3. On Frobeniusean algebras. III.

By Tadasi NAKAYAMA.

(Received June 13, 1941.)

Introduction. An (associative) algebra A possessing a unit element, and of a finite rank over a certain field F , is called *Frobeniusean* if its left and right regular representations $S(a)$ and $R(a)$ are equivalent to each other. If however A satisfies a weaker condition that the totality of distinct directly indecomposable components of $S(a)$ coincides with that of $R(a)$, then we call A *quasi-Frobeniusean*.⁽¹⁾ We have proved that, among others, if A is Frobeniusean then $\alpha)$ $l(r(I)) = I$, $r(l(r)) = r$ and $\beta)$ $(I:F) + (r(I):F) = (A:F)$, $(r:F) + (l(r)) = (A:F)$ for every left ideal I and right ideal r in A , and conversely, if $\alpha)$ and $\beta)$ are the case for (not necessarily all the ideals, but) every nilpotent simple left ideal I and every nilpotent simple right ideal r as well as for the radical and the zero ideal of A , then A is Frobeniusean; where $l(S)$ and $r(S)$ ($S \subseteq A$) denote respectively the sets of left and right annihilators of S in A . Similarly, if A is quasi-Frobeniusean then $\alpha)$ (but not $\beta)$) holds for every I and r , and conversely, if $\alpha)$ is valid for the above mentioned ideals then A is quasi-Frobeniusean. In particular, A is certainly quasi-Frobeniusean if $\alpha)$ is valid for every I and r , and furthermore, A is Frobeniusean if $\beta)$ (whence $\alpha)$) is the case for every I and r . Now, in the present note⁽²⁾ we want to, first, view these milder interpretations of the converse parts of the above main theorems from a somewhat different point. Namely, we drop the provision of being given by the annihilation relation, and just look at the dual correspondence between left and right ideals. In fact, we shall prove that *if the lattices of completely reducible left and right ideals of A are dual-isomorphic respectively to the lattices of the right and left ideals of A containing the radical, then A is quasi-Frobeniusean. Moreover, if in (at least) one of these dual-isomorphisms the corresponding left and right ideals have dual ranks over F , then A is Frobeniusean* (§§ 1, 2; Theorems 1, 2, 3 and 4). In particular, *if the whole lattices of left and right ideals in A are dual-isomorphic to each other, then A is quasi-Frobeniusean, and if this dual-isomorphism*

(1) On Frobeniusean algebras. I, Ann. Math. 40 (1939), p. 611; II, ibid. 42 (1941), p. 1. These papers we shall refer to as Part I and Part II.

(2) A brief account of a part of the present work has been published in Nakayama (36).

is such that corresponding left and right ideals have dual ranks over F , then A is Frobeniusean. (We shall, as in Part II, deal with not only the special case of algebras but also rings with minimum condition for ideals).

However, a similar modification of our previous characterization of Frobeniusean algebras by two-sided ideals and their annihilators (Part I, Theorem 8) does not hold (§3).

Further, not very essential is the following generalisation: Let A and A' be two algebras over F such that the left and the right ideal lattices of A are dual-isomorphic, respectively, to the right and the left ideal lattices of A' . Then both A and A' are quasi-Frobeniusean. If, moreover, the ranks of two ideals corresponding to each other in one of these dual-isomorphisms are dual, then A and A' are Frobeniusean (§5).

Group algebras of finite groups (over a field modular or non-modular⁽³⁾) and Grassmann-Cartan's algebras of outer forms⁽⁴⁾ constitute useful examples of Frobeniusean algebras; Group algebras are in fact symmetric.⁽⁵⁾ But these algebras possess a well-known second one-one correspondence between left and right ideals—namely, a (direct) isomorphism of the left and right ideal lattices. This shows, combined with our dual-isomorphism, that both the left and the right ideal lattices of these algebras are self-dual. Indeed, an algebra with self-dual left and right ideal lattices is necessarily quasi-Frobeniusean. And, if the dual-automorphisms satisfy the condition of dual ranks, then the algebra is Frobeniusean too. However, the converse is certainly not the case, and the left and right ideal lattices of (not only Frobeniusean but) symmetric algebras are not self-dual in general even when the underlying field is algebraically closed; In fact a much weaker symmetry between upper and lower Loewy parts of directly indecomposable components of the regular representations fails to prevail (§6).

The note contains also a remark concerning a theorem⁽⁶⁾ on two-sided principal ideals in Frobeniusean algebras which however does not hold for Frobeniusean rings (§7), as well as an elementary remark on regular (adjoint) representations of Lie-algebras (§4).

(3) We are particularly interested in the modular case.

(4) More generally, homology algebras of (oriented closed) manifolds are Frobeniusean. Their left and right ideal lattices are isomorphic to each other, and therefore, self-dual. This fact may be considered as an algebraic supplement to Poincaré's duality.

(5) In the sense of R. Brauer and C. Nesbitt. See Brauer-Nesbitt (3) and Part I, §9.

(6) Corollary to Part I, Theorem 4.

1. Frobeniusean and quasi-Frobeniusean rings (Preliminaries).

Let A be, first, an algebra (of finite rank) over a field F , and N its radical. Let

$$A/N = \bar{A} = \bar{A}_1 + \bar{A}_2 + \cdots + \bar{A}_k$$

be the decomposition of the semi-simple residue class algebra $\bar{A} = A/N$ into simple algebras $\bar{A}_\kappa$. There exists a system of mutually orthogonal primitive idempotent elements $e_{\kappa, i}$ ($\kappa = 1, 2, \dots, k$; $i = 1, 2, \dots, f(\kappa)$) in A such that the residue class $\bar{E}_\kappa \pmod{N}$ of $E_\kappa = e_{\kappa, 1} + e_{\kappa, 2} + \cdots + e_{\kappa, f(\kappa)}$ is the unit element of $\bar{A}_\kappa$. Put $E = \sum_\kappa E_\kappa$; if A has a unit element 1 then E coincides with 1. A is a direct sum

