15. Nevanlinna's fundamental theorems and Ahlfors' theorem on the number of asymptotic values.

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(Received March 12, 1942.)

PART I. NEVANLINNA'S FUNDAMENTAL THEOREMS.

1. First Fundamental Theorem.

Let K be the Riemann sphere of diameter 1, which touches the w -plane at $w = 0$ and $[a, b] = \frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}$ and Δ be a domain on the z -plane, which contains ∞ on its boundary and $w = w(z) = f(z)$ be one-valued and meromorphic in and at the boundary points ($\neq \infty$) of Δ , such that $[f(z), w_0] < \delta$ in Δ and $[f(z), w_0] = \delta$ at the boundary points ($\neq \infty$) of Δ , so that the inverse function $z = \varphi(w)$ of $w = f(z)$ is defined on the Riemann surface F , spread over $[w, w_0] < \delta$.

A δ -neighbourhood U of w_0 is defined as the connected part of F , which lies above $[w, w_0] < \delta$ and has w_0 as an inner point or as a boundary point. The boundary Δ of Δ consists of two classes of curves. Namely the ones which extend to infinity and the others which are closed curves and form holes in Δ . We add all such holes to Δ and the resulting simply connected domain be denoted by $\bar{\Delta}$. If $\bar{\Delta}$ has a boundary curve which extends to infinity, let Δ_0 be the outermost such curve and the simply connected domain bounded by Δ_0 be denoted by $\bar{\Delta}$. Let z_0 be a fixed point. We denote the part of Δ , which lies in $|z - z_0| \leq r$ by Δ_r and the common part of $|z - z_0| = r$ and Δ , $\bar{\Delta}$, $\bar{\Delta}$ by θ_r , $\bar{\theta}_r$, $\bar{\bar{\theta}}_r$ and its total length by $r\theta(r)$, $r\bar{\theta}(r)$, $r\bar{\bar{\theta}}(r)$ respectively.

We put

$L(r) = L(r, w; \Delta)$ = the total length of the image of θ_r by $w = f(z)$ on K ,
 $A(r) = A(r, w; \Delta)$ = the total area of the image of Δ_r by $w = f(z)$ on K ,
 $S(r) = S(r, w; \Delta) = A(r)/\pi\delta^2$, where $\pi\delta^2$ is the area of $[w, w_0] < \delta$ on K ,
 $n(r, a) = n(r, a, w; \Delta)$ = the number of zero points of $f(z) - a$ in Δ_r ,

$$N(r, a) = N(r, a, w; \Delta) = \int_{r_0}^r \frac{n(r, a)}{r} dr,$$

$$m(r, a) = m(r, a, w; \Delta) = \frac{1}{2\pi} \int_{\theta_r} \log \frac{1}{[w(re^{i\theta}), a]} d\theta,$$

$$T(r, a) = T(r, a, w; \Delta) = N(r, a) + m(r, a).$$

Then the following theorem⁽¹⁾ corresponds to Nevanlinna's first fundamental theorem.

Theorem I (FIRST FUNDAMENTAL THEOREM). *For any point a in $[w, w_0] < \delta$,*

$$T(r, a) = T(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right), \quad (1)$$

where

$$T(r) = T(r, w; \Delta) = \int_{r_0}^r \frac{S(r)}{r} dr,$$

$$\int_{r_0}^r \frac{L(r)}{r} dr = O(\sqrt{T(2r) \log r}) \text{ for all } r, \quad (2)$$

$$= O(\sqrt{T(r)} \log T(r)), \quad (3)$$

except certain intervals I_n , such that $\sum_n \int_{I_n} d \log \log r < \infty$ ⁽²⁾.

We will call $T(r)$ the characteristic function of $f(z)$ in Δ and $\lim_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \rho$ the order of $f(z)$ in Δ . If $\rho < \infty$, then we have from (1), (2),

$$T(r, a) = T(r) + O(r^{\rho'}) \quad (\rho' < \rho) \quad (1^*)$$

2. Some lemmas.

We will prove some lemmas.

