak{E})$ may imply $|q(f * h) - q(f)| < \mathfrak{E}$ for any class f in $\mathfrak{R}$.

Proof. It is sufficient to prove that the existence of a positive number $\mathfrak{E}_0$ and two sequence $\{f_n\}$, $\{h_n\}$ such that $f_n \in \mathfrak{R}$, $\rho(h_n, e) < \frac{1}{n}$ and $|q(f_n * h_n) - q(f_n)| \geq \mathfrak{E}_0$ may be reduced to a contradiction. As $\mathfrak{R}$ is totally ρ -bounded, there exists a ρ -convergent subsequence $\{f_{n_k}\}$ of $\{f_n\}$. Let f_0 denote the limit of $\{f_{n_k}\}$. But as $\rho(h_n, e) < \frac{1}{n}$, h_{n_k} will ρ -converge to e . By the ρ -continuity of the function q and the convolution, we have $|q(f * e) - q(f)| \geq \mathfrak{E}_0$, i.e. $0 \geq \mathfrak{E}_0$, which is contradictory.

III. A relation between infinitely divisible laws and $\mathfrak{L}$ -processes.

Definition 3.1. Let R be a "resolution" of $F: F = F_1 * F_2 * F_3 * \dots * F_n$. By the degree of divisibility $\Delta(F, R)$ of a distribution F in R we understand $(\max_{1 \leq i \leq n} \rho(F_i, E))^{-1}$. The degree of divisibility of F is defined as the least upper bound of $\Delta(F, R)$, R running over all resolutions of F . In the case $\Delta(F) = \infty$, F will be called an infinitely divisible law.

Corollary 3.1. Gaussian distributions and Poissonian distributions are infinitely divisible.

Let $\mathfrak{F}_{ab}$ denote the set of all real-valued functions $\{f(t)\}$ defined on the t -interval $[ab]$ satisfying the following conditions:

(1) there exist $f(t-0)$ and $f(t+0)$ at any t -point,

(2) $f(t+0) = f(t)$,

(3) $f(0) = 0$.

Let $\mathfrak{B}_{ab}$ denote the least completely additive class of subsets on $\mathfrak{S}_{ab}$ containing all cylinder subsets⁽¹⁵⁾ of $\mathfrak{S}_{ab}$. Let x be an $(\mathfrak{S}_{ab}, \mathfrak{B}_{ab})$ -valued random variable, and y the value of x at t . Then y will be a real-valued random variable. We shall denote this variable by x_t . Then x is clearly the combination of x_t .

Definition 3.2. An $\mathfrak{L}$ -process x over $[ab]$ is defined as an $(F_{ab}, \mathfrak{B}_{ab})$ -valued random variable satisfying the following conditions:

1°. $x_t = x_{t-0}$ holds with probability 1⁽¹⁶⁾ for any t in $[ab]$.

2°. For any system of non-overlapping t -intervals $(t_1s_1), (t_2s_2), \dots, (t_ns_n)$, $x_{s_i} - x_{t_i}$ ($i = 1, 2, \dots, n$) are independent, that is to say, $x_{s_i} - x_{t_i}$ are independent of the combination of x_τ for $0 \leq \tau \leq t$, where $a \leq t \leq s \leq b$; we shall say in this case that x_t is differential.

Note. If x is an $\mathfrak{L}$ -process over $[ab]$, then $(x_t - x_a; a \leq t \leq \beta)$ will also be an $\mathfrak{L}$ -process over $[a\beta]$ where $a \leq a \leq \beta \leq b$.

Definition 3.3. An $\mathfrak{L}$ -process x on (Ω, P) is called a $\mathfrak{G}$ -process, if, $x_t(\omega)$ is continuous in t for any element ω in Ω .

Definition 3.4. An $\mathfrak{L}$ -process x on (Ω, P) is called a $\mathfrak{P}$ -process, if, for any element ω in Ω , $x_t(\omega)$ is a step-function of t which increases only by jumps with the height 1.

As is well-known, the variation between two t -points of an $\mathfrak{L}$ -process is subject to an infinitely divisible law, and that of a $\mathfrak{G}$ -process to a Gaussian distribution, and that of a $\mathfrak{P}$ -process to a Poissonian distribution. Conversely we shall obtain the following theorem.

Theorem 3.1. Let F be an infinitely divisible law (or a Gaussian distribution, or a Poissonian distribution). We can constitute an $\mathfrak{L}$ -process $(x_t; 0 \leq t \leq 1)$ (or a $\mathfrak{G}$ -process, or a $\mathfrak{P}$ -process) such that x_1 may be subject to the given law F .

⁽¹⁵⁾ See A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung, p. 25 (Ergebnisse der Mathematik Vol. 2, No. 3).

⁽¹⁶⁾ Let $M_\alpha(\omega)$ be a proposition where α runs over a parameter domain A , and ω over a probability field (Ω, P) . If $P(\bigcap_{\alpha \in A} M_\alpha) = 1$, then we say that M_α holds for any α in A with probability 1, or that, with probability 1, M_α holds for any α in A . If $P(M_\alpha) = 1$ for any α in A , then we say that M_α holds with probability 1 for any α in A , or that, for any α in A , M_α holds with probability 1.

Proof. We shall discuss only the case of “ $\mathcal{Q}$ -process”, as the other cases can be also treated analogously. First we shall show the existence of F_{st} ($0 \leq s \leq t \leq 1$) satisfying the following conditions:

$$\begin{aligned} (4) \quad & F_{01} = F \\ (5) \quad & F_{st} * F_{tu} = F_{su} \\ (6) \quad & \rho\text{-}\lim_{s \rightarrow u} F_{su} = F_{uu} (= E). \end{aligned}$$

Let $\mathfrak{R}$ denote the set of all classes lower order-bounded by the class f containing the distribution F . By theorem 2.1 we can see that $\mathfrak{R}$ is a ρ -compact set. F being infinitely divisible, there exists a sequence of the resolutions of f :

$$(7) \quad f = f_1^k * f_2^k * \cdots * f_{n_k}^k \quad (k = 1, 2, 3, \dots)$$

with the property:

$$(8) \quad \rho(f_i^k, e) < \frac{1}{k} \quad (i = 1, 2, \dots, n_k).$$

Let f_{ml}^k denote the convolution of all classes such as f_{σ}^k in the k -th resolution (7) satisfying the condition:

$$(9) \quad 1 - (1 - q(f))^{\frac{l-1}{2^m}} > q(f_1^k * f_2^k * \cdots * f_{\sigma}^k) \geq 1 - (1 - q(f))^{\frac{l}{2^m}},$$

where $k, m = 1, 2, 3, \dots, l = 1, 2, 3, \dots, 2^m$. If such class does not exist, we shall define f_{ml}^k as e . All classes thus obtained will belong to $\mathfrak{R}$. Therefore any sequence such as $\{f_{ml}^1, f_{ml}^2, \dots\}$ will have a ρ -convergent subsequence. By means of the diagonal method we can find a subsequence $\{k'\}$ of $\{k\}$ independent of the suffix (ml) such that $\{f_{ml}^{k'}, f_{ml}^{k'}, \dots\}$ may ρ -converge to a limit f_{ml} , since all the pairs $\{(ml)\}$ appearing in the suffix form an enumerable set.

For $t = \frac{l}{2^m}$ and $s = \frac{k}{2^m}$ we shall define g_{ts}^m by

$$(10) \quad g_{ts}^m = f_{m, l+1} * f_{m, l+2} * \cdots * f_{m, k}.$$

g_{ts}^m is clearly independent of m on account of the identity $f_{m+1, 2\sigma-1} * f_{m+1, 2\sigma} = f_{m\sigma}$ which follows at once from the definition; we can denote g_{ts}^m by g_{ts} with no confusion. Concerning the system $\{g_{ts}\}$ thus obtained, we shall verify the following properties:

$$(11) \quad g_{ts} * g_{su} = g_{tu} \quad (0 \leq t \leq s \leq u \leq 1),$$

$$(12) \quad g_{01} = g$$

$$(13) \quad q(g_{0t}) = 1 - (1 - q(f))t.$$

(11) and (12) are clear. In order to prove (13), we shall first remark

$$(14) \quad q(g_{0t}) = \lim_{k \rightarrow \infty} q(f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{ml}^{k'}) \left(t = \frac{l}{2^m} \right),$$

which is clear on account of the ρ -continuity of q (Cf. the first part of the proof of Theorem 2.3). But we have by the definition of $\{f_{ml}^{k'}\}$

$$(15) \quad q(f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{ml}^{k'}) \geq 1 - (1 - q(f)) \frac{l}{2^m} = 1 - (1 - q(t))t$$

$$(15') \quad q(f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{ml}^{k'} * f_{\sigma+1}^{k'}) < 1 - (1 - q(f)) \frac{l}{2^m} = 1 - (1 - q(f))t,$$

where σ is the number of the factors of the k' -th resolution (7) used to constitute $f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{ml}^{k'}$. On account of $f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{ml}^{k'} \in \mathfrak{R}$ and $\rho(f_{\sigma+1}^{k'}, e) < \frac{1}{k'}$, we have, by Theorem 2.4.

$$(16) \quad |q(f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{ml}^{k'}) - q(f_{m1}^{k'} * f_{m2}^{k'} * \dots * f_{\sigma+1}^{k'})| < \mathfrak{E}(k),$$

where $\mathfrak{E}(k)$ is a positive number tending to 0 with $\frac{1}{k}$. From (14), (15), (15'), and (16) follows the identity (13).

We shall now define g_{ts} for every pair of real numbers (ts) with $0 \leq t < s \leq 1$ so that this definition may be compatible with the above definition. Let $\{t_i\} \{s_i\}$ be sequences of rational numbers with the form: $\frac{l}{2^m}$ such that

$$(17) \quad t_1 \geq t_2 \geq t_3 \geq \dots \rightarrow t, \text{ and}$$

$$(18) \quad s_1 \leq s_2 \leq s_3 \leq \dots \rightarrow s.$$

The existence of $\rho\text{-}\lim_{n \rightarrow \infty} g_{t_n, s_n}$ is clear by Corollary 2.1, and furthermore this limit is independent of the choice of the sequence $\{t_n\} \{s_n\}$; we can denote it by $g_{t+0, s-0}$. Analogously we can define $g_{t-0, s+0}$ in the case $0 \leq t \leq s \leq 1$. We shall prove

$$(19) \quad g_{t+0, s-0} = g_{t-0, s+0} \quad (t < s), \text{ and}$$

$$(20) \quad g_{t-0, t+0} = e.$$

From (13) follows

$$(21) \quad q(g_{0,t-0}) = q(g_{0,t+0}) \quad (= 1 - (1 - q(f))t).$$

But we have, by the definition,

$$(22) \quad g_{0,t-0} * g_{t-0,t+0} = g_{0,t+0}.$$

From (21) and (22) follows (20) by Theorem 2.2, and accordingly we have

$$g_{t-0,s+0} = g_{t-0,t+0} * g_{t+0,s-0} * g_{s-0,s+0} = e * g_{t+0,s-0} * e = g_{t+0,s-0}.$$

If we define g_{ts} as $g_{t-0,s+0} (= g_{t+0,s-0})$, the system $\{g_{ts}\}$ satisfies evidently the conditions:

$$(23) \quad g_{ts} * g_{su} = g_{tu},$$

$$(24) \quad g_{01} = g,$$

$$(25) \quad \rho\text{-}\lim_{\substack{t \rightarrow u \\ s \rightarrow u}} g_{ts} = g_{uu} (= e).$$

Let G_t be the Gaussian distribution with the mean value 0 and with the standard deviation t where $t \in [0, 1]$, and h_t be the class containing G_t . Now choose from $g_{0t} * h_t$ the distribution $F_{0t} * G_t$ whose median is that of $F * G_1$ multiplied by t ; we can see that the median of $F_{0t} * G_t$ is uniquely determined and that F_{0t} is also uniquely determined. Both $F_{00} = E$ and $F_{01} = F$ are clear. As $g_{tu} = g_{ts} * g_{su}$, we have $F_{tu} \sim F_{ts} * F_{su}$ i.e.

$$(26) \quad F_{tu} = F_{ts} * F_{su} * E_m,$$

E_m being the distribution which attributes the probability 1 to m . Therefore we have

$$F_{0u} = F_{0t} * F_{tu} = F_{0t} * F_{ts} * F_{su} * E_m = F_{0u} * E_m,$$

and so $m = 0$, by which (5) follows at once from (26). We can easily see

$$(27) \quad \rho\text{-}\lim_{s \rightarrow t} F_{0s} * G_s = F_{0t} * G_t$$

and so

$$(28) \quad \rho\text{-}\lim_{s \rightarrow t} F_{ts} * G_t = F_{0t} * G_t$$

But

$$(29) \quad F_{ts} * G_t * F_{st} = F_{0t} * G_t$$

In considering (25), (28) and (29), we can easily prove (6) by reductio ad absurdum.