$$(1) \quad \begin{aligned} A &= Ae_{1,1} + \cdots + Ae_{1,f(1)} + Ae_{2,1} + \cdots + Ae_{k,f(k)} + l_0 \\ [A &= e_{1,1}A + \cdots + e_{1,f(1)}A + e_{2,1}A + \cdots + e_{k,f(k)}A + r_0], \end{aligned}$$

where $l_0 [r_0]$ is a left [right] ideal $\subseteq N$ consisting of all left [right] annihilators of E . Further, the directly indecomposable left [right] ideals $Ae_{\kappa, i} [e_{\kappa, i}A]$ with one and the same first suffix κ , and only those, are operator-isomorphic to each other:

$$(2) \quad \begin{aligned} Ae_{\kappa, 1} &\cong Ae_{\kappa, 2} \cong \cdots \cong Ae_{\kappa, f(\kappa)} \\ [e_{\kappa, 1}A &\cong e_{\kappa, 2}A \cong \cdots \cong e_{\kappa, f(\kappa)}A]. \end{aligned}$$

For the sake of brevity, let us write e_κ for $e_{\kappa, 1}$, a representative of $e_{\kappa, i}$. We have seen (Part I, Lemma 2)

Lemma 1. *Let A have a unit element. A is quasi-Frobeniusean if and only if there is a permutation $(\pi(1), \pi(2), \dots, \pi(k))$ of $(1, 2, \dots, k)$ such that for each κ*

i) $e_\kappa A$ has a unique simple right subideal r_κ and $r_\kappa \cong \bar{e}_{\pi(\kappa)} \bar{A} = e_{\pi(\kappa)} A / e_{\pi(\kappa)} N$,

ii) $Ae_{\pi(\kappa)}$ has a unique simple left subideal $l_{\pi(\kappa)}$ and $l_{\pi(\kappa)} \cong \bar{A} \bar{e}_\kappa = Ae_\kappa / Ne_\kappa$.

Furthermore, A is Frobeniusean if and only if there exists a permutation $(\pi(\kappa))$ which satisfies

$$\text{iii) } f(\kappa) = f(\pi(\kappa))$$

besides i) and ii).

Now, the above structural theory applies not only to algebras but to rings satisfying the minimum condition (for left and right ideals)⁽⁷⁾⁽⁸⁾ as well; The notations will be retained throughout in the sequel. In view of the above criteria, which connect, as a matter of fact, the representation theory to the structural one, we introduced in Part II the following

Definition. Let a ring A satisfying the minimum condition possess a unit element. We call A *quasi-Frobeniusean* or *Frobeniusean* if there is a permutation $(\pi(\kappa))$ of (κ) which satisfies i) and ii) or i), ii) and iii).

Further, as a substitute for "rank over F ", we define d_l and d_r as follows: Let first m be a simple left module of A . Then $d_l(m)$ is by definition the rank of m relative to the quasi-field of its (operator-) automorphisms. Let next m be not simple but have a composition series $m = m_0 > m_1 > \dots > m_s = 0$. Then we define $d_l(m) = \sum_{i=1}^s d_l(m_{i-1}/m_i)$. The function $d_r(n)$ is defined similarly for right moduli n .

2. A ring whose left and right ideal lattices are dual.

Let A be, as above, a ring satisfying the minimum condition for ideals. We prove

Theorem 1. Let the lattices A_0 and P_0 of completely reducible left and right ideals in A be dual-isomorphic⁽⁹⁾ respectively to the lattices P_1 and A_1 of right and left ideals of A containing the radical.⁽¹⁰⁾⁽¹¹⁾ Then A is quasi-Frobeniusean. If moreover the dual-isomorphism of A_0 and P_1 , chosen suitably, is such that $d_l(l) = d_r(A/r)$ when $l \in A_0$ corresponds to $r \in P_1$, then A is Frobeniusean.

Proof. Let N be the radical of A . The lattices A_0 and P_1 are modular and complemented, and are, by assumption, dual-isomorphic to each other. This implies that the maximal number of independent simple left ideals of A is equal to the number of components in an irredundant

⁽⁷⁾ It may possess an operator domain Ω such that $\alpha(a+b) = \alpha a + \alpha b$, $\alpha(ab) = (\alpha a)b = \alpha(\alpha b)$ when $\alpha \in \Omega$ and $a, b \in A$.

⁽⁸⁾ For the structure of rings satisfying the minimum condition only cf. besides C. Hopkins (24), cited before, K. Asano (31) and J. Levitzky (34).

⁽⁹⁾ As a matter of fact, we have only to assume that A_0 and P_0 are dual-homomorphic to P_1 and A_1 . Then the dual-isomorphisms follow automatically, since the dimensions of P_1 and A_1 are at most equal to those of A_0 and P_0 , as the decomposition (1) shows. Cf. also the proof below.

⁽¹⁰⁾ They are intersections (meets) of maximal ideals. Thus, they can properly be called *completely meet-reducible*.

⁽¹¹⁾ This is equivalent to saying that A_0 and P_0 are (directly) isomorphic to A_1 and P_1 respectively, since A_1 and P_1 are dual-isomorphic to each other.

representation of N as an intersection of maximal right ideals. But this latter is equal to $\sum_{\kappa} f(\kappa)$. We conclude therefore that the maximal completely reducible left subideals of the $\sum_{\kappa} f(\kappa)$ left ideals $Ae_{\kappa,i}$ ($\kappa = 1, 2, \dots, k$; $i = 1, 2, \dots, f(\kappa)$), which are independent, must be all simple, and moreover, the left ideal l_0 in the decomposition (1) must vanish. Thus every $Ae_{\kappa,i}$ has a unique simple left subideal, which we shall denote by $l_{\kappa,i}$. It is evident that $l_{\kappa,1} \cong l_{\kappa,2} \cong \dots \cong l_{\kappa,f(\kappa)}$. We maintain that $l_{\kappa,i} \not\cong l_{\lambda,j}$ if $\kappa \neq \lambda$. To prove this, observe that P_1 can be decomposed into a product of exactly k (irreducible) projective geometries,⁽¹²⁾ and consequently, the same must be the case for A_0 . But, two simple left ideals are perspective (in A_0) if (and only if) they are operator-isomorphic. Hence there must exist exactly k non-isomorphic simple left ideals, and therefore, necessarily $l_{\kappa,i} \cong l_{\lambda,j}$ when $\kappa \neq \lambda$.