Lemma. *Let $w = w(z) = w(re^{i\theta})$ satisfy the condition in §1 and $U(w)$ be a linear transformation, which makes $[w, w_0] < \delta$ invariant, then*

$$\int_{[b, w_0] \leq \delta} \left| \frac{d}{dr} \log \frac{[U(w), b]}{[w, b]} \right| d\omega(b) \leq \text{const.} \frac{|w'|}{1 + |w|^2},$$

(¹) M. TSUJI: On the behaviour of an inverse function of a meromorphic function at its transcendental singular point, III. Proc. Imp. Acad. **18** (1942). Y. TUMURA: Sur le problème de M. KUNUGUI. Proc. Imp. Acad. **18** (1942) and Sur le première théorème dans la théorie des fonctions méromorphes. Proc. Imp. Acad. **18** (1924).

(²) A. DINGHAS: Eine Bemerkung zur Ahlforsschen Theorie der Überlagerungsflächen. Math. Z. **44** (1939).

$$\left| \frac{d}{dr} \log \frac{[U(w), U(b)]}{[w, b]} \right| \leq \text{const.} \frac{|w'|}{1 + |w|^2},$$

where $[b, w_0] < \delta$ and $d\omega(b)$ is the surface element on K at b .

Proof. We may assume that $[w, w_0] < \delta$ is a circle $|w| < \rho$ and $U(w) = \frac{\rho(w-a)}{\rho^2 - \bar{a}w}$ ($|a| < \rho$), since by the rotation of K , we can bring $[w, w_0] < \delta$ to $|w| < \rho$ and $\frac{|w'|}{1 + |w|^2}$ is invariant by the rotation.

Now

$$\left| \frac{d}{dr} \log [w, b] \right| \leq \left| \frac{d}{dr} \log |w - b| \right| + \frac{1}{2} \left| \frac{d}{dr} \log (1 + |w|^2) \right|, \quad (4)$$

$$\left| \frac{d}{dr} \log |w - b| \right| \leq \left| \frac{d}{dr} \log (w - b) \right| = \frac{|w'|}{|w - b|},$$

so that

$$\int_{|b| \leq \rho} \left| \frac{d}{dr} \log |w - b| \right| d\omega(b) \leq \text{const.} |w'| \leq \text{const.} \frac{|w'|}{1 + |w|^2}. \quad (5)$$

If we put $w = u + iv$, then

$$\frac{1}{2} \left| \frac{d}{dr} \log (1 + |w|^2) \right| = \frac{|uu_r + vv_r|}{1 + |w|^2} \leq \frac{\sqrt{(u^2 + v^2)(u_r^2 + v_r^2)}}{1 + |w|^2} \leq \text{const.} \frac{|w'|}{1 + |w|^2}. \quad (6)$$

Hence from (4), (5), (6),

$$\int_{|b| \leq \rho} \left| \frac{d}{dr} \log [w, b] \right| d\omega(b) \leq \text{const.} \frac{|w'|}{1 + |w|^2}. \quad (7)$$

Similarly

$$\int_{|b| \leq \rho} \left| \frac{d}{dr} \log [U(w), b] \right| d\omega(b) \leq \text{const.} \left| \frac{d}{dz} U(w) \right| \leq \text{const.} \frac{|w'|}{1 + |w|^2}. \quad (8)$$

Hence we have from (7), (8),

$$\int_{|b| \leq \rho} \left| \frac{d}{dr} \log \frac{[U(w), b]}{[w, b]} \right| d\omega(b) \leq \text{const.} \frac{|w'|}{1 + |w|^2},$$

which proves the first part. Next

$$\begin{aligned} \left| \frac{d}{dr} \log \frac{[U(w), U(b)]}{[w, b]} \right| &\leq \left| \frac{d}{dr} \log \left| \frac{U(w) - U(b)}{w - b} \right| \right| \\ &+ \frac{1}{2} \left| \frac{d}{dr} \log (1 + |U(w)|^2) \right| + \frac{1}{2} \left| \frac{d}{dr} \log (1 + |w|^2) \right|. \end{aligned} \quad (9)$$