Next we shall define an $\mathcal{Q}$ -process $(x_t; 0 \leq t \leq 1)$ so that $x_1 - x_0$ may be subject to F . Let $\mathcal{Q}^*$ denote the space of all real-valued function of t defined on the interval $(0, 1)$. We shall define $x_t^*(\omega^*)$ by

$$(30) \quad x_t^*(\omega^*) = f(t) - f(0) \quad (\omega^* = f(t)).$$

We shall introduce a probability distribution $P_{s_1 s_2 \dots s_n}^*$ into $\mathcal{Q}^*$ as follows:

$$(31) \quad P_{s_1 s_2 \dots s_n}^* \left((x_{s_1}^* - x_0^* \in E_1)(x_{s_2}^* - x_{s_1}^* \in E_2) \dots (x_{s_n}^* - x_{s_{n-1}}^* \in E_n) \right)^{(17)} \\ = F_{0s_1}(E_1) F_{s_1 s_2}(E_2) \dots F_{s_{n-1} s_n}(E_n).$$

Then $(x_t^*; t = s_1, s_2, \dots, s_n)$ is a stochastic differential process on $(\mathcal{Q}^*, P_{s_1 s_2 \dots s_n}^*)$ depending on a discontinuous parameter t , and $x_{s_i}^* - x_{s_{i-1}}^*$ is subject to $F_{s_{i-1} s_i}$. Accordingly we shall have, by Corollary 1.5,

$$(32) \quad P_{s_1 s_2 \dots s_n}^* (x_{s_{i+1}}^* - x_{s_i}^* \in E) = (F_{s_{i-1} s_i} * F_{s_i s_{i+1}})(E) = F_{s_{i-1} s_{i+1}}(E).$$

In accordance with the choice of the system $(s_1, s_2, \dots, s_n)$, we can obtain different probability distributions on $\mathcal{Q}^*$, but we can prove, by virtue of (32), that these distributions are coherent to each other, namely that they attribute to the same subset of $\mathcal{Q}^*$ always the same probability; we can thus omit the suffix and write P^* . P^* is an additive (not necessarily completely additive) measure on $\mathcal{Q}^*$, which is completely additive on the Borel system of the cylinder sets⁽¹⁸⁾ over any assigned finite system of t -values. By a theorem of Kolmogoroff⁽¹⁹⁾ we can extend this measure and constitute a probability distribution on $\mathcal{Q}^*$; we shall rewrite it as P^* . The process $(x_t^*; 0 \leq t \leq 1)$ proves to be a stochastic differential process on $(\mathcal{Q}^*, P^*)$. As $x_s^* - x_t^*$ is evidently subject to F_{ts} , x_t^* is continuous in t with regard to the topology of "convergence in probability" on account of (6). Let $\mathcal{Q}$ denote the set of all elements of $\mathcal{Q}$ satisfying (1), (2), and (3). By virtue of a theorem of Doob⁽²⁰⁾ we can define a probability distribution P on $\mathcal{Q}$ so that $P(E^* \mathcal{Q}) = P^*(E^*)$ for any P^* -measurable set E^* . If we define a random variable x by $x(\omega) = \omega$, x is the required $\mathcal{Q}$ -process.

(17) $(x_{s_{i+1}}^* - x_{s_i}^* \in E)$ means the ω^* -set $E(\omega^*; x_{s_{i+1}}^* - x_{s_i}^* \in E)$. Cf. foot-note(5).

(18) See foot-note (15).

(19) See A. Kolmogoroff: loc. cit. (15) p. 25. Cf. J. L. Doob: loc. cit. (9) Th. 1.1.

(20) See J. L. Doob: loc. cit. (2) Th. 3.8.

Theorem 3.2. Let $\{F_k\}$ be a ρ -convergent sequence of distributions, and F denote its limit. If the degree of divisibility $\Delta(F_k)$ increases indefinitely with k , then F is infinitely divisible.

Proof. Let $f_k (k = 1, 2, \dots)$ and f denote respectively the classes containing the distributions $F_k (k = 1, 2, \dots)$ and F . We shall need only prove that the set $\sum_{k=1}^{\infty} E(g; g \geq f_k)$ is totally ρ -bounded, namely that any sequence of classes in this set may have a ρ -convergent subsequence; if this is established, we can prove the existence of $\{F_{st}\}$ satisfying (4), (5), (6) in the same manner as in the proof of the preceding theorem.

Let $\{g_k\}$ be any sequence in $\sum_{k=1}^{\infty} E(g; g \geq f_k)$. If there is one (infinite) subsequence of $\{g_k\}$ contained in anyone of the sets $E(g; g \geq f_k) (k = 1, 2, \dots)$, this subsequence may have a ρ -convergent subsequence by Theorem 2.1. In the contrary case we can find a subsequence $\{h_k\}$ of $\{g_k\}$ such that $h_k \geq f_{k'}$ ($k = 1, 2, \dots$). Choose from each of $\{h_k\}$ the distribution whose median is equal to 0, and denote it by $H_k (k = 1, 2, \dots)$.

Let $\{\lambda_i\}$ be an enumerable set everywhere dense in $(-\infty, +\infty)$. As $H_k(\lambda_i)$ are real numbers between 0 and 1, we can find a convergent subsequence of $\{H_k(\lambda_i)\}$. By means of the diagonal method we can find a subsequence $\{\bar{k}\}$ of $\{k\}$ such that the sequence $\{H_{\bar{k}}(\lambda_i); k = 1, 2, \dots\}$ may be convergent for any natural number i . Denote the limit by $H(\lambda_i)$. Then we shall see that $\lambda_i < \lambda_j$ implies $H_{\bar{k}}(\lambda_i) \leq H_{\bar{k}}(\lambda_j)$ and so $H(\lambda_i) \leq H(\lambda_j)$. Define $\bar{H}(\lambda)$ as g.l.b. $H(\lambda_i)$ $\lambda_i < \lambda$. Then it is easy to see that $\bar{H}(\lambda)$ is continuous except for a finite or enumerable set of λ -values, and that the sequence $H_{\bar{k}}(\lambda)$ converges to $\bar{H}(\lambda)$ at any continuity λ -point of $\bar{H}(\lambda)$. We shall now prove

$$(33) \quad \bar{H}(+\infty) = 1 \quad \bar{H}(-\infty) = 0,$$

or

$$(33') \quad \lim_{\lambda \rightarrow \infty} (\bar{H}(\lambda) - \bar{H}(-\lambda)) = 1.$$

Assume that

$$(34) \quad \lim_{\lambda \rightarrow \infty} (\bar{H}(\lambda) - \bar{H}(-\lambda)) = a < 1.$$

We shall now prove that there exists, for any positive number l , a natural number $K(l)$ such that $k > K(l)$ may imply

$$(35) \quad H_{\bar{k}}(l+0) - H_{\bar{k}}(-l-0) \leq \frac{a+1}{2} < 1$$

The discontinuity points of $\bar{H}(\lambda)$ forming a finite or enumerable set, we can choose the continuity points of $\bar{H}(\lambda)$ from each of $(-\infty, -l)$ and

(l, ∞) Let m, M denote these points respectively. There exists a positive number $K(l)$ such that $k > K(l)$ may imply $|H_{\bar{k}}(m) - \bar{H}(m)| < \frac{1-\alpha}{4}$ and $|H_{\bar{k}}(M) - \bar{H}(M)| < \frac{1-\alpha}{4}$, and accordingly

$$(36) \quad H_{\bar{k}}(l+0) - H_{\bar{k}}(-l-0) \leq H_{\bar{k}}(M) - H_{\bar{k}}(m) \leq \bar{H}(M) - \bar{H}(m) \\ + \frac{1-\alpha}{2} < \lim_{\lambda \rightarrow \infty} (\bar{H}(\lambda) - \bar{H}(-\lambda)) + \frac{1-\alpha}{2}.$$

From (34) and (36) follows (35) at once.

In the case $Q(H_{\bar{k}}, l) \geq \frac{1}{2}$, $k > K(l)$ implies

$$Q(H_{\bar{k}}, l) \leq H_{\bar{k}}(l+0) - H_{\bar{k}}(-l-0) < \frac{\alpha+1}{2}$$

on account of $H_{\bar{k}}(0) = \frac{1}{2}$, and so $k > K(l)$ always implies

$$Q(H_{\bar{k}}, l) \leq \max\left(\frac{1}{2}, \frac{\alpha+1}{2}\right) = \frac{\alpha+1}{2} \text{ i.e. } Q(h_{\bar{k}}, l) \leq \frac{\alpha+1}{2} < 1.$$

As we have $f_{(\bar{k})} \leq h_{\bar{k}}$ by the hypothesis, $k > K(l)$ implies $Q(f_{(\bar{k})}, l) \leq Q(h_{\bar{k}}, l) \leq \frac{\alpha+1}{2} < 1$. $f_{(\bar{k})}$ ρ -converging to f , we have finally, $Q(f, l-0) \leq \frac{\alpha+1}{2}$, whatever value l may take. This contradicts $Q(f, +\infty) = 1$. Therefore (33) must hold, that is to say, $\bar{H}$ must be a distribution. The class containing $\bar{H}$ is the limit class of the subsequence $\{h_{\bar{k}}\}$ of $\{h_k\}$ or of $\{g_k\}$.

Definition 3.5. Let λ be a positive number and F a distribution. We shall define $\frac{1}{\lambda}F$ by $\left(\frac{1}{\lambda}F\right)(E) = F(\lambda(\times)E)$ ⁽²¹⁾.

Let $\{F_k\}$ be a sequence of distributions and let $\{\lambda_k\}$ be a divergent sequence of positive numbers such that $\frac{\lambda_{n+1}}{\lambda_n}$ may tend to 1 as n increases indefinitely.

If $\frac{1}{\lambda_n}(F_1 * F_2 * \dots * F_n)$ be ρ -convergent, then the ρ -limit is called a *limit law* ⁽²²⁾.

Theorem 3.3. *Limit laws are infinitely divisible* ⁽²³⁾.

(21) $\lambda(\times)E$ means $E(a; a \in E)$.

(22), (23) See P. Lévy: loc. cit. (1) p. 192.

First we shall prove a lemma due to A. Wintner⁽²⁴⁾.

Lemma 3.1. *If $\{H_n * G_n\}$ and $\{H_n\}$ ρ -converge to the same limit H , $\{G_n\}$ will ρ -converge to E .*

Proof of the lemma. We shall make use of *reductio ad absurdum*. Assume that $\{G_n\}$ should not converge to E . Then $\{G_n\}$ will have a subsequence $\{G_{n'}\}$ such that

$$(37) \quad \rho(G_{n'}, E) \geq \varepsilon_0 \quad (n = 1, 2, 3, \dots).$$

Let h, g_n, h_n denote respectively the classes containing H, G_n, H_n . Then $h_n * g_n$ will ρ -converge to h . On account of $g_{n'} \geq h_{n'} * g_{n'}$ we can see that $\{g_{n'}\}$ has a ρ -convergent subsequence $\{g_{n''}\}$, if we recall what was stated at the beginning of the proof of Theorem 3.2. Let g denote the ρ -limit of $\{g_{n''}\}$. Then we have $h * g = h$ and so $g = e$, and so we can find a sequence $\{m_n\}$ of real numbers such that

$$(38) \quad G_{n''} * E - m_n \rightarrow E \quad (n \rightarrow \infty).$$

Accordingly we have

$$(39) \quad H_{n''} * G_{n''} * E - m_n \rightarrow H * E = H \quad (n \rightarrow \infty).$$

But by the hypothesis we have

$$(40) \quad H_{n''} * G_{n''} \rightarrow H.$$

By (39) and (40) we have $m_n \rightarrow 0$, and so finally by (38)

$$(41) \quad G_{n''} \rightarrow E \quad (n \rightarrow \infty),$$

in contradiction with (37).

Proof of the Theorem 3.3. It is sufficient to deduce, from the following three conditions, that F is infinitely divisible:

$$\left. \begin{aligned} (42) \quad & \frac{1}{\lambda_n} (F_1 * F_2 * \dots * F_n) \rightarrow F \\ (43) \quad & \frac{\lambda_{n+1}}{\lambda_n} \rightarrow 1 \\ (44) \quad & \lambda_n \rightarrow \infty \end{aligned} \right\} (n \rightarrow \infty).$$

⁽²⁴⁾ A. Wintner: Asymptotic distributions and infinite convolutions (1937-1938), Lemma 5.1.