Similarly, r_0 in (1) vanishes and every $e_{\kappa,i}A$ has a unique simple right subideal $r_{\kappa,i}$. $r_{\kappa,i} \cong r_{\lambda,j}$ if $\kappa \neq \lambda$.

That $l_0 = r_0 = 0$ means that the element E is a unit element in A . And, the first part of our theorem follows now immediately from the following lemma, which has been stated in Part II⁽¹³⁾ but which we want to re-establish here, because there was some confusion in our description:

Lemma 2. *Let A , a ring with the minimum condition, have a unit element, and let there be a permutation $(\pi(1), \pi(2), \dots, \pi(k))$ of $(1, 2, \dots, k)$ satisfying i) and the first half of ii) (See the preceding section). Then A is quasi-Frobeniusean; the second half of ii) follows automatically.*

Proof. Consider the left and the right annihilator ideals $l(N)$, $r(N)$ of the radical N . They are both two-sided. We have $l(N) = \sum_{\kappa} E_{\kappa} l(N) = \sum_{\kappa,i} e_{\kappa,i} l(N)$. Here $e_{\kappa,i} l(N) = e_{\kappa,i} A \cap l(N)$ is the maximal completely reducible right subideal of $e_{\kappa,i} A$, which is, by i), simple and $\cong e_{\pi(\kappa)} A / e_{\pi(\kappa)} N$. Hence $e_{\kappa,i} l(N) = e_{\kappa,i} l(N) E_{\pi(\kappa)}$, while $e_{\kappa,i} l(N) E_{\lambda} = 0$ for any $\lambda \neq \pi(\kappa)$. Thus $E_{\kappa} l(N) E_{\lambda} = E_{\kappa} l(N)$ or 0 according as $\lambda = \pi(\kappa)$ or not. This is the case for every κ , and consequently

$$l(N) E_{\pi(\kappa)} = \sum_{\mu} E_{\mu} l(N) E_{\pi(\kappa)} = E_{\kappa} l(N) E_{\pi(\kappa)} = E_{\kappa} l(N).$$

This shows that $E_{\kappa} l(N) = l(N) E_{\pi(\kappa)}$ is a two-sided ideal. Let d be its arbitrary non-zero element; $d = \sum_{i=1}^{f(\kappa)} e_{\kappa,i} d$ and at least one of $e_{\kappa,i} d$, say $e_{\kappa,p} d$, does not vanish. $e_{\kappa,p} d A = e_{\kappa,p} l(N)$ since the right ideal $e_{\kappa,p} l(N)$ is simple, and therefore $A d A = A e_{\kappa,p} l(N) = E_{\kappa} l(N)$. Hence the two-

⁽¹²⁾ G. Birkhoff (32).

⁽¹³⁾ See the second REMARK in Part II, § 4 and the CORRECTION at the end of Part II.

sided ideal $E_\kappa l(N) = l(N)E_{\pi(\kappa)}$ is simple. In particular, it is completely reducible as a left ideal too, or, what is the same, $E_\kappa l(N) \subseteq r(N)$. Hence $l(N) \subseteq r(N)$.⁽¹⁴⁾

Now, since $l_{\pi(\kappa)}$, the unique simple left subideal of $Ae_{\pi(\kappa)}$, is equal to $r(N)e_{\pi(\kappa)}$, we have

$$e_\kappa l_{\pi(\kappa)} = e_\kappa r(N)e_{\pi(\kappa)} \supseteq e_\kappa l(N)e_{\pi(\kappa)} = r_\kappa e_{\pi(\kappa)} \neq 0,$$

because of i). Consequently, $l_{\pi(\kappa)} \cong Ae_\kappa / Ne_\kappa$. Thus we have derived the second part of ii) from the other conditions, and the lemma is proved.

The latter half of our theorem is now also easy to prove. Denote, namely, our dual-isomorphism between A_0 and P_1 by $\sigma: l \leftrightarrow r = \sigma(l)$ ($l \in A_0, r \in P_1$). Thus $d_l(l) = d_r(A/\sigma(l))$. Consider in particular $\sigma(r(N)E_{\pi(\kappa)}) \in P_1$ corresponding to $r(N)E_{\pi(\kappa)} \in A_0$. We know, from what we have seen, that $r(N)E_{\pi(\kappa)}$ is the sum of all the simple left ideals isomorphic to $Ae_\kappa / Ne_\kappa \cong r(N)e_{\pi(\kappa)}$. That is, $r(N)E_{\pi(\kappa)}$ is the join of all the atoms in A_0 perspective to $r(N)e_{\pi(\kappa)}$. Hence $\sigma(r(N)E_{\pi(\kappa)})$ must be the intersection of a maximal set of right ideals r such that the quotients of A over r are all mutually projective. Consequently, as one easily sees,

$$\sigma(r(N)E_{\pi(\kappa)}) = N \vee (1 - E_\lambda)A$$

with a suitable λ . Since $r(N)E_{\pi(\kappa)}$ is, as we have seen, a join of $f(\pi(\kappa))$ independent simple left ideals, and $N \vee (1 - E_\lambda)A$ is an intersection of $f(\lambda)$ maximal right ideals independent under A , it follows that

$$(3) \quad f(\pi(\kappa)) = f(\lambda).$$

On the other hand,

$$\begin{aligned} (4) \quad f(\kappa)f(\pi(\kappa)) &= d_l(r(N)E_{\pi(\kappa)}) = d_r(A/\sigma(r(N)E_{\pi(\kappa)})) \\ &= d_r(A/(N \vee (1 - E_\lambda)A)) = f(\lambda)^2. \end{aligned}$$

These two equalities, combined, give iii) $f(\kappa) = f(\pi(\kappa))$. Thus our A is Frobeniusean and the theorem is proved.