Since

$$\left| \frac{U(w) - U(b)}{w - b} \right| = \frac{\rho(\rho^2 - |a|^2)}{|\rho^2 - \bar{a}w| \cdot |\rho^2 - \bar{a}b|},$$

we have

$$\left| \frac{d}{dr} \log \left| \frac{U(w) - U(b)}{w - b} \right| \right| \leq \text{const.} \frac{|w'|}{1 + |w|^2}. \quad (10)$$

By (6),

$$\frac{1}{2} \left| \frac{d}{dr} \log (1 + |U(w)|^2) \right| \leq \text{const.} \left| \frac{d}{dz} U(w) \right| \leq \text{const.} \frac{|w'|}{1 + |w|^2}. \quad (11)$$

From (9), (10), (11), (6), we have

$$\left| \frac{d}{dr} \log \frac{[U(w), U(b)]}{[w, b]} \right| \leq \text{const.} \frac{|w'|}{1 + |w|^2},$$

which proves the second part.

Theorem II. Let $w = w(z)$ satisfy the condition in §1 and $U(w)$ be a linear transformation, which makes $[w, w_0] < \delta$ invariant, then

$$S(r, w; \Delta) - S(r, U(w); \Delta) = O(L(r)).$$

Proof. Let Γ_r be the whole boundary of Δ_r and $\Gamma_r = \theta_r + \gamma_r$ and a, b be any two points in $[w, w_0] < \delta$, then

$$\begin{aligned} \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{[w, a]}{[w, b]} d\theta &= \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{|w-a|}{|w-b|} d\theta \\ &= \frac{1}{2\pi r} \int_{\theta_r} \text{darg} \frac{w-a}{w-b} = \\ \frac{1}{2\pi r} \int_{\Gamma_r} \text{darg} \frac{w-a}{w-b} - \frac{1}{2\pi r} \int_{\gamma_r} \text{darg} \frac{w-a}{w-b} &= \frac{n(r, a, w; \Delta) - n(r, b, w; \Delta)}{r} \\ &\quad - \frac{1}{2\pi r} \int_{\gamma_r} \text{darg} \frac{w-a}{w-b}. \end{aligned} \quad (12)$$

Now γ_r consists of two classes of curves: $\gamma_r = \gamma_r^{(1)} + \gamma_r^{(2)}$, where $\gamma_r^{(1)}$ are closed curves and form holes of Δ_r and $\gamma_r^{(2)}$ are not closed curves and meet θ_r . Evidently

$$\int_{\gamma_r^{(1)}} \text{darg} \frac{w-a}{w-b} = 0. \quad (13)$$

Let Δ_r consists of a certain number of connected domains $\Delta_r^{(i)} (i = 1, 2, \dots)$ and $\gamma_{r,i}^{(2)} + \theta_{r,i}$ be the part of $\gamma_r^{(2)} + \theta_r$, which belongs to the boundary of $\Delta_r^{(i)}$ and let its image on K be of the form :

$$(\alpha_1, \beta_1) + \theta_1 + (\alpha_2, \beta_2) + \theta_2 + \dots + (\alpha_n, \beta_n) + \theta_n, \quad (14)$$

where $\sum_{v=1}^n (\alpha_v, \beta_v)$ is the image of $\gamma_{r,i}^{(2)}$ and consists of circular arcs on $[w, w_0] = \delta$ and $\sum_{v=1}^n \theta_v$ is the image of $\theta_{r,i}$ where θ_v connects β_v and α_{v+1} in $[w, w_0] < \delta$, ($\alpha_{n+1} = \alpha_1$). We replace θ_v by a circular arc (β_v, α_{v+1}) on $[w, w_0] = \delta$ and put

$$C = (\alpha_1, \beta_1) + (\alpha_2, \beta_2) + \dots + (\alpha_n, \beta_n),$$

$$C' = (\beta_1, \alpha_2) + (\beta_2, \alpha_3) + \dots + (\beta_n, \alpha_1),$$