(42) means

$$(45) \quad \frac{\lambda_{n-1}}{\lambda_n} \cdot \frac{1}{\lambda_{n-1}} (F_1 * F_2 * \dots * F_{n-1}) * \frac{1}{\lambda_n} F_n \rightarrow F.$$

By Lemma 3.1 we can deduce from (43) and (45) that $\frac{1}{\lambda_n} F_n$ converges to E , namely that there exists a natural number $K(\mathfrak{E})$ for any positive number $\mathfrak{E}$ such that $n > K(\mathfrak{E})$ may imply $\rho\left(\frac{1}{\lambda_n} F_n, E\right) < \mathfrak{E}$. Let $\{n'\}$ denote the subsequence of $\{n\}$ such that $m < n'$ may imply $\lambda_m < \lambda_{n'}$; the existence of $\{n'\}$ follows at once from (3). Then we can see that $n' \geq k > K(\mathfrak{E})$ implies

$$(46) \quad \rho\left(\frac{1}{\lambda_{n'}} F_k, E\right) \leq \rho\left(\frac{1}{\lambda_k} F_k, E\right) < \mathfrak{E}.$$

$\lambda_{n'}$ increasing indefinitely with n , we can find a natural number $N(\mathfrak{E}) (> K(\mathfrak{E}))$ such that $n > N(\mathfrak{E})$ may imply

$$(47) \quad \rho\left(\frac{1}{\lambda_{n'}} F_k, E\right) < \mathfrak{E} \quad (k = 1, 2, 3, \dots, K(\mathfrak{E})).$$

From (46) and (47) we can deduce that $n > N(\mathfrak{E})$ implies

$$(48) \quad \Delta\left(\frac{1}{\lambda_{n'}} (F_1 * F_2 * \dots * F_{n'})\right) > \frac{1}{\mathfrak{E}}.$$

According to Theorem 3.2, (48) shows that the ρ -limit F of $\left\{\frac{1}{\lambda_{n'}} (F_1 * \dots * F_{n'})\right\}$ is infinitely divisible.

IV. A canonical form of $\mathfrak{L}$ -processes.

Let $\mathfrak{L} \equiv (\mathfrak{L}_t; 0 \leq t \leq 1)$ be an $\mathfrak{L}$ -process on (Ω, P) . If $\mathfrak{L}_{t+0}(\omega) - \mathfrak{L}_t(\omega) \equiv u \neq 0$, then we say that $\mathfrak{L}(\omega) \equiv (\mathfrak{L}_t(\omega); 0 \leq t \leq 1)$ has a *jump* with the *height* u at t . We shall represent such jump of $\mathfrak{L}(\omega)$ by the point (tu) in the (tu) -plane. By *jumps in a certain (tu) -set E* we shall understand the jumps which may be represented by the points in E .

In this paper we shall often need to indicate the property of a set of real numbers that the set is not only bounded but also separated from 0 by a positive distance. For this purpose we shall make use of the sign $*$; for example, "*set**" means "set with the above property". Terms such as "*interval**" "*Borel set**" are to be understood similarly.

Let $\mathfrak{P}$ be the combination of $\mathfrak{P}$ -processes $\mathfrak{P}^U = (\mathfrak{P}_t^U; 0 \leq t \leq 1)$ on (Ω, P) , U running over all Borel sets*. $\mathfrak{P}$ will be called a *differential system of $\mathfrak{P}$ -processes*, if it satisfies the following two conditions:

(1) for any system of disjointed Borel sets* $U_k (k = 1, 2, \dots)$ such that their sum U may be also a Borel set*, we have $\mathfrak{P}^U = \sum_{k=1}^{\infty} \mathfrak{P}^{U_k}$, namely $\mathfrak{P}_t^U(\omega) = \sum_{k=1}^{\infty} \mathfrak{P}_t^{U_k}(\omega)$ for any t in $[0, 1]$ and for any ω in Ω .

(2) for any system of disjointed Borel sets* $U_k (k = 1, 2, \dots)$, $\mathfrak{P}^{U_1}, \mathfrak{P}^{U_2}, \dots, \mathfrak{P}^{U_n}$ are independent.

We can consider $\mathfrak{P}$ as a measure defined on Borel sets* which takes values of $\mathfrak{P}$ -processes. We can consider the integration based on this measure.

Theorem 4.1. Let $\mathfrak{P}$ be a differential system of $\mathfrak{P}$ -processes $\mathfrak{P}^U (\equiv (\mathfrak{P}_t^U; 0 \leq t \leq 1))$. Then $\int_a^b u \mathfrak{P}^{du(25)}(ab > 0)$ is an $\mathfrak{P}$ -process the number of whose jumps in a (tu) -set $[0t] \times U$ is equal to $\mathfrak{P}_t^U$ for any number t in $[0, 1]$ and for any Borel set U contained in (ab) .

Proof. We shall prove this theorem in the case $a > 0, b > 0$.

Let $t_1, t_2, \dots, t_n$ denote the positions of the jumps of $\mathfrak{P}^{(ab)}(\omega)$. Then we have

$$(3) \quad \mathfrak{P}_{t_i+0}^{(ab)}(\omega) - \mathfrak{P}_{t_i-0}^{(ab)}(\omega) = 1 \quad (i = 1, 2, \dots, n).$$

On account of (1), $\mathfrak{P}_{t_i+0}^{(au)}(\omega)$, as well as $\mathfrak{P}_{t_i-0}^{(au)}$, is a function of u continuous on the right hand, and so it is the case with $\mathfrak{P}_{t_i+0}^{(au)}(\omega) - \mathfrak{P}_{t_i-0}^{(au)}$. By the definition this function is monotone increasing and takes 0 or 1. Therefore we can find a number u_i in $[ab]$ for $i = 1, 2, \dots, n$ such that

$$(4) \quad \begin{aligned} \mathfrak{P}_{t_i+0}^{(au)}(\omega) - \mathfrak{P}_{t_i-0}^{(au)}(\omega) &= 0 \quad (u < u_i) \\ &= 1 \quad (u \geq u_i). \end{aligned}$$

Consequently we have

$$(5) \quad \mathfrak{P}_t^U(\omega) = \sum_{\substack{u_i \in U \\ 0 \leq t_i \leq t}} 1,$$

$$(6) \quad \int_a^b u \mathfrak{P}_t^{du} = \sum_{0 \leq t_i \leq t} u_i.$$

(25) $\int_a^b$ means $\int_{a+0}^{b+0}$.

For any fixed point t , $\int_a^b u \mathfrak{P}_t^{du}(\omega)$ is a P -measurable function of ω , i.e. a random variable, since it is the limit of a bounded and monotone sequence of the random variables:

$$(7) \quad \sum_{a < \frac{K}{2^n} < b} \frac{K}{2^n} \mathfrak{P}_t \left(\frac{K}{2^n}, \frac{K+1}{2^n} \right] (\omega) \quad (n = 1, 2, \dots).$$

It follows at once from (2) that $\int_a^b u \mathfrak{P}^{du}$ is a stochastic differential process. The above expression (6) shows that $\int_a^b u \mathfrak{P}_t^{du}(\omega)$ increases only by jumps when t increases. We can see by (5) and (6) that

$$(8) \quad E \left(\omega ; \int_a^b u \mathfrak{P}_{t+0}^{du}(\omega) - \int_a^b u \mathfrak{P}_{t-0}^{du}(\omega) \neq 0 \right) \\ = E \left(\omega ; \mathfrak{P}_{t+0}^{(ab)}(\omega) - \mathfrak{P}_{t-0}^{(ab)}(\omega) \neq 0 \right).$$

As the right hand side of (8) has P -measure 0 on account of the definition of $\mathfrak{P}$ -process, so does also the left hand side. Consequently we shall see on account of (5) and (6) that $\int_a^b u \mathfrak{P}^{du}$ is an $\mathfrak{L}$ -process the number of whose jumps in the (tu) -set $[0t] \times U$ is equal to $\mathfrak{P}_t^U$.

Next we shall discuss the behaviour of the integral in the vicinity of $u = \infty$.

Theorem 4.2. *Let $\mathfrak{P}$ be a differential system of $\mathfrak{P}$ -processes satisfying the condition:*

$$(9) \quad \int_a^\infty \overline{\mathfrak{P}_1^{du}} < \infty \text{ (or } \int_{-\infty}^{-a} \overline{\mathfrak{P}_1^{du}} < \infty) \quad (a > 0)$$

Then $\int_a^\infty u \mathfrak{P}^{du}$ (or $\int_{-\infty}^{-a} u \mathfrak{P}^{du}$)⁽²⁶⁾ is an $\mathfrak{L}$ -process in the larger sense, the number of whose jumps in a (tu) -set $[0t] \times U$ is equal to $\mathfrak{P}_t^U$ for any number t in $[01]$ and for any Borel set U in (a, ∞) (or $(-\infty, -a]$) with probability 1.

Proof. On account of Theorem 4.1, it is sufficient to prove that $\mathfrak{P}_1^{(a\infty)}$ is finite with probability 1. It is clear that

$$(10) \quad P(\mathfrak{P}_1^{(ab)} \leq k) = \sum_{i=0}^k e^{-\tau(b)} \frac{(\tau(b))^i}{i!} \quad (k = 1, 2, \dots),$$

(26) $\int_a^\infty u \mathfrak{P}^{du} = \left(\lim_{b \rightarrow \infty} \int_a^b u \mathfrak{P}_t^{du}(\omega); \omega \in \Omega; 0 \leq t \leq 1 \right)$

where $\eta(b)$ denotes the mean value of $\mathfrak{P}_1^{[ab]}$. $\mathfrak{P}_1^{[ab]}$ is monotone increasing in b and so $\mathfrak{P}_1^{(a\infty)}(\omega)$ will exist, if it is permitted to take infinity. Accordingly we have

$$(11) \quad P(\mathfrak{P}_1^{(a\infty)} \leq k) = \lim_{b \rightarrow \infty} P(\mathfrak{P}_1^{[ab]} \leq k) = \sum_{i=0}^k e^{-\eta(\infty)} \frac{(\eta(\infty))^i}{i!}$$

since $\eta(\infty) < \infty$ by virtue of (9). Consequently we have

$$(12) \quad P(\mathfrak{P}_1^{(a\infty)} < \infty) = \sum_{i=0}^{\infty} e^{-\eta(\infty)} \frac{(\eta(\infty))^i}{i!} = 1.$$

We shall now discuss the behaviour of the integral in the vicinity of $u = 0$. In the case that $\int_0^a \mathfrak{P}_1^{du}(a > 0)$ is finite, we can see, as in the above proof, that $\mathfrak{P}_1^{[0a]}$ is finite with probability 1, and so we can easily define $\int_0^a u \mathfrak{P}^{du}$. In a more general case that $\int_0^a \overline{\mathfrak{P}}^{du}(a > 0)$ is finite, we can also define $\int_0^a u \mathfrak{P}^{du}$, but we shall consider here the most general case:

$$(13) \quad \int_{0 < |u| \leq 1} u^2 \overline{\mathfrak{P}}^{du} < \infty.$$

In such a case $\int_0^a u \mathfrak{P}^{du}$ itself is not always finite, but we can establish the following

Theorem 4.3. *Let $\mathfrak{P}$ be a differential system of $\mathfrak{P}$ -processes satisfying (13). Then the sequence of $\mathfrak{L}$ -processes:*

$$(14) \quad \int_{\frac{1}{n}}^1 (u \mathfrak{P}^{du} - u \overline{\mathfrak{P}}^{du}) = \left(\int_{\frac{1}{n}}^1 u \mathfrak{P}_t^{du} - \int_{\frac{1}{n}}^1 u \overline{\mathfrak{P}}_t^{du}; 0 \leq t \leq 1 \right) \quad (n = 1, 2, \dots)$$

will be convergent uniformly in t with probability 1. The limit is an $\mathfrak{L}$ -process in the larger sense, the number of whose jumps in a (tu) -set $([0t] \times U)$ is equal to $\mathfrak{P}_t^U$ for any number t in $[01]$ and for any Borel set U contained in $[01]$ with probability 1.*

Proof. If we put

$$(15) \quad x_n(t) = \int_{\frac{1}{n+1}}^{\frac{1}{n}} u \mathfrak{P}_t^{du} - \int_{\frac{1}{n+1}}^{\frac{1}{n}} u \overline{\mathfrak{P}}_t^{du},$$

we can easily see by the hypothesis that $x_n \equiv (x_n(t); 0 \leq t \leq 1)$ ($n = 1, 2, \dots$) are independent $\mathfrak{L}$ -processes and satisfy

$$(16) \quad \overline{x_n(t)} = 0, \quad \text{and}$$

$$(17) \quad \sum_{n=1}^{\infty} \left(\sigma(x_n(1)) \right)^2 < \infty.$$

In order to prove the first part of the theorem, it is sufficient to show

$$(18) \quad P(\lim_{n \rightarrow \infty} \text{l.u.b.}_{l > m \geq n} \text{l.u.b.}_{0 \leq t \leq 1} \left| \sum_m^l x_i(t) \right| > 0) = 0.$$