Corollary. *Let the left and the right ideal lattices A and P of A be dual-isomorphic to each other. Then A is quasi-Frobeniusean. If moreover the dual-isomorphism, chosen suitably, is such that $d_l(l) = d_r(A/r)$ when l corresponds to r , then A is Frobeniusean.*

⁽¹⁴⁾ After the present lemma is established, we shall be able to assert, by symmetry, the converse inclusion (whence the equality) too.

Theorem 2. In case of an algebra A , over a field F , we can replace the condition $d_l(I) = d_r(A/r)$ in Theorem 1 (and its corollary) with the relation

$$(I : F) + (r : F) = (A : F) .$$

Proof. We start with a quasi-Frobeniusean algebra A . Then the division algebras $\bar{e}_\kappa \bar{A} \bar{e}_\kappa = e_\kappa A e_\kappa / e_\kappa N e_\kappa$ and $\bar{e}_{\pi(\kappa)} \bar{A} e_{\pi(\kappa)}$, over F , are isomorphic to each other.⁽¹⁵⁾ For, denote $r(N) = l(N)$ by M , and let m be a non-zero element in $e_\kappa M e_{\pi(\kappa)}$. Then the correspondence $\bar{a} (\in \bar{e}_\kappa \bar{A} \bar{e}_\kappa) \leftrightarrow \bar{a}' (\in \bar{e}_{\pi(\kappa)} \bar{A} e_{\pi(\kappa)})$, such that $am = ma'$, gives really an isomorphism between the two division algebras. Put for the sake of brevity

$$m(\kappa) = (\bar{e}_\kappa \bar{A} \bar{e}_\kappa : F) .$$

Thus

$$(5) \quad m(\kappa) = m(\pi(\kappa)) .$$

Now, let σ be a dual-isomorphism between A_0 and P_1 satisfying our rank condition $(I : F) + (\sigma(I) : F) = (A : F)$, or

$$(I : F) = (A / \sigma(I) : F) \quad (I \in A_0) .$$

We have seen in the proof of Theorem 1 that $\sigma(ME_{\pi(k)})$ is of a form

$$\sigma(ME_{\pi(k)}) = N \cup (1 - E_\lambda)A$$

with a suitable λ . λ is a function of $\pi(k)$, whence of k , and we write $\lambda = \lambda(k)$. Then $(\lambda(1), \lambda(2), \dots, \lambda(k))$ is a permutation of $(1, 2, \dots, k)$. We have seen also that

$$(6) \quad f(\pi(k)) = f(\lambda(k)) .$$

On the other hand, $(ME_{\pi(k)} : F) = (A / (N \cup (1 - E_{\lambda(k)})A) : F)$, or, what is the same,

$$f(k)m(\kappa)f(\pi(k)) = f(\lambda(k))m(\lambda(k))f(\lambda(k)) .$$

Combined with (6), this gives

$$(7) \quad f(\kappa)m(\kappa) = f(\lambda(\kappa))m(\lambda(\kappa)) .$$

Now, let us forget our algebra A for a moment and just look at two permutations $(\pi(1), \pi(2), \dots, \pi(k))$, $(\lambda(1), \lambda(2), \dots, \lambda(k))$ of $(1, 2, \dots, k)$, and two functions $f(k)$ and $m(k)$, taking natural numbers as values, which satisfy the three relations (5), (6) and (7). We then want to prove

(15) Cf. Part I, § 4, vii) or § 5, Theorem 5. See also Part II, Theorem 3.

$f(\kappa) = f(\pi(\kappa)) = f(\lambda(\kappa))$ (whence $m(\kappa) = m(\pi(\kappa)) = m(\lambda(\kappa))$) for every κ . To do so, consider those κ , for which the product $f(\kappa)m(\kappa)$ takes the smallest value. Choose among them an index κ_0 for which $m(\kappa_0)$ is smallest. Then observe the relation ((7))

$$f(\kappa_0)m(\kappa_0) = f(\lambda(\kappa_0))m(\lambda(\kappa_0)).$$

Thus $\lambda(\kappa_0)$ is among our specified κ 's, and therefore, $m(\kappa_0) \leq m(\lambda(\kappa_0))$ because of our choice of κ_0 . On the other hand $f(\lambda(\kappa_0)) = f(\pi(\kappa_0))$ ((6)), while $m(\kappa_0) = m(\pi(\kappa_0))$ ((5)). Consequently,

$$f(\pi(\kappa_0))m(\pi(\kappa_0)) = f(\lambda(\kappa_0))m(\kappa_0) \leq f(\lambda(\kappa_0))m(\lambda(\kappa_0)) = f(\kappa_0)m(\kappa_0).$$

Hence, according to our construction, $\pi(\kappa_0)$ is also among our specified κ 's, and the above inequalities are necessarily equalities. Thus

$$f(\kappa_0) = f(\pi(\kappa_0)) = f(\lambda(\kappa_0)), \quad m(\kappa_0) = m(\pi(\kappa_0)) = m(\lambda(\kappa_0)).$$

Repeating this argument, we find that $f(\kappa)$ and $m(\kappa)$ are constant on those indices κ which are obtained from κ_0 by repeated application of π and λ ; Our assertion is valid for those indices.

Consider now the remaining indices. They form, with π , λ , f and m restricted on them, a second system of the described type, which however consists of a smaller number of indices. Thus, our assertion can be proved by a simple induction,

Now that the relation $f(\kappa) = f(\pi(\kappa))$ is established, our algebra A is Frobeniusean, and the theorem is proved.