$$C'' = C + C',$$

then C'' is an integral multiple of the whole circumference of $[w, w_0] = \delta$, so that $\int_{C''} \text{darg} \frac{w-a}{w-b} = 0$, hence

$$\begin{aligned} \int_{\gamma_{r,i}^{(2)}} \text{darg} \frac{w-a}{w-b} &= \int_C \text{darg} \frac{w-a}{w-b} = \int_{C''} \text{darg} \frac{w-a}{w-b} - \int_{C'} \text{darg} \frac{w-a}{w-b} \\ &= - \int_{C'} \text{darg} \frac{w-a}{w-b}. \end{aligned}$$

If a, b lie in $[w, w_0] \leq \delta_1 < \delta$, then

$$\left| \int_{C'} \text{darg} \frac{w-a}{w-b} \right| \leq K(\delta_1) \times \text{sum of lengths of } (\theta_1 + \dots + \theta_n),$$

where $K(\delta_1)$ depends on δ_1 only. Hence summing up for $i = 1, 2, \dots$,

$$\left| \int_{\gamma_r^{(2)}} \text{darg} \frac{w-a}{w-b} \right| \leq K(\delta_1) \cdot L(r), \quad (15)$$

so that from (13), (15),

$$\left| \frac{1}{2\pi r} \int_{\gamma_r} \text{darg} \frac{w-a}{w-b} \right| \leq K(\delta_1) \cdot \frac{L(r)}{r}. \quad (16)$$

Hence from (12), (16),

$$\begin{aligned} \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{[w, a]}{[w, b]} d\theta \\ = \frac{n(r, a, w; \Delta) - n(r, b, w; \Delta)}{r} + O\left(\frac{L(r)}{r}\right), \end{aligned} \quad (17)$$

or

$$\begin{aligned} \frac{n(r, a, w; \mathcal{A})}{r} + \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{1}{[w, a]} d\theta \\ = \frac{n(r, b, w; \mathcal{A})}{r} + \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{1}{[w, b]} d\theta + O\left(\frac{L(r)}{r}\right). \end{aligned}$$

Let $d\omega(b)$ be the surface element on K at b , then since $\pi\delta_1^2$ is the area of $[w, w_0] < \delta_1$ on K , taking the integral mean over $[w, w_0] \leq \delta_1 < \delta$, we have

$$\begin{aligned} \frac{n(r, a, w; \mathcal{A})}{r} + \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{1}{[w, a]} d\theta = \frac{S_1(r, w; \mathcal{A})}{r} \\ + \frac{1}{2\pi^2\delta_1^2} \int_{\theta_r} d\theta \int_{[b, w_0] \leq \delta_1} \frac{d}{dr} \log \frac{1}{[w, b]} d\omega(b) + O\left(\frac{L(r)}{r}\right), \quad (18) \end{aligned}$$

where $S_1(r, w; \mathcal{A}) = \frac{A_1(r, w; \mathcal{A})}{\pi\delta_1^2}$, $A_1(r, w; \mathcal{A})$ being the total area on K over $[w, w_0] \leq \delta_1$, which is covered by $w = w(z)$, when z varies in $\mathcal{A}$.

By Ahlfors' first covering theorem⁽³⁾,

$$S(r, w; \mathcal{A}) - S_1(r, w; \mathcal{A}) = O(L(r)), \quad (19)$$

hence

$$\begin{aligned} \frac{n(r, a, w; \mathcal{A})}{r} + \frac{1}{2\pi} \int_{\theta_r} \log \frac{1}{[w, a]} d\theta = \frac{S(r, w; \mathcal{A})}{r} \\ + \frac{1}{2\pi^2\delta_1^2} \int_{\theta_r} d\theta \int_{[b, w_0] \leq \delta_1} \frac{d}{dr} \log \frac{1}{[w, b]} d\omega(b) + O\left(\frac{L(r)}{r}\right). \quad (20) \end{aligned}$$

Similarly for $U(w)$,

$$\begin{aligned} \frac{n(r, U(a), U(w); \mathcal{A})}{r} + \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{1}{[U(w), U(a)]} d\theta = \frac{S(r, U(w); \mathcal{A})}{r} \\ + \frac{1}{2\pi^2\delta_1^2} \int_{\theta_r} d\theta \int_{[b, w_0] \leq \delta_1} \frac{d}{dr} \log \frac{1}{[U(w), b]} d\omega(b) + O\left(\frac{L(r)}{r}\right). \quad (21) \end{aligned}$$

(3) L. AHLFORS: Zur Theorie der Überlagerungsflächen. Acta. Math. 65 (1935).