On account of (17) there exists a natural number $N(\varepsilon)$ for any assigned positive number ε such that $m > n \geq N(\varepsilon)$ may imply

$$(19) \quad \sum_n^m \left(\sigma(x_i(1)) \right)^2 < \varepsilon^3.$$

x_i ($i = 1, 2, \dots$) being $\mathfrak{L}$ -processes, we have, by 2° of Definition 3.2,

$$(20) \quad \left(\sigma(x_i(t)) \right)^2 + \left(\sigma(x_i(1) - x_i(t)) \right)^2 = \left(\sigma(x_i(1)) \right)^2.$$

From (19), (20) follows that $m > n \geq N(\varepsilon)$ implies

$$(21) \quad \sum_n^m \left(\sigma(x_i(t)) \right)^2 < \varepsilon^3$$

and

$$(22) \quad \sum_n^m \left(\sigma(x_i(1) - x_i(t)) \right)^2 < \varepsilon^3$$

for any number t in $[0, 1]$. x_i ($i = 1, 2, \dots$) being independent, we can deduce from (21) and (22) respectively

$$(21') \quad \left(\sigma\left(\sum_n^m x_i(t)\right) \right)^2 < \varepsilon^3$$

$$(22') \quad \left(\sigma\left(\sum_n^m (x_i(1) - x_i(t))\right) \right)^2 < \varepsilon^3.$$

By the inequality of Bienaymé (21') and (22') will imply respectively

$$(21'') \quad P\left(\left|\sum_n^m x_i(t)\right| \geq \varepsilon\right) \leq \varepsilon$$

$$(22'') \quad P\left(\left|\sum_n^m (x_i(1) - x_i(t))\right| \geq \varepsilon\right) \leq \varepsilon.$$

We shall now prove that $m > n > N(\varepsilon)$ may imply

$$(23) \quad P(\text{l.u.b.}_{0 \leq t \leq 1} \left| \sum_n^m x_i(t) \right| \geq 2\varepsilon) \leq \frac{\varepsilon}{1 - \varepsilon}.$$

For brevity we shall denote $\sum_n^m x_i(t)$ by $y(t)$. Then $y \equiv (y(t); 0 \leq t \leq 1)$ is an $\mathfrak{L}$ -process and satisfies

$$(21''') \quad P(|y(t)| \geq \varepsilon) < \varepsilon, \quad \text{and}$$

$$(22''') \quad P(|y(1) - y(t)| \geq \varepsilon) < \varepsilon.$$

But (23) may be written also as

$$(23') \quad P(\text{l.u.b.}_{0 \leq t \leq 1} |y(t)| \geq 2\varepsilon) < \frac{\varepsilon}{1 - \varepsilon}.$$

Let $\{t_i\}$ be an enumerable set dense in the interval $[0, 1]$. y being an $\mathfrak{L}$ -process, we can see, by 1° of Definition 3.2, that

$$(24) \quad \text{l.u.b.}_{0 \leq t \leq 1} |y(t)| = \text{l.u.b.}_{i=1, 2, \dots} |y(t_i)| = \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} |y(t_i)|$$

It is sufficient to prove, instead of (23'), that

$$(25) \quad P(\max_{1 \leq i \leq n} |y(t_i)| \geq 2\varepsilon) \leq \frac{\varepsilon}{1 - \varepsilon},$$

since the ω -set $(\max_{1 \leq i \leq n} |y(t_i)| \geq 2\varepsilon)$ increases with n .

We shall arrange the sequence $t_1, t_2, \dots, t_n$, according to the order of magnitude and write as $s_1, s_2, \dots, s_n$. Then $y(1) - y(s_i)$ is independent of $y(s_1), y(s_2), \dots, y(s_i)$. Therefore, if we put

$$(26) \quad A_i = (|y(s_i)| \geq 2\varepsilon) \prod_{\sigma=1}^{i-1} (|y(s_\sigma)| < 2\varepsilon) \quad (i = 1, 2, \dots, n)$$

$$(27) \quad B_i = (|y(1) - y(s_i)| \leq \varepsilon) \quad (i = 1, 2, \dots, n),$$

we have

$$(28) \quad P(A_i B_i) = P(A_i)P(B_i) \quad (i = 1, 2, \dots, n).$$

As $(A_1, A_2, \dots, A_n)$ is a disjointed system of ω -sets on account of (26,) so is $(A_1 B_1, A_2 B_2, \dots, A_n B_n)$. Therefore we have

$$\begin{aligned} (29) \quad & P(A_1 B_1 + A_2 B_2 + \dots + A_n B_n) \\ &= P(A_1 B_1) + P(A_2 B_2) + \dots + P(A_n B_n) \\ &= P(A_1)P(B_1) + P(A_2)P(B_2) + \dots + P(A_n)P(B_n) \quad (\text{by (28)}) \\ &> (1 - \varepsilon)(P(A_1) + P(A_2) + \dots + P(A_n)) \quad (\text{by (22''')}) \\ &= (1 - \varepsilon)P(A_1 + A_2 + \dots + A_n). \end{aligned}$$

But we have, by (26) and (27),

$$(30) \quad A_1 B_1 + A_2 B_2 + \cdots + A_n B_n < (|y(1)| \geq \varepsilon)$$

$$(31) \quad A_1 + A_2 + \cdots + A_n = (\max_{1 \leq i \leq n} |y(s_i)| \geq 2\varepsilon) = (\max_{1 \leq i \leq n} |y(t_i)| \geq 2\varepsilon)$$

From (21'''), (29), (30) and (31) follows (25).

As $x_1, x_2, \dots$ are independent $\mathfrak{L}$ -processes, we can analogously deduce from (23) that $n > N(\varepsilon)$ implies

$$(32) \quad P(\text{l.u.b.}_{m>n} \text{ l.u.b.}_{0 \leq t \leq 1} |\sum_n^m x_i(t)| \geq 4\varepsilon) \leq \frac{\frac{\varepsilon}{1-\varepsilon}}{1-\frac{\varepsilon}{1-\varepsilon}} = \frac{\varepsilon}{1-2\varepsilon},$$

if we recall the well-known inequality:

$$\text{l.u.b.}_t |a_t + b_t| \geq \text{l.u.b.}_t |a_t| - \text{l.u.b.}_t |b_t|.$$

From (32) follows

$$(33) \quad P(\text{l.u.b.}_{l>m>n} \text{ l.u.b.}_{0 \leq t \leq 1} |\sum_m^l x_i(t)| \geq 8\varepsilon) \leq \frac{2\varepsilon}{1-2\varepsilon},$$

and accordingly

$$(34) \quad P(\lim_{n \rightarrow \infty} \text{l.u.b.}_{l>m>n} \text{ l.u.b.}_{0 \leq t \leq 1} |\sum_m^l x_i(t)| \geq 8\varepsilon) \leq \frac{2\varepsilon}{1-2\varepsilon},$$

from which follows (18).

In order to prove the second part of the theorem it will be sufficient to show that the number of the jumps of the process $\int_{\frac{1}{n}}^1 u \mathfrak{P}^{du} - \int_{\frac{1}{n}}^1 u \overline{\mathfrak{P}}^{du}$ in the (tu) -set $[0t] \times U$ is equal to $\mathfrak{P}_t^U$ whenever the integer n is so great that $[\frac{1}{n}, 1]$ may contain U ; if this fact is established for such n , it is clear that it holds also with regard to the limit process $\lim_{n \rightarrow \infty} \left(\int_{\frac{1}{n}}^1 u \mathfrak{P}^{du} - \int_{\frac{1}{n}}^1 u \overline{\mathfrak{P}}^{du} \right)$ on account of the uniform convergency just proved above. The above statement being verified for the first term of the above process $\int_{\frac{1}{n}}^1 u \mathfrak{P}^{du} - \int_{\frac{1}{n}}^1 u \overline{\mathfrak{P}}^{du}$ by virtue of Theorem 4.1, we need only prove that the second term $-\int_{\frac{1}{n}}^1 u \overline{\mathfrak{P}}^{du}$ is continuous in t . But this follows at once from the fact that $\mathfrak{P}_t^{[\frac{1}{n}, 1]}$ is continuous in t because of the condition 1° stated in Definition 3.2. Thus the theorem is proved.

Theorem 4.4. Let $\mathfrak{L}$ be a differential system of $\mathfrak{P}$ -processes $(\mathfrak{P}_t^U; 0 \leq t \leq 1)$ on (Ω, P) satisfying the conditions (9), (13), and $\mathfrak{G} \equiv (\mathfrak{G}_t; 0 \leq t \leq 1)$ a $\mathfrak{G}$ -process on (Ω, P) independent of the system $\mathfrak{P}$. Then the process:

$$\left(\mathfrak{G}_t + \lim_{\substack{n \rightarrow \infty \\ n > |u| > \frac{1}{n}}} \int \left(u \mathfrak{P}_t^{du} - \frac{u}{1+u^2} \overline{\mathfrak{P}_t^{du}} \right); 0 \leq t \leq 1 \right)$$

is an $\mathfrak{L}$ -process in the larger sense, the number of whose jumps in the (tu) -set $[0t] \times U$ is equal to $\mathfrak{P}_t^U$ for any number t in $(0, 1)$ and for any Borel set* U with probability 1.

Proof. Since the sequence of $\int \frac{u}{1+u^2} \overline{\mathfrak{P}_t^{du}} (n=2, 3, \dots)$ converges uniformly in t on account of (9), the limit $\int \frac{u}{1+u^2} \overline{\mathfrak{P}_t^{du}}$ is continuous in t , and $\int \left(u - \frac{u}{1+u^2} \right) \overline{\mathfrak{P}_t^{du}}$ is also continuous on account of (13). Therefore

we shall obtain this theorem by Theorem 4.1, 4.2, and 4.3.

Next we shall prove the converse of theorem 4.4:

Theorem 4.5. Let $\mathfrak{L} \equiv (\mathfrak{L}_t; 0 \leq t \leq 1)$ be an $\mathfrak{L}$ -process on (Ω, P) . Then there exists a differential system $\mathfrak{P}$ of $\mathfrak{P}$ -processes $(\mathfrak{P}_t^U; 0 \leq t \leq 1)$ with the properties (9), (13), and a $\mathfrak{G}$ -process $\mathfrak{G}$ independent of the system $\mathfrak{P}$ such that

$$(35) \quad \mathfrak{L}_t = \mathfrak{G}_t + \lim_{\substack{n \rightarrow \infty \\ n > |u| > \frac{1}{n}}} \int \left(u \mathfrak{P}_t^{du} - \frac{u}{1+u^2} \overline{\mathfrak{P}_t^{du}} \right)$$

may hold for any number t with probability 1.

Note. According to this theorem it may be justified to call the $\mathfrak{G}$ -process appearing here the continuous part of $\mathfrak{L}$, and the differential system $\mathfrak{P}$ the discontinuous part of $\mathfrak{L}$.