3. A remark on the two-sided ideal lattice.

The above Theorem 1 (as well as Theorem 2) is closely related to the converse parts of our previous main theorems (Part II, Theorems 6, 7; Part I, Theorem 3), though none covers the other. A similar modification of our characterization of Frobeniusean rings by their two-sided ideals and their annihilators (Parts II, Th. 10, 11; Part I, Th. 7, 8) does *not* hold. Namely, an algebra, whose two-sided ideal lattice is self-dual and has a dual-automorphism such that the corresponding two two-sided ideals have dual ranks over the ground field F , is neither Frobeniusean nor quasi-Frobeniusean in general.⁽¹⁶⁾

Example. Let $A = Fe_1 + Fe_2 + Fn_1 + Fn_2$ and its multiplication table be

(16) As to algebras with self-dual left and right ideal lattices, see § 5 below.

	e_1	e_2	n_1	n_2
e_1	e_1	0	n_1	0
e_2	0	e_2	0	n_2
n_1	0	n_1	0	0
n_2	0	n_2	0	0

Two-sided ideals in this (associative) algebra A are, as one easily verifies,

$$\mathfrak{J}_0 = A = \{e_1, e_2, n_1, n_2\},$$

$$\mathfrak{J}_1 = \{e_2, n_1, n_2\}, \quad \mathfrak{J}_2 = \{e_1, n_1, n_2\},$$

$$\mathfrak{J}_3 = \{e_1, n_1\}, \quad \mathfrak{J}_4 = \{n_1, n_2\},$$

$$\mathfrak{J}_5 = \{n_2\}, \quad \mathfrak{J}_6 = \{n_1\},$$

$$\mathfrak{J}_7 = 0.$$

Their lattice possesses the above mentioned property. But A is not quasi-Frobeniusean (since Ae_2 has a completely reducible left subideal which is not simple).

4. Regular (adjoint) representations of a Lie-algebra.

In this section we consider a Lie-algebra L over a field F . Let $a_1, a_2, \dots, a_n$ be its basis. Its left and right regular representations $S(a), R(a)$ ($a \in L$) are given by

$$a \circ (a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_n) S(a),$$

$$(a_1, a_2, \dots, a_n) \circ a = (a_1, a_2, \dots, a_n) R(a)'.$$

Here $S(a) = -R(a)'$. That a matrix $P = (\pi_{\rho\sigma})$ intertwines $R(a)$ with $S(a)$ (that is, $R(a)P = PS(a)$) is equivalent to that the bilinear form $P(x, y) = \sum \pi_{\rho\sigma} \xi_\rho \eta_\sigma$ ($x = \sum \xi_\rho a_\rho, y = \sum \eta_\sigma a_\sigma \in L$) is an invariant:

$$P(x, a \circ y) + P(a \circ x, y) = 0. \quad (17)$$

Hence $S(a)$ and $R(a)$ are equivalent to each other if and only if there exists a non-singular invariant bilinear form $P(x, y)$ of L . If this is the

(17) Further, if this is the case and if $a \rightarrow M(a)$ is an arbitrary representation of L , then the matrix $C = \sum \pi_{\rho\sigma} M(a_\rho) M(a_\sigma)$ (generalized Casimir matrix) is commutative with all the representation matrices $M(a)$.

case, the lattice of ideals in L is self-dual and has a dual-automorphism satisfying the condition of dual ranks. Indeed,⁽¹⁸⁾ such a dual-automorphism can be given by letting an ideal $\mathfrak{z}$ correspond to the ideal $\sigma(\mathfrak{z})$ consisting of all the elements y such that $P(x, y) = 0$ for all x in $\mathfrak{z}$.

However, a Lie-algebra whose ideal lattice has a dual-automorphism such that the corresponding ideals have dual ranks does not necessarily have mutually equivalent left and right regular representations.

Example. Let

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and let their multiplication be the commutator-multiplication: $a_1 \circ a_2 = -a_2$. There are three ideals in $L = a_1F + a_2F$; namely L , a_2F and 0 . Their lattice certainly satisfies the above condition. But $R(a_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $R(a_2) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, and $R(a)$ is not equivalent to $S(a) = -R(a)'$, as one readily sees.

It is perhaps of some interest to note that there exists a certain connection between symmetric algebras and Lie-algebras with mutually equivalent left and right regular representations. Namely, if A is a *symmetric algebra* and if we consider it as a Lie-algebra under commutator multiplication, then the two regular representations of the Lie-algebra thus obtained can be transformed to each other by the same non-singular symmetric matrix which transforms the two regular representations of the original (associative) algebra A . If furthermore a non-singular symmetric hyperplane⁽¹⁹⁾ H in A does not contain the unit element 1 of A , then also the regular representations of the Lie-algebra H , under commutator multiplication, are transformed to each other by a non-singular symmetric matrix. To see this last, one has only to observe that if $a_1 = 1, a_2, \dots, a_n$ is a basis of A such that $a_2, a_3, \dots, a_n$ span H , then the symmetric, and non-singular, parastrophic matrix corresponding to H and to this basis has only zeros in the first row and first column except at their intersection $(1, 1)$ -place.

5. A generalization concerning two algebras.

We now return to associative rings. Let A and A' be two rings satisfying the minimum condition, N and N' their radicals. Let $A_0[P_0]$ and $A'_0[P'_0]$ be the lattices of completely reducible left [right] ideals of

⁽¹⁸⁾ See Part I, § 8, Lemma 6.

⁽¹⁹⁾ See Brauer-Nesbitt (3) and Part I, §§ 2 and 9.

A and A' respectively. Further, denote the lattices of left [right] ideals of A and A' containing N and N' , respectively, by $\Lambda_1[P_1]$ and $\Lambda'_1[P'_1]$.

Theorem 3. Suppose $\Lambda_0, \Lambda'_0, P_0$ and P'_0 are dual⁽²⁰⁾-isomorphic⁽²¹⁾ respectively with P'_1, P_1, Λ'_1 and Λ_1 . Then both A and A' are quasi-Frobeniusean. If moreover the dual-isomorphism $\sigma: I \leftrightarrow \sigma(I) = I'$ ($I \in \Lambda_0, I' \in \Lambda'_0$) of Λ_0 and Λ'_0 satisfies $d_l(I) = d_r(A'/\sigma(I))$ (for every $I \in \Lambda_0$), then A is Frobeniusean.