Since $n(r, a, w; \Delta) = n(r, U(a), U(w); \Delta)$, we have from (20), (21),

$$\begin{aligned} \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{[U(w), U(a)]}{[w, a]} d\theta &= \frac{S(r, w; \Delta) - S(r, U(w); \Delta)}{r} \\ &+ \frac{1}{2\pi^2 \delta_1^2} \int_{\theta_r} d\theta \int_{[b, w_0] \leq \delta_1} \frac{d}{dr} \log \frac{[U(w), b]}{[w, b]} d\omega(b) + O\left(\frac{L(r)}{r}\right). \end{aligned}$$

Since $L(r) = r \int_{\theta_r} \frac{|w'|}{1 + |w|^2} d\theta$, we have by the Lemma,

$$S(r, w; \Delta) - S(r, U(w); \Delta) = O(L(r)), \quad \text{q. e. d.}$$

3. Proof of Theorem I.

Let θ_r consist of a certain number of circular arcs on $|z - z_0| = r$, whose end points are $re^{i\theta_1(r)}$, $re^{i\theta_2(r)}$ ($\theta_2(r) > \theta_1(r)$) and put $\theta(r) = \sum (\theta_2(r) - \theta_1(r))$. We put $w_1 = w(re^{i\theta_1})$, $w_2 = w(re^{i\theta_2})$, then w_1, w_2 lie on $[w, w_0] = \delta$, so that $[w_1, w_0] = [w_2, w_0] = \delta$.

Let a be a point in $[w, w_0] \leq \delta_1 < \delta$, then by (17),

$$\begin{aligned} \frac{d}{dr} m(r, w_0, w; \Delta) - \frac{d}{dr} m(r, a, w; \Delta) &= \frac{1}{2\pi} \frac{d}{dr} \int_{\theta_r} \log \frac{[w, a]}{[w, w_0]} d\theta = \\ \frac{1}{2\pi} \sum \left(\log \frac{[w_2, a]}{[w_2, w_0]} \frac{d\theta_2}{dr} - \log \frac{[w_1, a]}{[w_1, w_0]} \frac{d\theta_1}{dr} \right) &+ \frac{1}{2\pi} \int_{\theta_r} \frac{d}{dr} \log \frac{[w, a]}{[w, w_0]} d\theta \\ = \frac{1}{2\pi} \sum \left(\log \frac{[w_2, a]}{\delta} \frac{d\theta_2}{dr} - \log \frac{[w_1, a]}{\delta} \frac{d\theta_1}{dr} \right) &+ \\ + \frac{n(r, a, w; \Delta) - n(r, w_0, w; \Delta)}{r} &+ O\left(\frac{L(r)}{r}\right). \end{aligned} \quad (22)$$

Hence

$$\begin{aligned} T(r, w_0, w; \Delta) &= T(r, a, w; \Delta) + \\ + \frac{1}{2\pi} \int_{r_0}^r \sum \left(\log \frac{[w_2, a]}{\delta} \frac{d\theta_2}{dr} - \log \frac{[w_1, a]}{\delta} \frac{d\theta_1}{dr} \right) dr &+ O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right). \end{aligned} \quad (23)$$