Proof. Let $\mathfrak{P}_t^U$ denote the number of the jumps of $\mathfrak{L}$ in the (tu) -set $[0t] \times U$, where U is a Borel set*. Then it is clear that $\mathfrak{P}_t^U(\omega)$ is always finite. First we shall prove that $\mathfrak{P}_{t_2}^U - \mathfrak{P}_{t_1}^U$ is a B -measurable function⁽²⁷⁾ of the $\mathfrak{L}$ -process $(\mathfrak{L}_s - \mathfrak{L}_{t_1}; t_1 < s \leq t_2)$. For this purpose it is sufficient to consider the special case that U is a set such as $[\lambda^\infty) (\lambda > 0)$. By the definition we have

$$(36) \quad (\mathfrak{P}_{t_2}^{[\lambda^\infty)} - \mathfrak{P}_{t_1}^{[\lambda^\infty)} \geq 1) = \prod_n \sum_{\substack{r_i - r_j < \frac{1}{n} \\ t_1 < r_j < r_i < t_2}} \left(\mathfrak{L}_{r_i} - \mathfrak{L}_{r_j} > \lambda - \frac{1}{n} \right) + N_1,$$

(27) See Definition 1.2.

where $(r_1, r_2, \dots)$ is an enumerable set dense in (01), and N_1 denotes the set of all ω -elements for which $\mathfrak{L}_t$ has a jump at t_2 . $P(N_1) = 0$ is clear on account of the condition 2° of Definition 3.2. Similarly we have

$$(37) \quad (\mathfrak{P}_{t_2}^{[\lambda\infty]} - \mathfrak{P}_{t_1}^{[\lambda\infty]} \geq 2) = \sum_k \prod_n \sum_{\left(\begin{array}{l} t_1 < r_j < r_i < r_{j'} < r_{i'} < t_2 \\ r_i - r_j < \frac{1}{n} \\ r_{i'} - r_{j'} < \frac{1}{n} \\ r_{j'} - r_i > \frac{1}{k} \end{array} \right)} \left(\mathfrak{L}_{r_i} - \mathfrak{L}_{r_j} > \lambda - \frac{1}{n} \right) \wedge \left(\mathfrak{L}_{r_{i'}} - \mathfrak{L}_{r_{j'}} > \lambda - \frac{1}{n} \right) + N_2,$$

where $P(N_2) = 0$. The analogous identity will hold also for $(\mathfrak{P}_{t_2}^{[\lambda\infty]} - \mathfrak{P}_{t_1}^{[\lambda\infty]} \geq k)$ ($k = 3, 4, \dots$). As $\mathfrak{P}_{t_2}^{[\lambda\infty]} - \mathfrak{P}_{t_1}^{[\lambda\infty]}$ takes only 0 or natural numbers, those identities show that $\mathfrak{P}_{t_2}^{[\lambda\infty]} - \mathfrak{P}_{t_1}^{[\lambda\infty]}$ is a B-measurable function of $(\mathfrak{L}_s - \mathfrak{L}_{t_1}; t_1 < s \leq t_2)$. As $\mathfrak{L}$ is differential, the $\mathfrak{L}$ -processes $(\mathfrak{L}_s - \mathfrak{L}_{t_i}; t_i < s \leq s_i)$ ($i = 1, 2, \dots, n$) will be independent for any system of non-overlapping intervals (t_i, s_i) ($i = 1, 2, \dots, n$), and accordingly $\mathfrak{P}_{s_i}^U - \mathfrak{P}_{t_i}^U$ ($i = 1, 2, \dots, n$) will also be independent, that is to say, $(\mathfrak{P}_t^U; 0 \leq t \leq 1)$ is a differential process. It is clear that $\mathfrak{P}^U = (\mathfrak{P}_t^U; 0 \leq t \leq 1)$ satisfies the condition stated in Definition 3.4. The condition 2° in Definition 3.2 will hold for $\mathfrak{P}^U$ as it holds for $\mathfrak{L}$. Therefore $\mathfrak{L}^U$ is a $\mathfrak{P}$ -process.

Next we shall prove that $x = (\mathfrak{L}_t - \int_U^t u \mathfrak{P}_t^{du}; 0 \leq t \leq 1)$ and $\mathfrak{P}^U$ are independent for any Borel set* U . x and $\mathfrak{P}^U$ being both differential, we need only prove that

$$(38) \quad P\left((x_s - x_t \in B)(\mathfrak{P}_s^U - \mathfrak{P}_t^U = k)\right) = P(x_s - x_t \in B)P(\mathfrak{P}_s^U - \mathfrak{P}_t^U = k)$$

holds for $t < s$, for any Borel set B , and for $k = 0, 1, 2, \dots$.

Now, $\mathfrak{P}_s^U - \mathfrak{P}_t^U$ being subject to a Poissonian distribution, we have

$$(39) \quad P(\mathfrak{P}_s^U - \mathfrak{P}_t^U = k) = e^{-\eta} \frac{\eta^k}{|k|} \quad (\eta = \overline{\mathfrak{P}_s^U} - \overline{\mathfrak{P}_t^U})$$

$$= 1 \quad (\text{if } \eta = 0, k = 0)$$

In case $\eta = 0$, (38) is clear. In case $\eta \neq 0$ we have

$$(40) \quad P(\mathfrak{P}_s^U - \mathfrak{P}_t^U = k) \neq 0 \quad (k = 0, 1, 2, \dots).$$

In order to prove (38) it is sufficient to show that the conditional probability:

$$(41) \quad P_k(x_s - x_t \in B) \equiv P(x_s - x_t \in B / \mathfrak{P}_s^U - \mathfrak{P}_t^U = k) \\ = \frac{P((x_s - x_t \in B)(\mathfrak{P}_s^U - \mathfrak{P}_t^U = k))}{P(\mathfrak{P}_s^U - \mathfrak{P}_t^U = k)}$$

is independent of k , namely, that

$$(42) \quad P_0(x_s - x_t \in B) = P_k(x_s - x_t \in B)$$

holds for $k = 1, 2, \dots$. We shall prove (42) in case $k = 1$, as we can do analogously in the other cases. We can assume $t = 0$ and $s = 1$ with no loss of generality. For brevity we shall adopt the following notations:

$$(43) \quad \mathcal{Q}^0 \equiv (\mathfrak{P}_1^U = 0)$$

$$(44) \quad \mathcal{Q}^1 \equiv (\mathfrak{P}_1^U = 1)$$

$$(45) \quad \mathcal{Q}_{ni}^0 \equiv \left(\mathfrak{P}_{\frac{i}{n}}^U - \mathfrak{P}_{\frac{i-1}{n}}^U = 0 \right) \quad (i = 1, 2, \dots, n)$$

$$(46) \quad \mathcal{Q}_{ni}^1 \equiv \left(\mathfrak{P}_{\frac{i}{n}}^U - \mathfrak{P}_{\frac{i-1}{n}}^U = 1 \right) \quad (i = 1, 2, \dots, n),$$

$$(47) \quad \overline{\mathcal{Q}}_{ni}^0 \equiv \mathcal{Q}_{n1}^0 \mathcal{Q}_{n2}^0 \dots \mathcal{Q}_{n, i-1}^0 \mathcal{Q}_{n, i+1}^0 \dots \mathcal{Q}_{nn}^0 \quad (i = 1, 2, \dots, n).$$

Then we have

$$(48) \quad \mathcal{Q}^0 = \mathcal{Q}_{n1}^0 \mathcal{Q}_{n2}^0 \dots \mathcal{Q}_{nn}^0 = \mathcal{Q}_{ni}^0 \overline{\mathcal{Q}}_{ni}^0 \quad (i = 1, 2, \dots, n),$$

and

$$(49) \quad \mathcal{Q}^1 = \sum_{i=1}^n \mathcal{Q}_{ni}^1 \overline{\mathcal{Q}}_{ni}^0.$$

If we put for $\omega \in \mathcal{Q}_{ni}^1 \overline{\mathcal{Q}}_{ni}^0$

$$(50) \quad y^{(n)}(\omega) = \mathfrak{Z}_1(\omega) - \mathfrak{Z}_{\frac{i}{n}}(\omega) + \mathfrak{Z}_{\frac{i-1}{n}}(\omega) \quad (i = 1, 2, \dots, n),$$

then we have, for $\omega \in \mathcal{Q}^1$,

$$(51) \quad \lim_{n \rightarrow \infty} y^{(n)}(\omega) = x_1(\omega) \left(\equiv \mathfrak{Z}_1 - \int_U u \mathfrak{P}_1^{du} \right),$$

and so

$$(52) \quad F_{x_1}^1 = \lim_{n \rightarrow \infty} F_{y^{(n)}}^1$$

$F_{x_1}^1$ and $F_{y^{(n)}}^1$ being respectively the probability law of x_1 and $y^{(n)}$ considered as a random variable on $(\mathcal{Q}^1 P_1)$.

Let A_i denote the ω -set $\left(\mathfrak{L}_1 - \mathfrak{L}_{\frac{i}{n}} + \mathfrak{L}_{\frac{i-1}{n}} \in B\right)$ ($i = 1, 2, \dots, n$). Then we have

$$(53) \quad P_1(y^{(n)} \in B) = \frac{1}{P(\mathcal{Q}^1)} \sum_{i=1}^n P(\mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0) \frac{P(A_i \mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0)}{P(\mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0)},$$

and

$$(54) \quad \frac{P(A_i \mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0)}{P(\mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0)} = \frac{P(A_i \bar{\mathcal{Q}}_{ni}^0) P(\mathcal{Q}_{ni}^1)}{P(\mathcal{Q}_{ni}^1) P(\bar{\mathcal{Q}}_{ni}^0)} = \frac{P(A_i \bar{\mathcal{Q}}_{ni}^0)}{P(\bar{\mathcal{Q}}_{ni}^0)};$$

the first half of (54) follows from the hypothesis that $\mathfrak{L}$ is differential. Similarly we shall obtain

$$(55) \quad P_0(A_i) = \frac{P(A_i \bar{\mathcal{Q}}_{ni}^0)}{P(\bar{\mathcal{Q}}_{ni}^0)}.$$

From (53), (54) and (55) follows

$$(56) \quad P_1(y^{(n)} \in B) = \frac{1}{P(\mathcal{Q}^1)} \sum_{i=1}^n P(\mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0) P_0(A_i)$$

i.e.

$$(57) \quad F_{y^{(n)}}^1 = \frac{1}{P(\mathcal{Q}^1)} \sum_{i=1}^n P(\mathcal{Q}_{ni}^1 \bar{\mathcal{Q}}_{ni}^0) F_{\mathfrak{L}_1 - \mathfrak{L}_{\frac{i}{n}} + \mathfrak{L}_{\frac{i-1}{n}}}^0,$$

where $F_{\mathfrak{L}_1 - \mathfrak{L}_{\frac{i}{n}} + \mathfrak{L}_{\frac{i-1}{n}}}^0$ is the probability law of $\mathfrak{L}_1 - \mathfrak{L}_{\frac{i}{n}} + \mathfrak{L}_{\frac{i-1}{n}}$ considered as a random variable on $(\mathcal{Q}^0, P_0)$.

Now, we shall prove that $\mathfrak{L}$ is an $\mathfrak{L}$ -process on $(\mathcal{Q}^0, P_0)$. If we put

$$(58) \quad \mathcal{Q}_{\tau\sigma} = (\mathfrak{P}_\sigma^U - \mathfrak{P}_\tau^U = 0),$$

we shall have

$$(59) \quad \mathcal{Q}^0 = \mathcal{Q}_{0t_1} \mathcal{Q}_{t_1s_1} \mathcal{Q}_{s_1t_2} \mathcal{Q}_{t_2s_2} \mathcal{Q}_{s_21}.$$

Let c_i denote the ω -set $(\mathfrak{L}_{s_i} - \mathfrak{L}_{t_i} \in B_i)$ ($i = 1, 2$). Then we can obtain

$$(60) \quad P_0(c_1 c_2) = \frac{P(c_1 c_2 \mathcal{Q}^0)}{P(\mathcal{Q}^0)} = \frac{P(c_1 c_2 \mathcal{Q}_{0t_1} \mathcal{Q}_{t_1s_1} \mathcal{Q}_{s_1t_2} \mathcal{Q}_{t_2s_2} \mathcal{Q}_{s_21})}{P(\mathcal{Q}_{0t_1} \mathcal{Q}_{t_1s_1} \mathcal{Q}_{s_1t_2} \mathcal{Q}_{t_2s_2} \mathcal{Q}_{s_21})} \\ = \frac{P(c_1 \mathcal{Q}_{t_1s_1}) P(c_2 \mathcal{Q}_{t_2s_2}) P(\mathcal{Q}_{0t_1}) P(\mathcal{Q}_{s_1t_2}) P(\mathcal{Q}_{s_21})}{P(\mathcal{Q}_{0t_1}) P(\mathcal{Q}_{t_1s_1}) P(\mathcal{Q}_{s_1t_2}) P(\mathcal{Q}_{t_2s_2}) P(\mathcal{Q}_{s_21})}$$

(as $\mathfrak{L}$ is differential).

$$\begin{aligned}
&= \frac{P(c_1 \mathcal{Q}_{t_1 s_1}) P(c_2 \mathcal{Q}_{t_2 s_2})}{P(\mathcal{Q}_{t_1 s_1}) P(\mathcal{Q}_{t_2 s_2})} \\
&= \frac{P(\mathcal{Q}_{0 t_1}) P(c_1 \mathcal{Q}_{t_1 s_1}) P(\mathcal{Q}_{s_1})}{P(\mathcal{Q}_{0 t_1}) P(\mathcal{Q}_{t_1 s_1}) P(\mathcal{Q}_{s_1})} \cdot \frac{P(\mathcal{Q}_{0 t_2}) P(c_2 \mathcal{Q}_{t_2 s_2}) P(\mathcal{Q}_{s_2})}{P(\mathcal{Q}_{0 t_2}) P(\mathcal{Q}_{t_2 s_2}) P(\mathcal{Q}_{s_2})} \\
&= \frac{P(\mathcal{Q}^0 c_1)}{P(\mathcal{Q}^0)} \frac{P(\mathcal{Q}^0 c_2)}{P(\mathcal{Q}^0)} \quad (\text{as } \mathcal{Q} \text{ is differential}) \\
&= P_0(c_1) P_0(c_2) .
\end{aligned}$$