Proof. Let idempotent elements $e'_{\mu,l}$ ($\mu = 1, 2, \dots, k'; l = 1, 2, \dots, f'(\mu)$) in A' have the similar significance as $e_{\kappa,i}$ in A . Then the numbers of independent atoms in the complemented modular lattices P'_1 and P_1 are $\sum_{\mu=1}^{k'} f'(\mu)$ and $\sum_{\kappa=1}^k f(\kappa)$ respectively, while those of Λ_0 and Λ'_0 are, as the decomposition (1) and the similar decomposition of A' show, at least $\sum f(\kappa)$ and $\sum f'(\mu)$. We have therefore both $\sum f(\kappa) \leq \sum f'(\mu)$ and $\sum f'(\mu) \leq \sum f(\kappa)$. Hence $\sum f(\kappa) = \sum f'(\mu)$, and the left ideals $Ae_{\kappa,i}, A'e'_{\mu,l}$ have unique simple left subideals, while the left annihilators $l(\sum e_{\kappa,i}), l(\sum e'_{\mu,l})$ of $\sum e_{\kappa,i}, \sum e'_{\mu,l}$ in A, A' vanish. Moreover, if we consider the number of mutually non-perspective atoms in P'_1, P_1, Λ_0 and Λ'_1 , then it follows readily that $k \geq k'$ and $k \leq k'$. Hence $k = k'$, and the unique simple left subideals in the left ideals $Ae_{\kappa,i}[A'e'_{\mu,l}]$ are isomorphic (if and) only if the first indices $\kappa[\mu]$ are the same. ²

The same situation prevails for right ideals of A and A' . Thus both A and A' are quasi-Frobeniusean, because of Lemma 2.

Further, one sees readily that

$$\sigma(ME_{\pi(\kappa)}) = N' \cup (1 - E'_{\mu})A'$$

with a suitable $\mu = \mu(\kappa)$, where $E'_{\mu} = \sum_l e'_{\mu,l}$ and, as before, $M = r(N) = l(N)$. Consequently

$$(8) \quad f(\pi(\kappa)) = f'(\mu(\kappa)).$$

Now, suppose the condition of the last half of the theorem is fulfilled. Then

$$f(\kappa)f(\pi(\kappa)) = d_l(ME_{\pi(\kappa)}) = d_r(A'/(N' \cup (1 - E'_{\mu})A')) = f'(\mu(\kappa))^2.$$

Hence, combined with (8), $f(\kappa) = f(\pi(\kappa))$, and A is Frobeniusean.

(20) Here one can also replace "dual" with "direct", on considering a ring anti-isomorphic to A' . (Of course the assumption of the latter half should then be modified correspondingly).

(21) We have only to assume that $\Lambda_0, \Lambda'_0, P_0, P'_0$ are dual-homomorphic to P'_1, P_1, Λ'_1 and Λ_1 . Then the dual-isomorphisms follow automatically, as one readily sees from the decomposition (1) and the similar decomposition of A' . See also the proof below.

Corollary. Suppose the left and the right ideal lattices A, P of A are dual-isomorphic to the right and the left ideal lattices P', A' of A' . Then both A and A' are quasi-Frobeniusean. If moreover $d_l(I) = d_r(A'/r')$ when $I \leftrightarrow r'$ in the first, say, of these dual-isomorphisms, then A and A' are Frobeniusean,

Theorem 4. In case of algebras A and A' , over a field F , the condition of the latter half of Theorem 5 can be replaced by

$$(I : F) = (A' / \sigma(I) : F) \quad (I \in \Lambda_0),$$

$$(I' : F) = (A / \sigma'(I') : F) \quad (I' \in \Lambda'_0);$$

where σ and σ' denote the dual-isomorphisms of Λ_0, Λ'_1 and Λ'_0, Λ_1 . (Then not only A but also A' is Frobeniusean).

Proof. Put $m(\kappa) = (\bar{e}_\kappa \bar{A} \bar{e}^\kappa : F)$ and $m'(\mu) = (\bar{e}'_\mu \bar{A}' \bar{e}'_\mu : F)$ ($e'_\mu = e'_{\mu,1}$). Then

$$m(\kappa) = m(\pi(\kappa)), \quad f(\pi(\kappa)) = f'(\mu(\kappa)), \quad f(\kappa)m(\kappa) = f'(\mu(\kappa))m'(\mu(\kappa)),$$

the last equation being due to our new assumption; and similarly,

$$m'(\mu) = m'(\pi'(\mu)), \quad f'(\pi'(\mu)) = f(\lambda(\mu)), \quad f'(\mu)m'(\mu) = f(\lambda(\mu))m(\lambda(\mu))$$

where the permutation $(\pi'(1), \pi'(2), \dots, \pi'(k'))$ of $(1, 2, \dots, k')$ is such that $E'_\mu M' E'_{\pi'(\mu)} = E'_\mu M' = M' E'_{\pi'(\mu)}$ ($M' = l(N') = r(N')$), and where

$$\sigma'(E'_\mu M') = N \cup (1 - E_{\lambda(\mu)})A.$$

Now, denote the range of the indices μ (of $e'_\mu = e'_{\mu,1}$) by $1', 2', \dots, k'$ instead of $1, 2, \dots, k' (=k)$, to distinguish it from that of κ . And, put

$$n(\rho) = \begin{cases} m(\rho) \\ m'(\rho) \end{cases}, \quad g(\rho) = \begin{cases} f(\rho) \\ f'(\rho) \end{cases}, \quad \tau(\rho) = \begin{cases} \pi(\rho) \\ \pi'(\rho) \end{cases}, \quad \nu(\rho) = \begin{cases} \mu(\rho) \\ \lambda(\rho) \end{cases},$$

according as, in each case, ρ is among $1, 2, \dots, k$ or among $1', 2', \dots, k'$. Then the two permutations $(\tau(1), \tau(2), \dots, \tau(k), \tau(1'), \dots, \tau(k'))$, $(\nu(1), \nu(2), \dots, \nu(k), \nu(1'), \dots, \nu(k'))$ of the $2k = k + k'$ indices $(1, 2, \dots, k, 1', \dots, k')$ and the two functions $n(\rho)$ and $g(\rho)$, taking natural numbers as values, satisfy

$$n(\rho) = n(\tau(\rho)), \quad g(\tau(\rho)) = g(\nu(\rho)), \quad g(\rho)n(\rho) = g(\nu(\rho))n(\nu(\rho)).$$

Thus we again come across our old combinatorial problem, and find that $g(\rho) = g(\tau(\rho))$ ($\rho = 1, 2, \dots, k, 1', \dots, k'$), or, what is the same,

$$f(\kappa) = f(\pi(\kappa)), \quad f'(\mu) = f'(\pi'(\mu)) \quad (\kappa, \mu = 1, 2, \dots, k).$$

Hence A and A' are Frobeniusean.