Multiplying $d\omega(a)$ and taking the integral mean over $[w, w_0] \leq \delta_1 < \delta$, we have

$$\begin{aligned}
T(r, w_0, w; \Delta) &= \frac{1}{\pi \delta_1^2} \int_{[a, w_0] \leq \delta_1} T(r, a, w; \Delta) d\omega(a) \\
&+ \frac{1}{2\pi^2 \delta_1^2} \int_{r_0}^r \left(\int \sum_{[a, w_0] \leq \delta_1} \left(\log \frac{[w_2, a]}{\delta} \frac{d\theta_2}{dr} - \log \frac{[w_1, a]}{\delta} \frac{d\theta_1}{dr} \right) d\omega(a) \right) dr \\
&+ O\left(\int_{r_0}^r \frac{L(r)}{r} dr \right). \quad (24)
\end{aligned}$$

Now by (19),

$$\begin{aligned}
\frac{1}{\pi \delta_1^2} \int_{[a, w_0] \leq \delta_1} T(r, a, w; \Delta) d\omega(a) &= \int_{r_0}^r \frac{S_1(r, w; \Delta)}{r} dr + O(1) \\
&= \int_{r_0}^r \frac{S(r, w; \Delta)}{r} dr + O\left(\int_{r_0}^r \frac{L(r)}{r} dr \right). \quad (25)
\end{aligned}$$

Since, w_1, w_2 lying on $[w, w_0] = \delta$,

$$\frac{1}{2\pi^2 \delta_1^2} \int_{[a, w_0] \leq \delta_1} \log \frac{[w_1, a]}{\delta} d\omega(a) = \frac{1}{2\pi^2 \delta_1^2} \int_{[a, w_0] \leq \delta_1} \log \frac{[w_2, a]}{\delta} d\omega(a) = A = \text{const.},$$

the second term of (24) becomes

$$A \int_{r_0}^r \frac{d}{dr} \sum (\theta_2(r) - \theta_1(r)) dr = A \int_{r_0}^r \frac{d\theta(r)}{dr} dr = A(\theta(r) - \theta(r_0)) = O(1). \quad (26)$$

Hence from (24), (25), (26), we have

$$T(r, w_0, w; \Delta) = \int_{r_0}^r \frac{S(r, w; \Delta)}{r} dr + O\left(\int_{r_0}^r \frac{L(r)}{r} dr \right). \quad (27)$$

Let a be any point in $[w, w_0] < \delta$ and $U(w)$ be the linear transformation, which makes $[w, w_0] < \delta$ invariant and bring a to w_0 , so that $w_0 = U(a)$, then

$$\begin{aligned}
T(r, w_0, U(w); \Delta) &= T(r, U(a), U(w); \Delta) \\
&= N(r, U(a), U(w); \Delta) + \frac{1}{2\pi} \int_{\theta_r} \log \frac{1}{[U(w), U(a)]} d\theta = \\
N(r, a, w; \Delta) &+ \frac{1}{2\pi} \int_{\theta_r} \log \frac{1}{[w, a]} d\theta + O(1) = T(r, a, w; \Delta) + O(1). \quad (28)
\end{aligned}$$

Hence from (27)

$$\begin{aligned} T(r, a, w; \Delta) &= T(r, w_0, U(w); \Delta) + O(1) \\ &= \int_{r_0}^r \frac{S(r, U(w); \Delta)}{r} dr + O\left(\int_{r_0}^r \frac{L_1(r)}{r} dr\right), \end{aligned}$$

where $L_1(r)$ is the length of the image of θ_r on K by $w = U(w(z))$. Since $L_1(r) = O(L(r))$, we have by Theorem II,

$$T(r, a, w; \Delta) = \int_{r_0}^r \frac{S(r, w; \Delta)}{r} dr + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right),$$

which proves the first part.

$$\text{Since } A(r) = \int_{r_0}^r \int_{\theta_r} \frac{|w'|^2}{(1+|w|^2)^2} r dr d\theta, \quad L(r) = r \int_{\theta_r} \frac{|w'|}{1+|w|^2} d\theta,$$

we have

$$[L(r)]^2 \leq 2\pi r^2 \int_{\theta_r} \frac{|w'|^2}{(1+|w|^2)^2} d\theta = 2\pi r \frac{dA(r)}{dr}, \quad (29)$$

so that

$$\int_{r_0}^r \frac{L(r)}{r} dr \leq \sqrt{\log r} \int_{r_0}^r \frac{[L(r)]^2}{r} dr \leq \sqrt{2\pi \log r \cdot A(r)} = O(\sqrt{T(2r) \log r}),$$

which proves (2).