Generally we shall have, for non-overlapping t -intervals $(t_i s_i) (i = 1, 2, \dots, n)$,

$$(61) \quad P_0(c_1 c_2 \dots c_n) = P_0(c_1) P_0(c_2) \dots P_0(c_n) ,$$

where $c_i \equiv (\mathcal{Q}_{s_i} - \mathcal{Q}_{t_i} \in B_i) (i = 1, 2, \dots, n)$. This shows that $\mathcal{Q}$ is differential on $(\mathcal{Q}^0, P_0)$. $\mathcal{Q}$ is clearly an $(\mathfrak{F}_{01}, \mathfrak{B}_{01})$ -valued random variable⁽²⁸⁾. The condition 1° in Definition 3.2 can be verified as follows:

$$P_0(|\mathcal{Q}_s - \mathcal{Q}_t| > \varepsilon) \leq \frac{P(|\mathcal{Q}_s - \mathcal{Q}_t| > \varepsilon)}{P(\mathcal{Q}_0)} \rightarrow 0 \quad (\text{if } s \rightarrow t) .$$

We have thus established that $\mathcal{Q}$ is an $\mathcal{Q}$ -process on $(\mathcal{Q}^0, P_0)$. Therefore we can see that

$$(62) \quad F_{\mathcal{Q}_1 - \mathcal{Q}_s + \mathcal{Q}_t}^0 = F_{\mathcal{Q}_1 - \mathcal{Q}_s}^0 * F_{\mathcal{Q}_t}^0 \quad (t < s) ,$$

and that $F_{\mathcal{Q}_t}$ is ρ -continuous in t and consequently that $F_{\mathcal{Q}_1 - \mathcal{Q}_s + \mathcal{Q}_t}^0$ is also continuous in t in the (ts) -region: $0 \leq t \leq s \leq 1$. By the covering theorem of Heine-Borel we have, in the (ts) -region:

$$\begin{aligned}
&0 \leq s - t \leq \frac{1}{n} \quad 0 \leq s \leq t \leq 1 , \\
(63) \quad &\rho(F_{\mathcal{Q}_1 - \mathcal{Q}_s + \mathcal{Q}_t}^0, F_{\mathcal{Q}_1}^0) < \varepsilon_n ,
\end{aligned}$$

where ε_n is independent of t, s and converges to 0 with $\frac{1}{n}$. From (57) and (63) follows

$$(64) \quad \rho(F_{y^{(n)}}^1, F_{\mathcal{Q}_1}^0) < \varepsilon_n .$$

As $x_1 = \mathcal{Q}_1$ on $\mathcal{Q}_0$, we have, when n tends to infinity,

⁽²⁸⁾ See the beginning of Chapter III.

$$(65) \quad \rho(F_{x_1}^1, F_{x_1}^0) = 0 \quad \text{i.e.} \quad F_{x_1}^1 = F_{x_1}^0$$

by virtue of (52).

Now, we shall prove that $\mathfrak{P}^{U_1}, \mathfrak{P}^{U_2}, \dots, \mathfrak{P}^{U_n}$ and

$$\mathfrak{L} - \int_U u \mathfrak{P}^{du} \quad (U = U_1 + U_2 + \dots + U_n)$$

are independent if $(U_1, U_2, \dots, U_n)$ is a system of disjointed Borel sets*. By the above statement we can see that $\mathfrak{P}^{U_1}$ and $\mathfrak{P} - \int_{U_1} u \mathfrak{P}^{du}$ are independent and that $\mathfrak{P}^{U_2}, \mathfrak{P}^{U_3}, \dots, \mathfrak{P}^{U_n}$ and $\mathfrak{L} - \int_U u \mathfrak{P}^{du}$ are B -measurable functions of $\mathfrak{L} - \int_{U_1} u \mathfrak{P}^{du}$. And so $\mathfrak{P}^{U_1}$ and the combination $(\mathfrak{P}^{U_2}, \mathfrak{P}^{U_3}, \dots, \mathfrak{P}^{U_n}, \mathfrak{L} - \int_U u \mathfrak{P}^{du})$ are independent. Similarly we shall see that $\mathfrak{P}^{U_i}$ and $(\mathfrak{P}^{U_{i+1}}, \mathfrak{P}^{U_{i+2}}, \dots, \mathfrak{P}^{U_n}, \mathfrak{L} - \int_U u \mathfrak{P}^{du})$ are independent ($i = 2, 3, \dots, n$). Therefore $\mathfrak{P}^{U_1}, \mathfrak{P}^{U_2}, \dots, \mathfrak{P}^{U_n}, \mathfrak{L} - \int_U u \mathfrak{P}^{du}$ are independent on account of Corollary 1.2. Consequently we can conclude that the combination of $\mathfrak{P}^U$ for U running over all Borel sets* is a differential system of $\mathfrak{P}$ -processes. This system satisfies (9) on account of the fact that $\mathfrak{P}^{(\lambda, \infty)}$ and $\mathfrak{P}^{(-\infty, -\lambda)}$ ($\lambda > 0$) are also $\mathfrak{P}$ -processes. We shall prove that (13) holds for this differential system of $\mathfrak{P}$ -processes. The following notations will be adopted here for brevity:

F_n : the probability law of $\int_{1 \geq |u| > \frac{1}{n+1}} u \mathfrak{P}_1^{du} \quad (n = 1, 2, 3, \dots)$

f_n : the class containing $F_n \quad (n = 1, 2, \dots)$

g : the class containing the probability law of $\mathfrak{L}_1$.

As $\int_{\frac{1}{i} \geq |u| > \frac{1}{i+1}} u \mathfrak{P}_1^{du} \quad (i = 1, 2, \dots, n-1)$ and $\mathfrak{L} - \int_{1 \geq |u| > \frac{1}{n+1}} u \mathfrak{P}_1^{du}$ are independent, we have

$f_1 \supseteq f_2 \supseteq \dots \supseteq g$. By Corollary 2.1 we can easily show the existence of $\rho\text{-}\lim_{n \rightarrow \infty} f_n$ and accordingly that of $\rho\text{-}\lim_{n \rightarrow \infty} F_n * E_{-m_n}$, where $\{m_n\}$ is a convenient sequence of real numbers. We shall denote this limit by F . Then we have

$$(66) \quad \lim_{n \rightarrow \infty} \varphi_{F_n * E_{-m_n}}(z) = \varphi_F(z).$$

$\mathfrak{P}_1^U$ being subject to a Poissonian distribution, we have

$$(67) \quad \varphi_{F_n * E_{-m_n}}(z) = \exp \left(\int_{\frac{1}{n+1}}^1 (e^{izu} - 1) \overline{\mathfrak{P}_1^{du}} - i m_n z \right),$$

and so

$$(68) \quad \left| \varphi_{F_n * E_{-m_n}}(z) \right| = \exp \left(\Re \int_{\frac{1}{n}}^1 (e^{izu} - 1) \overline{\mathfrak{P}}^{du} \right)^{(29)} \\ \leq \exp \left(-\frac{z^2}{4} \int_{\frac{1}{n}}^1 u^2 \overline{\mathfrak{P}}_1^{du} \right)$$

Assume that $\int u^2 \overline{\mathfrak{P}}_1^{du} = \infty$. Then by virtue of (66) and (68) we have

$$(69) \quad |\varphi_F(z)| = 0 \quad \text{for } |z| > 0.$$

$\varphi_F(z)$ being continuous in z on account of the definition, we shall have

$$(70) \quad \varphi_F(0) = 0.$$

This is contradictory, for $\varphi_F(0) = 1$ follows at once from the definition. Thus we have proved that $\mathfrak{P}$ satisfies the conditions (9) and (13). By Theorem 4.4 we can see that $\left(\lim_{n \rightarrow \infty} \int_{n > |u| > \frac{1}{n}} u \mathfrak{P}_t^{du} - \frac{u}{1+u^2} \overline{\mathfrak{P}}_t^{du} \right); 0 \leq t \leq 1$ is an

$\mathfrak{Q}$ -process in the larger sense, the number of whose jumps in a (tu) -set $[0t] \times U$ is equal to $\mathfrak{P}_t^U$ for any number t in $[01]$ and for any Borel set* U on the ω -set $\mathcal{Q}'$ of P -measure 1. Furthermore on $\mathcal{Q}'$, the set of the (tu) -points corresponding to the jumps of $\mathfrak{Q}(\omega)$ is coincident with that of $\lim_{n \rightarrow \infty} \int_{n > |u| > \frac{1}{n}} \left(u \mathfrak{P}^{du}(\omega) - \frac{u}{1+u^2} \overline{\mathfrak{P}}^{du}(\omega) \right)$. We shall define a $\mathfrak{G}$ -process $\mathfrak{G}$ by

$$(71) \quad \mathfrak{G}(\omega) = \mathfrak{Q}(\omega) - \lim_{n \rightarrow \infty} \int_{n > |u| > \frac{1}{n}} \left(u \mathfrak{P}^{du}(\omega) - \frac{u}{1+u^2} \overline{\mathfrak{P}}^{du} \right) \quad (\text{for } \omega \in \mathcal{Q}') \\ = 0 \quad (\text{for } \omega \in \mathcal{Q}')$$

The independence of $\mathfrak{G}$ and $\mathfrak{P}$ can be easily seen by virtue of the fact that, for any system of disjointed Borel sets* $U_1, U_2, \dots, \mathfrak{P}^{U_1}, \mathfrak{P}^{U_2}, \dots, \mathfrak{P}^{U_n}$ and $\mathfrak{Q} - \int_U \left(u \mathfrak{P}^{du} - \frac{u}{1+u^2} \overline{\mathfrak{P}}^{du} \right)$ ($U = U_1 + U_2 + \dots + U_n$) are independent. Thus Theorem 4.5 is proved.

Theorem of Lévy. Let F be an infinitely divisible law. Then the characteristic function $\varphi_F(z)$ of F can be expressed by

$$(72) \quad \varphi_F(z) = \exp \left(imz - \frac{\sigma^2}{2} z^2 + \lim_{n \rightarrow \infty} \int_{|u| > \frac{1}{n}} \left(e^{izu} - 1 - \frac{izu}{1+u^2} \right) n(du) \right)$$

(29) $\Re$ means the real part.

where

(73) m and σ are real numbers, and σ is positive.

(74) n is a completely additive non-negative set-function defined for Borel sets*, and satisfies

$$\int_{|u|>1} n(du) < \infty, \quad \int_{0<|u|<1} u^2 n(du) < \infty.$$

Proof. By Theorem 3.1 we can define an $\mathfrak{L}$ -process $\mathfrak{L} \equiv (\mathfrak{L}_t; 0 \leq t \leq 1)$ such that $\mathfrak{L}_1$ may be subject to F . As $\varphi_F(z) = e^{\overline{\mathfrak{L}}_1 z}$ is clear by virtue of Corollary 1.4, we shall obtain at once this theorem from Theorem 4.5.

V. Homogeneous $\mathfrak{L}$ -processes.

Definition 5.1. A stochastic differential process $x \equiv (x_t; a \leq t \leq b)$ is termed *temporally homogeneous* (or simply homogeneous) if and only if $s-t = s'-t'$ implies $x_s - x_t \sim x_{s'} - x_{t'}$.

Corollary 5.1. If x is a homogeneous stochastic differential process, we have

$$(x_{s_i} - x_{t_i}; i = 1, 2, \dots, n) \sim (x_{s'_i} - x_{t'_i}; i = 1, 2, \dots, n)$$

for any system of pairs $(t_1 s_1), (t_2 s_2), \dots, (t_n s_n), (t'_1 s'_1), \dots, (t'_n s'_n)$ such that the set $(t_1, s_1, t_2, s_2, \dots, t_n, s_n)$ may be transformed to the set $(t'_1, s'_1, t'_2, s'_2, \dots, t'_n, s'_n)$ by translation.

Theorem 5.1. Let $\mathfrak{L}(\mathfrak{L}')$ be a temporally homogeneous $\mathfrak{L}$ -process over (01) on $(\Omega, P) ((\Omega', P'))$. Then $\mathfrak{L}_1 \sim \mathfrak{L}'_1$ implies $\mathfrak{L} \sim \mathfrak{L}'$.