6. Loewy parts of regular representations and algebras whose left and right ideal lattices are self-dual.

Group algebras (modular or non-modular) and homology algebras of (oriented closed) manifolds⁽²²⁾ are useful examples of Frobeniusean algebras; indeed, group algebras are even symmetric. But these algebras possess a well-known second 1-1 correspondence between left and right ideals, namely, a (direct) isomorphism between left and right ideal lattices. In fact, a group algebra has an involutorial anti-automorphism which maps group elements to their inverses.⁽²³⁾ Also an algebra of a manifold has an involutorial anti-automorphism mapping an i -dimensional homology class z_i onto $(-1)^{\frac{i(i-1)}{2}} z_i$. Thus, combined with our dual-isomorphisms of left and right ideal lattices, it follows that *both the left and the right ideal lattices of these algebras are self-dual.*⁽²⁴⁾

However, such is evidently not the case for general (not only Frobeniusean but) symmetric algebras. For instance, the left [right] ideal lattice of a simple algebra A is not self-dual, as soon as A is of capacity 3 at least and the belonging division algebra does not possess an anti-automorphism. In this example, however, the lattice of left subideals of a directly indecomposable left ideal in a decomposition (1) is self-dual. An example of a symmetric algebra, for which this last is not the case, can readily be obtained by considering a fully primary (= vollprimär) symmetric algebra A such that, when N denotes its radical, A/N has no anti-automorphism and N^i/N^{i+1} is of rank at least 3 over A/N for a certain i . But this example does not apply to an algebraically closed ground field, and moreover, there prevails in this example still a symmetry between the upper and the lower successive largest completely reducible parts (= so-called Loewy parts) of a directly indecomposable component in the regular representation. So, we finally turn to the following example, which applies to any ground field, algebraically closed

(22) See Lefschetz (33). That a homology algebra of a manifold is Frobeniusean, can readily be seen from the fact that 0-dimensional homology classes constitute a unique simple left ideal as well as a unique simple right ideal.

(23) Also semi-linear group algebras (See Nakayama-Shoda (12) and Part I, § 9) are anti-isomorphic to themselves, and their ideal lattices are self-dual.

(24) The fact, that the left (say) ideal lattices of homology algebras of manifolds are self-dual, may be looked upon as an algebraic supplement to Poincaré's duality.

or, not, and which shows that *even such a symmetry does not prevail in a general symmetric algebra.*⁽²⁵⁾

Example. Let F be any field, and let A be an algebra of rank 18 over F , consisting of matrices S of the form⁽²⁶⁾

$$S = \begin{pmatrix} \boxed{\begin{matrix} a_1 & & & & & \\ a_2 & \beta_1 & & & & \\ a_3 & 0 & \beta_1 & & & \\ a_4 & 0 & 0 & \beta_1 & & \\ a_5 & \beta_2 & \beta_3 & 0 & \gamma_1 & \\ a_6 & \beta_4 & \beta_5 & \beta_6 & \gamma_2 & a_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} \beta_1 & & & & & \\ \beta_2 & \gamma_1 & & & & \\ \beta_3 & 0 & \gamma_1 & & & \\ \beta_4 & \gamma_2 & 0 & a_1 & & \\ \beta_5 & 0 & \gamma_2 & 0 & a_1 & \\ \beta_6 & 0 & 0 & 0 & 0 & a_1 \\ \beta_7 & \gamma_3 & \gamma_4 & a_2 & a_3 & a_4 & \beta_1 \end{matrix}} & \\ & & \boxed{\begin{matrix} \gamma_1 & & & & & \\ \gamma_2 & a_1 & & & & \\ \gamma_3 & a_2 & \beta_1 & & & \\ \gamma_4 & a_3 & 0 & \beta_1 & & \\ \gamma_5 & a_5 & \beta_2 & \beta_3 & \gamma_1 & \end{matrix}} \end{pmatrix}$$

where α 's, β 's and γ 's run independently over F . By putting one of these parameters to be equal to 1 and all the others to be zero, we

(25) But, if $G_\kappa(a)$ is an irreducible representation of a symmetric (or weakly symmetric) algebra A and if $U_\kappa(a)$ is the directly indecomposable component of the regular representation of A corresponding to $G_\kappa(a)$, then the multiplicity of G_κ in the i -th upper Loewy part to U_κ is equal to that of G_κ in the i -th lower Loewy part of U_κ . Because

$$(e_\kappa N^{i-1} e_\kappa / e_\kappa N^i e_\kappa : F) = (e_\kappa l(N^i) e_\kappa / e_\kappa l(N^{i-1}) e_\kappa : F).$$

(26) An actual computation, that a product of two matrices of this form is again of the same form, will perhaps show the nature of the example clearly.