Remark. Instead of $[w, w_0] < \delta$, we consider a domain D on K , which is bounded by an analytic Jordan curve. Let $w = w(z)$ be meromorphic in Δ and takes in Δ the values which belong to D and the boundary of Δ corresponds to the boundary of D . We define $T(r, a, w; \Delta)$ etc as in §1. We map D conformally on $[v, v_0] < \delta$ by $w = \psi(v)$ and let $w = w(z)$ become $v = v(z)$. Let $L(r)$, $L_1(r)$ be the length of the images of θ_r by $w = w(z)$ and $v = v(z)$ on K , then $L_1(r) = O(L(r))$. By Theorem I, for any two points α, β in $[v, v_0] < \delta$,

$$\begin{aligned} T(r, \alpha, v; \Delta) &= T(r, \beta, v; \Delta) + O\left(\int_{r_0}^r \frac{L_1(r)}{r} dr\right) \\ &= T(r, \beta, v; \Delta) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right). \end{aligned}$$

Since $T(r, a, v; \Delta) = T(r, a, w; \Delta) + O(1)$, where $a = \psi(a)$, we have for any two points a, b in $D_1 \subset D$,

$$T(r, a, w; \Delta) = T(r, b, w; \Delta) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right).$$

Multiplying $d\omega(b)$ and taking the integral mean over $D_1 \subset D$, we have by (19)

$$T(r, a, w; \Delta) = \int_{r_0}^r \frac{S(r, w; \Delta)}{r} dr + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right). \quad (1^{**})$$

Hence *Theorem I holds, if $[w, w_0] < \delta$ is replaced by any domain D on K bounded by an analytic Jordan curve.*

4. Theorems on $T(r)$.

We will prove some theorems on $T(r)$.

Theorem III. *For any point a in $[w, w_0] < \delta$,*

$$(i) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, a)}{\log r},$$

$$(ii) \quad \overline{\lim}_{r \rightarrow \infty} \frac{T(r)}{r^\rho} = \overline{\lim}_{r \rightarrow \infty} \frac{T(r, a)}{r^\rho}, \quad (\rho > 0),$$

(iii) $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr$ and $\int_{r_0}^{\infty} \frac{T(r, a)}{r^{\rho+1}} dr$ ($\rho > 0$) are finite or infinite at the same time.⁽⁴⁾

Proof. (i) We put $\overline{\lim}_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \rho$, $\overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, a)}{\log r} = \rho_1$.

(a) If $\rho < \infty$, then $\rho = \rho_1$ follows from (1*).

(b) If $\rho_1 < \infty$, then from (1), (3), we have outside $I_n = (r_1^{(n)}, r_2^{(n)})$,

$$T(r, a) = T(r) + O(\sqrt{T(r)} \log T(r)) \geq (1 - \varepsilon)T(r) \text{ or } T(r) \leq r^{\rho_1 + \varepsilon} \text{ for } r \geq r_0. \quad (30)$$

(4) c. f. K. KUNUGUI: Sur la théorie des fonctions méromorphes et uniformes. Jap. Journ. Math. 18 (1942).

Especially $T(r_2^{(n)}) \leq [r_2^{(n)}]^{p_1+\varepsilon}$. Since $\sum_n \int d \log \log r < \infty$, $r_2^{(n)} \leq [r_1^{(n)}]^K$ ($K = \text{const.}(>1)$). Hence for $r_1^{(n)} \leq r \leq r_2^{(n)}$,

$$T(r) \leq T(r_2^{(n)}) \leq [r_2^{(n)}]^{p_1+\varepsilon} \leq [r_1^{(n)}]^{K(p_1+\varepsilon)} \leq r^{K(p_1+\varepsilon)}.$$

Consequently we have from (30), $T(r) \leq r^{K(p_1+\varepsilon)}$ for all $r \geq r_0$, so that $\rho \leq K\rho_1 < \infty$. Hence if $\rho = \infty$, then $\rho_1 = \infty$.