Proof. It is sufficient to prove that, for any pair $t, s (t < s)$, $\mathfrak{L}_s - \mathfrak{L}_t$ and $\mathfrak{L}'_s - \mathfrak{L}'_t$ are equivalent in law. Let $F_{ts}(F'_{ts})$ denote the probability law of $\mathfrak{L}_s - \mathfrak{L}_t (\mathfrak{L}'_s - \mathfrak{L}'_t)$ and $F_n(F'_n)$ denote $F_{\frac{0.1}{2^n}}(F'_{\frac{0.1}{2^n}})$. From the hypothesis that $\mathfrak{L}$ and $\mathfrak{L}'$ are temporally homogeneous, we can deduce

$$(1) \quad F_{st} = \rho\text{-}\lim_{n \rightarrow \infty} \bigstar_{s < \frac{k}{2^n} < t} F_{\frac{k}{2^n} \frac{k+1}{2^n}} = \rho\text{-}\lim_{n \rightarrow \infty} \bigstar_{s < \frac{k}{2^n} < t} F_n, \quad \text{and}$$

$$(2) \quad F'_{st} = \rho\text{-}\lim_{n \rightarrow \infty} \bigstar_{s < \frac{k}{2^n} < t} F'_{\frac{k}{2^n} \frac{k+1}{2^n}} = \rho\text{-}\lim_{n \rightarrow \infty} \bigstar_{s < \frac{k}{2^n} < t} F'_n.$$

Therefore we need only prove

$$(3) \quad F_n = F'_n \quad (n = 1, 2, \dots).$$

By the hypothesis we have

$$(4) \quad F = F_1 * F_1 = F'_1 * F'_1,$$

and so

$$(5) \quad \varphi_F(z) = (\varphi_{F_1}(z))^2 = (\varphi_{F'_1}(z))^2.$$

As F is infinitely divisible, we have

$$(6) \quad \varphi_F(z) \neq 0,$$

according to the theorem of Lévy cited in the preceeding chapter. From (5) and (6) follows

$$(7) \quad \varphi_{F_1}(z) \neq 0.$$

By virtue of (5) and (7) we can see that $\frac{\varphi_{F_1}(z)}{\varphi_{F'_1}(z)}$ is either 1 or -1 . As $\varphi_F(z)$ and $\varphi_{F'_1}(z)$ are continuous by the definitions, $\frac{\varphi_{F_1}(z)}{\varphi_{F'_1}(z)}$ is also continuous, namely $\frac{\varphi_{F_1}(z)}{\varphi_{F'_1}(z)}$ is identically equal to 1 or to -1 . Therefore we have $\frac{\varphi_{F_1}(z)}{\varphi_{F'_1}(z)} = 1$ i.e. $F_1 = F'_1$, as $\frac{\varphi_{F_1}(0)}{\varphi_{F'_1}(0)}$ is clearly equal to 1. Similarly we can obtain $F_2 = F'_2$, $F_3 = F'_3$, ..., and finally $F_n = F'_n$.

Theorem 5.2. *Let m, σ, n satisfy the conditions (73) (74) stated in the preceeding chapter. Then there exists a temporally homogeneous $\mathfrak{L}$ -process whose continuous part $\mathfrak{G}$ satisfies*

$$(8) \quad \sigma(\mathfrak{G}_1) = \sigma, \quad m(\mathfrak{G}_1) = m$$

and whose discontinuous part $\mathfrak{P}$ satisfies

$$(9) \quad \overline{\mathfrak{P}_1^U} = n(U).$$

Proof. Let $\mathcal{Q}_0$ denote the set of all continuous functions of t . By a theorem of Doob⁽³⁰⁾ we can introduce a P -measure P_0 on $\mathcal{Q}_0$ such that

$$P_0\left(\prod_{i=1}^n E(f(t); f(s_i) - f(t_i) \in B_i)\right) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi(s_i - t_i)\sigma}} \\ \times \int_{B_i} \exp\left(-\frac{(\lambda - ms_i + mt_i)^2}{2\sigma^2(s_i - t_i)}\right) d\lambda$$

(30) See J. L. Doob: loc. cit. (2) Th. 3.9.

may hold for $0 \leq t_1 < s_1 \leq t_2 < s_2 \leq \dots \leq t_n < s_n \leq 1$ and for any Borel sets $B_1, B_2, \dots, B_n$. If we put

$$\mathbb{G}_t(\omega_0) = f(t) - f(0) \quad \text{for } \omega_0 = f(t) \in \Omega_0,$$

then $\mathbb{G} \equiv (\mathbb{G}_t; 0 \leq t \leq 1)$ is evidently a $\mathbb{G}$ -process on (Ω_0, P_0) .

Let D denote the set $E((tu); 0 \leq t \leq 1)$ of the (tu) -plane. We shall introduce a measure N on D as follows:

$$(10) \quad N(E) = \int \int_E n(du) dt$$

E being a Borel subset of D . We can easily see that $N(E)$ is finite for any Borel set E separated from the t -axis by a positive distance.

Let $C^{\alpha\beta}$ denote the (tu) -set $(01] \times (a\beta]$ ($\alpha < \beta$, $\alpha\beta > 0$). We shall now define a sequence of partitions of $C^{\alpha\beta}$. First, let C_1, C_2, C_3, C_4 be defined respectively as $(0, \frac{1}{2}] \times (a, \frac{\alpha+\beta}{2}]$, $(0, \frac{1}{2}] \times (\frac{\alpha+\beta}{2}, \beta]$, $(\frac{1}{2}, 1] \times (a, \frac{\alpha+\beta}{2}]$, $(\frac{1}{2}, 1] \times (\frac{\alpha+\beta}{2}, \beta]$. Then we have

$$(11) \quad C^{\alpha\beta} = C_1 + C_2 + C_3 + C_4.$$

In the exactly same manner C_i may be decomposed into $C_{i1}C_{i2}C_{i3}C_{i4}$ ($i = 1, 2, 3, 4$). Continuing this decomposition we can define $C_{i_1 i_2 \dots i_n}$ ($i_1, i_2, \dots, i_n = 1, 2, 3, 4$). In this paper this sequence of partitions will be called *the chain of $C^{\alpha\beta}$* , and each set such as $C_{i_1 i_2 \dots i_n}$ a *chain-element*. For brevity we shall adopt here the following notations:

$$(12) \quad \nu = N(C^{\alpha\beta})$$

$$\nu_{i_1 i_2 \dots i_n} = N(C_{i_1 i_2 \dots i_n}).$$

An integral-valued set function defined for any chain-element of $C^{\alpha\beta}$ will be called a *distribution-chain on $C^{\alpha\beta}$* if it satisfies

$$(13) \quad m(C^{\alpha\beta}) = m(C_1) + m(C_2) + m(C_3) + m(C_4)$$

$$m(C_{i_1 i_2 \dots i_n}) = \sum_{i=1}^4 m(C_{i_1 i_2 \dots i_n i}).$$

Let $\Omega_{\alpha\beta}$ denote the set of all distribution-chains on $C^{\alpha\beta}$. We shall now introduce a sequence of P -measures $\{P^{(n)}\}$ of $\Omega_{\alpha\beta}$ as follows:

$$(14) \quad P^{(0)}(E(m; m(C^{\alpha\beta}) = k)) = e^{-\nu} \frac{\nu^k}{k!} \quad (k = 0, 1, 2, \dots),$$

$$(15) \quad P^{(n)}(E(m; m(C_{i_1 i_2 \dots i_n}) = k_{i_1 i_2 \dots i_n} \text{ for } i_1, i_2, \dots, i_n = 1, 2, 3, 4)) \\ = \prod_{i_1=1}^4 \prod_{i_2=1}^4 \dots \prod_{i_n=1}^4 e^{-\nu_{i_1 i_2 \dots i_n}} \frac{k_{i_1 i_2 \dots i_n}^{\nu_{i_1 i_2 \dots i_n}}}{k_{i_1 i_2 \dots i_n}}.$$

It can be easily verified by simple calculations that $P^{(0)}, P^{(1)}, P^{(2)}, \dots$ are coherent, namely that these measures are coincident for the same subset of $\mathcal{Q}_{\alpha\beta}$. We can thus obtain an additive (not necessarily completely additive) measure P on $\mathcal{Q}_{\alpha\beta}$ which is coincident with $P^{(n)}$ for any $P^{(n)}$ -measurable subset of $\mathcal{Q}_{\alpha\beta}$. We shall prove that we can generalize P and constitute a completely additive measure (probability) $P_{\alpha\beta}$. We need only prove that a monotone decreasing sequence $(M_1, M_2, \dots)$ of subsets of $\mathcal{Q}_{\alpha\beta}$ may have at least one common point, if $P(M_1), P(M_2), \dots$ are bounded lowerly by a positive number δ . Let E_k denote the set $(m(C^{\alpha\beta}); m \in M_k) (k = 1, 2, \dots)$. Then $(E_1, E_2, \dots)$ is a monotone decreasing sequence of the sets of non-negative integers. The following two cases may be possible.

(16) $E_1, E_2, \dots$ have at least one common element.

(17) the minimum μ_k of the numbers in E_k increases indefinitely with k . In the latter case $P_{\alpha\beta}(M_k)$, as it is smaller than $\sum_{i=\nu_k}^{\infty} e^{-\nu} \frac{\nu^i}{i!}$ by the definition, should tend to 0 with $\frac{1}{k}$ in contradiction with the assumption. Therefore (16) should hold. Let σ be an element in $\prod_{i=1}^{\infty} E_i$, and E'_k the set $E((m(C_1), m(C_2), m(C_3), m(C_4)); m \in M_k, m(C^{\alpha\beta}) = \sigma)$. Then $\{E'_k\}$ will be a monotone decreasing sequence of bounded sets of the points in the 4-dimensional space whose coordinates are all integers. It is clear that none of E'_k is empty. Therefore $\prod_{k=1}^{\infty} E'_k$ will have at least one element. Denote this element by $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. Continuing this operation, we can now define $\sigma_{i_1 i_2 \dots i_n} (i_1, i_2, \dots, i_n = 1, 2, 3, 4)$. If we define a distribution-chain m by

$$(18) \quad m(C_{i_1 i_2 \dots i_n}) = \sigma_{i_1 i_2 \dots i_n},$$

then m will belong to each of $M_1, M_2, \dots$. We have thus obtained a probability field $(\mathcal{Q}_{\alpha\beta}, P_{\alpha\beta})$.

We shall now define a system of random variables x^C by

$$(19) \quad x^C(m) = m(C) \quad (\text{for any distribution chain } m),$$

C running over all chain-elements on $C^{\alpha\beta}$. Then x^C is subject to a Poissonian distribution with the mean value $N(C)$, and for any system of

disjointed chain-elements $C_1, C_2, \dots, C_n$, the random variables $x^{C_1}, x^{C_2}, \dots, x^{C_n}$ are independent, and $x^C(m)$ is additive (not necessarily completely additive) in C , that is to say, for any two systems of disjointed chain-elements $(C_1, C_2, \dots, C_n)$ $(C'_1, C'_2, \dots, C'_n)$ such that $C_1 + C_2 + \dots + C_n = C'_1 + C'_2 + \dots + C'_n$, we have always

$$\begin{aligned} x^{C_1}(m) + x^{C_2}(m) + x^{C_3}(m) + \dots + x^{C_n}(m) \\ = x^{C'_1}(m) + x^{C'_2}(m) + x^{C'_3}(m) + \dots + x^{C'_n}(m). \end{aligned}$$

Now, let (Ω, P) denote the direct product⁽³¹⁾ of the probability fields (Ω_0, P_0) , (Ω_{2^i}, P_{2^i}) and $(\Omega_{(-2^i+1)(-2^i)}, P_{(-2^i+1)(-2^i)})$ for i running over all integers. We shall denote here the representations⁽³²⁾ of the processes $\mathfrak{G}$ (defined on (Ω_0, P_0)) and x^C (defined above) in (Ω, P) respectively by the same notations $\mathfrak{G}$ and x^C .

We shall now classify the chain-elements appearing in the course of defining (Ω, P) . Chain-elements such as $C^{2^i, 2^{i+1}}, C^{(-2^i+1), (-2^i)}$ will be called *chain-elements of the first kind*. The chain-elements which were constructed from each of the chain-elements of the $(n-1)$ -th kind in the same manner as C_1, C_2, C_3, C_4 were constructed from $C^{\alpha\beta}$ will be called *chain-elements of the n -th kind*.

For brevity we shall make use of " (tu) -set^o" to indicate a bounded subset of the (tu) -set $E((tu); 0 \leq t \leq 1)$ separated from the t -axis by a positive distance. Terms such as *interval*^o, *Borel* (tu) -set^o, etc. may be understood similarly.

Let G be an open (tu) -set^o. Let $C_1^n, C_2^n, \dots, C_k^n$ be the chain-elements of the n -th kind whose closure is contained in G . Then $\sum_{i=1}^k C_i^n$ will increase with n and tend to G . We shall define $y^G(\omega)$ as $\lim_{n \rightarrow \infty} \sum_{i=1}^k x^{C_i^n}(\omega)$.