obtain 18 elements which constitute a basis of A . In fact, S is the left regular representation of A with respect to this basis, and the three blocks are its directly indecomposable components. Further its parastrophic matrix P with respect to the same basis is :

$$\begin{pmatrix}
 \lambda_1 & 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 & 0 & \mu_4 & \mu_5 & \mu_6 & 0 & 0 & \nu_2 & 0 & 0 & 0 \\
 \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & 0 & 0 & \nu_3 & 0 & 0 & 0 \\
 \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & 0 & \nu_4 & 0 & 0 & 0 \\
 \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_5 & 0 & 0 & 0 \\
 \lambda_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & \lambda_2 & \lambda_3 & \lambda_4 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & \nu_3 & \nu_4 & 0 \\
 0 & \lambda_5 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_5 & 0 & 0 \\
 0 & 0 & \lambda_5 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_5 & 0 \\
 0 & \lambda_6 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \lambda_6 & 0 & 0 & 0 & \mu_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & \lambda_6 & 0 & 0 & \mu_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & \lambda_5 & 0 & 0 & \mu_2 & \mu_3 & 0 & 0 & 0 & 0 & \nu_1 & 0 & 0 & 0 & \nu_5 \\
 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 & \mu_4 & \mu_5 & 0 & 0 & 0 & 0 & \nu_2 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & 0 & 0 & 0 & \nu_3 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & 0 & 0 & 0 & \nu_4 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_5 & 0 & 0 & 0 & 0
 \end{pmatrix}$$

where λ 's, μ 's and ν 's are independent parameters. If we put $\lambda_6 = \mu_7 = \nu_5 \neq 0$ and put all other parameters to be 0, then we obtain a non-singular symmetric matrix. Thus A is a symmetric algebra. However, one readily sees that there is no symmetry even for the sizes of the upper and lower successive largest completely reducible parts of the first, say, component of the regular representation S .

Now, the converse (not with respect to symmetric algebras but with respect to Frobeniusean algebras) is, however, certainly true, though not very significant. Namely,

Theorem 5. *Let the lattices A_0 and P_0 of completely reducible left and right ideals of A be dual-isomorphic, respectively, to the lattices A_1 and P_1 of left and right ideals containing the radical of A . Then A is*

quasi-Frobeniusean. If moreover the dual-isomorphism, chosen suitably, of A_0 and A_1 satisfies $d_l(I) = d_l(A/I_1)$, where $I \in A_0$ and $I_1 \in A_1$ correspond to each other, then A is Frobeniusean.

Proof. The theorem can be verified in quite a similar manner as Theorem 1. It can also be reduced to Theorem 5; one has simply to consider A and a second ring A' which is anti-isomorphic to A .

Corollary. *Suppose both the left and the right ideal lattices Λ, P of A are self-dual. Then A is quasi-Frobeniusean. If moreover the dual-automorphism of Λ , suitably chosen, is such that $d_l(I) + d_l(I_1) = d_l(A)$ for every pair I, I_1 of mutually corresponding left ideals, then A is Frobeniusean.*

Theorem 6. *In case of an algebra A , over a field F , we can replace the condition in Theorem 3 and its corollary, expressed by d_l , with the relation $(I : F) + (I_1 : F) = (A : F)$.*

Proof. This can be seen in a similar way as in Theorem 2, or in Theorem 4.

Example of primary (Frobeniusean) algebras, whose left and right ideal lattices have dual-automorphisms such that the corresponding ideals have dual ranks, but which are not symmetric, can readily be given by homology algebras of manifold, or in particular, by Grassmann-Cartan's algebras of outer forms (Cf. Part II, § 1, Example). See also Part I, § 9, Example.

6. On principal two-sided ideals in Frobeniusean rings.

We have proved (Corollary to Part I, Theorem 4) that if a two-sided ideal $\mathfrak{z}$ in a Frobeniusean algebra A is a principal left ideal $\mathfrak{z} = Ac$ then $\mathfrak{z} = cA$ too. However, a similar theorem fails to be true in case of a general Frobeniusean ring, as the following example shows:

Example. Let $\Omega = F(\xi)$ be a rational function field of an indeterminate, ξ , over F . $\xi \rightarrow \xi^2$ generates an isomorphism ρ of the field Ω with its subfield $\rho(\Omega) = F(\xi^2)$. Now, introduce four elements e, n, m and p , which satisfy

$$ae = ea, \quad an = n\rho(a), \quad \rho(a)m = ma, \quad ap = pa$$

where $a \in \Omega$. Further, assume $e, n, n\xi, m$ and p to be left-linearly independent over Ω , while $e, n, m, \xi m$ and p are assumed to be right-linearly independent over Ω . Let moreover the multiplication among the four elements be given by

	e	n	m	p
e	e	n	m	p
n	n	0	p	0
m	m	p	0	0
p	p	0	0	0

Then $A = \Omega e + \Omega n + \Omega n\xi + \Omega m + \Omega p = e\Omega + n\Omega + m\Omega + \xi m\Omega + p\Omega$ form an (associative) ring, as one readily verifies. It has a unit element e , possesses a unique simple left ideal $p\Omega$ which is at the same time a unique simple right ideal. Hence A is a Frobeniusean ring.

Now, consider the two-sided ideal $\mathfrak{z} = n\Omega + p\Omega$. It is a principal right ideal $\mathfrak{z} = nA$. But it is not a principal left ideal. For, the left A -module $\mathfrak{z}/p\Omega$ is a direct sum of two simple submoduli, and consequently, the left A -module $\mathfrak{z}$ is not operator-homomorphic to A , since A is fully primary.

Remark. However, even in a general ring possessing a unit element and satisfying the minimum condition (whence the maximum condition either) a two-sided ideal $\mathfrak{z}$ which is both left- and right-principal can be generated by one and the same element on the left and right. Namely: if $\mathfrak{z} = cA = Ad$, then $\mathfrak{z} = Ac = dA$ too.⁽²⁷⁾

Bibliography

(Continued from Parts I and II).

- (31) K. Asano, Über Ringe mit Vielfachenkettensatz, Proc. Imp. Acad. 15 (1939).
- (32) G. Birkhoff, Combinatorial relations in projective geometries, Ann. Math. 36 (1935).
- (33) S. Lefschetz, Topology, New York (1930).
- (34) J. Levitzki, On rings which satisfy the minimum condition for the right-hand ideals, Compositio Math. (1940).
- (35) T. Nakayama, Note on uni-serial and generalized uni-serial rings, Proc. Imp. Acad. 16 (1940).
- (36) T. Nakayama, Algebras with anti-isomorphic left and right ideal lattices, Proc. Imp. Acad. 14 (1941).

Department of Mathematics,
Osaka Imperial University.

⁽²⁷⁾ See Nakayama (35).