Let p be a point in the (tu) -set $E((tu); 0 \leq t \leq 1, u \neq 0)$, and $U\left(p, \frac{1}{n}\right)$ the $\frac{1}{n}$ -neighbourhood of p . We shall define $y^{(p)}(\omega)$ by $\lim_{n \rightarrow \infty} y^{U(p, \frac{1}{n})}(\omega)$. Then it is clear that $y^{(p)}(\omega)$ is a non-negative integer. Therefore we shall obtain

$$(20) \quad y^G(\omega) = \sum_{p \in G} y^{(p)}(\omega).$$

We shall define $y^E(\omega)$ as $\sum_{p \in E} y^{(p)}(E)$ for any (tu) -set^o E .

(31), (32) See Definition 1.11.

We shall now prove that $y^E(\omega)$ is P -measurable in ω , namely that $y^E = (y^E(\omega); \omega \in \Omega)$ is a random variable on (Ω, P) for any Borel (tu) -set $^\circ$ E . By the definition y^G is evidently P -measurable in G for any open (tu) -set $^\circ$ G . Let F any closed (tu) -set $^\circ$, and U_n the $\frac{1}{n}$ -neighbourhood of F . Then we have

$$(21) \quad y^F(\omega) = \lim_{n \rightarrow \infty} y^{U_n}(\omega).$$

As $y^{U_n}(\omega)$ ($n = 1, 2, 3, \dots$) are P -measurable in ω , it is the case with $y^F(\omega)$. From the definition we can easily deduce that $y^E = (y^E(\omega); \omega \in \Omega)$ is a random variable subject to a Poissonian distribution with the mean value $N(E)$, if E is an open (tu) -set $^\circ$ or a closed (tu) -set $^\circ$.

Let E be any Borel (tu) -set $^\circ$. Then we have

$$(22) \quad N(E) = \text{l.u.b.}_{F \subset E} N(F) = \text{g.l.b.}_{G \supset E} N(G) \quad (F: \text{closed}, G: \text{open}).$$

This shows that there exists a sequence $\{F_k\}$ of closed (tu) -sets $^\circ$ and a sequence $\{G_k\}$ of open (tu) -sets $^\circ$ such that

$$(23) \quad F_1 \subset F_2 \subset F_3 \subset \dots \subset E \quad \lim_{n \rightarrow \infty} N(F_n) = N(E),$$

$$(24) \quad G_1 \supset G_2 \supset G_3 \supset \dots \supset E \quad \lim_{n \rightarrow \infty} N(G_n) = N(E)$$

Let G and F denote respectively $\bigcup_{n=1}^{\infty} G_n$ and $\bigcap_{n=1}^{\infty} F_n$. Then we have

$$(25) \quad F \subset E \subset G, \text{ and so } y^F(\omega) \leq y^E(\omega) \leq y^G(\omega), \text{ and}$$

$$(26) \quad N(F) = N(E) = N(G).$$

$y^F(y^G)$ is evidently a random variable subject to a Poissonian distribution with the mean value $N(F)$ ($N(G)$) on account of the definition, and y^{G-F} is also a random variable subject to a Poissonian distribution with the mean value $N(G-F)$. As $N(G-F) = 0$ by virtue of (25) and (26), we have, with probability 1,

$$(27) \quad y^{G-F} = 0 \quad \text{i.e.} \quad y^G = y^F$$

and by (25), (26), $y^F = y^E = y^G$, which shows that y^E is a random variable subject to a Poissonian distribution with the mean value $N(E)$. It is clear that $y^E(\omega)$ is completely additive in E . We shall next prove that $y^{E_1}, y^{E_2}, \dots, y^{E_n}$ are independent for any system of disjointed Borel (tu) -sets $^\circ$ $E_1, E_2, \dots, E_n$. In the case that $E_1, E_2, \dots, E_n$ are all open (tu) -sets $^\circ$ we can deduce this fact at once from the definition,

if we recall that the analogous fact holds with regard to x^E . Let $F_1, F_2, \dots, F_n$ be any disjointed closed (tu) -sets^o, and $U_1^{(m)}, U_2^{(m)}, \dots, U_n^{(m)}$ respectively the $\frac{1}{m}$ -neighbourhood of $F_1, F_2, \dots, F_n$ ($m = 1, 2, \dots$). Then there exists a natural number M such that $m > M$ may imply that $U_1^{(m)}, U_2^{(m)}, \dots, U_n^{(m)}$ are disjointed. We have, for $m > M$,

$$(28) \quad P\left(\prod_{i=1}^n (y^{U_i^{(m)}} \in B_i)\right) = \prod_{i=1}^n P(y^{U_i^{(m)}} \in B_i)$$

where $B_1, B_2, \dots, B_n$ are arbitrary Borel sets of real numbers. Therefore we shall obtain

$$\begin{aligned} P\left(\prod_{i=1}^n (y^{F_i} \in B_i)\right) &= \lim_{m \rightarrow \infty} P\left(\prod_{i=1}^n (y^{U_i^{(m)}} \in B_i)\right) = \prod_{i=1}^n \lim_{m \rightarrow \infty} P(y^{U_i^{(m)}} \in B_i) \\ &= \prod_{i=1}^n P(y^{F_i} \in B_i). \end{aligned}$$

Let $E_1, E_2, \dots, E_n$ be any system of disjointed Borel (tu) -sets^o. Then we have a sequence of system of closed (tu) -sets^o $\{F_{m1}, F_{m2}, \dots, F_{mn}\}$ ($m = 1, 2, \dots$) such that

$$(29) \quad F_{1i} \subset F_{2i} \subset F_{3i} \subset \dots \subset E_i \quad (i = 1, 2, \dots, n)$$

$$\lim_{m \rightarrow \infty} y^{F_{mi}} = y^{E_i} \quad (i = 1, 2, \dots, n) \quad (\text{with } P\text{-measure } 1).$$

As $(E_1, E_2, \dots, E_n)$ are a disjointed system, so is $(F_{m1}, F_{m2}, \dots, F_{mn})$. By virtue of the above discussion we can see that $y^{F_{m1}}, y^{F_{m2}}, \dots, y^{F_{mn}}$ are independent and so by (29) that $y^{E_1}, y^{E_2}, \dots, y^{E_n}$ are so independent.

Let L_t^n denote the (tu) -set $E\left((\tau u); \tau = t, |u| \geq \frac{1}{n}\right)$. We shall now prove that the set $\mathcal{Q}_n$ of the ω -elements, for which there exists at least one t -value satisfying

$$(30) \quad y^{L_t^n}(\omega) \geq 2,$$

has P -measure 0. Let L_{ts}^n denote the (tu) -set $E\left((\tau u); t \leq \tau \leq s, |u| \geq \frac{1}{n}\right)$. Then the ω -set $\mathcal{Q}_n$ will be contained in the ω -set:

$$(31) \quad \sum_{k=1}^m \left(y^{L_{\frac{k-1}{m} \frac{k}{m}}^n} \geq 2\right)$$

whose P -measure is smaller than

$$(32) \quad m \left\{ 1 - e^{-\eta \frac{1}{m} \left(1 + \eta \frac{1}{m} \right)} \right\}$$

where $\eta = N(L_{01}^n) = \int_0^1 dt \int_{|u| > \frac{1}{n}} n(du)$. As (32) tends to 0 with $\frac{1}{m}$, Ω_n should have P -measure 0. Consequently the set Ω_0 of the ω -elements for which there exists at least one t -value satisfying

$$(33) \quad yL_t(\omega) \geq 2 \quad (L_t = E((\tau u); \tau = t, u \neq 0))$$

will have also P -measure 0, since Ω_0 is equal to $\sum_{n=1}^{\infty} \Omega_n$. We shall now subtract Ω_0 from Ω ; this subtraction may cause no essential change on the consequences obtained above.

For any Borel set* U we shall define $\mathfrak{P}_t^U(\omega)$ as $y^{[0t] \times U}$. Then $\mathfrak{P}^U = (\mathfrak{P}_t^U(\omega); 0 \leq t \leq 1)$ will be a $\mathfrak{P}$ -process, and the combination $\mathfrak{P}$ of $\mathfrak{P}^U$ for U running over all Borel sets* will be a differential system of $\mathfrak{P}$ -processes. The $\mathfrak{G}$ -process $\mathfrak{G}$ obtained before and the system $\mathfrak{P}$ will satisfy the conditions stated in Theorem 4.4. Therefore we can see that the process $\mathfrak{G} + \lim_{n \rightarrow \infty} \int_{n > |u| > \frac{1}{n}} \left(u \mathfrak{P}^{du} - \frac{u}{1+u^2} \mathfrak{P}^{\bar{d}u} \right)$ is precisely the $\mathfrak{P}$ -process whose existence was to be proved.

Theorem of Lévy. *Let m, σ, n satisfy the conditions (73) (74) in Chapter IV. Then (72) is the characteristic function of a certain infinitely divisible law.*

Proof. We shall construct an $\mathfrak{P}$ -process $\mathfrak{P}$ as in Theorem 5.2. Then (72) is equal to $e^{\overline{iz\mathfrak{Q}_1}}$ i.e. $\mathcal{P}_{F_{\mathfrak{Q}_1}}(z)$. $\mathfrak{Q}$ being an $\mathfrak{Q}$ -process, $F_{\mathfrak{Q}_1}$ is an infinitely divisible law.

Uniqueness theorem of Lévy. *An infinitely divisible law F is expressed in the canonical form in only one way.*

Proof. Assume that F be expressed in the following two ways:

$$(34) \quad \varphi_F(z) = \exp \left(imz - \frac{\sigma^2}{2} z^2 + \lim_{n \rightarrow \infty} \int_{n > |u| > \frac{1}{n}} \left(e^{izu} - 1 - \frac{izu}{1+u^2} \right) n(du) \right),$$

$$(35) \quad \varphi_F(z) = \exp \left(im'z - \frac{\sigma'^2}{2} z^2 + \lim_{n \rightarrow \infty} \int_{n > |u| > \frac{1}{n}} \left(e^{izu} - 1 - \frac{izu}{1+u^2} \right) n'(du) \right).$$

Then we can construct temporally homogeneous $\mathfrak{Q}$ -processes $\mathfrak{Q}, \mathfrak{Q}'$ by Theorem 5.2 such that

$$(36) \quad \mathfrak{Z} = \mathfrak{G} + \lim_{\substack{n \rightarrow \infty \\ n > |u| > \frac{1}{n}}} \int \left(u \mathfrak{P}^{du} - \frac{u}{1+u^2} \overline{\mathfrak{P}^{du}} \right) \\ \left(\overline{\mathfrak{G}}_1 = m, \quad \sigma(\mathfrak{G}_1) = \sigma, \quad \overline{\mathfrak{P}_1^U} = n(U) \right)$$

$$(37) \quad \mathfrak{Z}' = \mathfrak{G}' + \lim_{\substack{n \rightarrow \infty \\ n > |u| > \frac{1}{n}}} \int \left(u \mathfrak{P}'^{du} - \frac{u}{1+u^2} \overline{\mathfrak{P}'^{du}} \right) \\ \left(\overline{\mathfrak{G}}'_1 = m', \quad \sigma(\mathfrak{G}'_1) = \sigma', \quad \overline{\mathfrak{P}_1'^U} = n'(U) \right)$$

On account of Theorem 5.1 we can deduce $\mathfrak{Z} \sim \mathfrak{Z}'$ from the hypothesis: $\mathfrak{Z}_1 \sim \mathfrak{Z}'_1$. Since $\mathfrak{G}(\mathfrak{G}')$ and $\mathfrak{G}(\mathfrak{G}')$ are respectively the continuous part and the discontinuous part of $\mathfrak{Z}(\mathfrak{Z}')$, we have $\mathfrak{G} \sim \mathfrak{G}'$, $\mathfrak{P} \sim \mathfrak{P}'$, a fortiori $m = m'$, $\sigma = \sigma'$, $n = n'$.

Mathematical Institute,
Faculty of Science, Imperial University of Tokyo.

(Added in proof)

As to the phrase “a probability field separable in P_* ” that I have used in Definition 1.11, some supplementary explanations seem to be necessary.

Let $(\mathcal{Q}, P)$ be a probability field. It is well-known that the distance between two P -measurable sets E_1 and E_2 means $P(E_1 - E_2) + P(E_2 - E_1)$. If the system of all P -measurable subsets of $\mathcal{Q}$ be separable with respect to this distance, we shall call this probability field $(\mathcal{Q}, P)$ to be separable in P .

In the proof of Theorem 5.2 also I have constructed a probability field with the aid of Definition 1.11 without examining whether this separability condition is satisfied or not. But it can be easily verified that the condition is well satisfied there.