From (a), (b), we have $\rho = \rho_1$ always, the both sides being finite or infinite.

(ii) We put $\lim_{r \rightarrow \infty} \frac{T(r)}{r^\rho} = A$, $\lim_{r \rightarrow \infty} \frac{T(r, a)}{r^\rho} = A_1$.

(a) If $A < \infty$, then $A = A_1$ from (1*).

(b) If $A_1 < \infty$, then by (i), $T(r) = O(r^{\rho+\varepsilon})$, so that $A = A_1$ from (1*). Hence if $A = \infty$, then $A_1 = \infty$.

From (a), (b), we have $A = A_1$ always, the both sides being finite or infinite.

(iii) (a) If $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$, then $T(r) = O(r^\rho)$, hence $\int_{r_0}^{\infty} \frac{T(r, a)}{r^{\rho+1}} dr < \infty$ from (1*).

(b) If $\int_{r_0}^{\infty} \frac{T(r, a)}{r^{\rho+1}} dr < \infty$, then $T(r, a) = O(r^\rho)$, so that by (i), $T(r) = O(r^{\rho+\varepsilon})$, hence $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$ from (1*).

From (a), (b), $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr$ and $\int_{r_0}^{\infty} \frac{T(r, a)}{r^{\rho+1}} dr$ are finite or infinite at the same time.

Theorem IV. Let $f(z)$ be regular in Δ , where $|f(z)| > \lambda$ in Δ and $|f(z)| = \lambda$ on the boundary of Δ and $M(r) = \text{Max.}_{\theta_r} |f(z)|$. Then

$$(i) \quad \lim_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \lim_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r},$$

(ii) $\lim_{r \rightarrow \infty} \frac{T(r)}{r^\rho}$ and $\lim_{r \rightarrow \infty} \frac{\log M(r)}{r^\rho}$ ($\rho > 0$) are zero, positive or infinite at the same time.

(iii) $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr$ and $\int_{r_0}^{\infty} \frac{\log M(r)}{r^{\rho+1}} dr$ ($\rho > 0$) are finite or infinite at the same time⁽⁴⁾.

Proof. (i) We put $\overline{\lim}_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \rho$, $\overline{\lim}_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} = \rho_1$.

(a) If $\rho < \infty$, then $\log M(r) \leq 3m(2r, \infty) + \text{const.}^{(5)}$, so that by (1*),

$$\log M(r) \leq 3T(2r, \infty) + \text{const.} = 3T(2r) + O(r^{\rho'}) \quad (\rho' < \rho). \quad (31)$$

Hence $\rho_1 \leq \rho$. From (1*),

$$T(r) = T(r, \infty) + O(r^{\rho'}) = m(r, \infty) + O(r^{\rho'}) \leq \log M(r) + O(r^{\rho'}) \quad (\rho' < \rho). \quad (32)$$

Hence $\rho \leq \rho_1$, so that $\rho = \rho_1$.

(b) If $\rho_1 < \infty$, then by (1), (3) we have outside I_n for $r > r_0$,

$$\begin{aligned} (1-\varepsilon)T(r) &\leq T(r) + O(\sqrt{T(r)} \log T(r)) = T(r, \infty) \\ &= m(r, \infty) \leq \log M(r) + O(1), \end{aligned}$$

so that $T(r) < r^{\rho_1 + \varepsilon}$. Similarly as in the proof of Theorem III (i), (b), we have $\rho \leq K\rho_1 < \infty$. Hence if $\rho = \infty$, then $\rho_1 = \infty$.

From (a), (b) we have $\rho = \rho_1$ always, the both sides being finite or infinite.

(ii) We put $\overline{\lim}_{r \rightarrow \infty} \frac{T(r)}{r^\rho} = A$, $\overline{\lim}_{r \rightarrow \infty} \frac{\log M(r)}{r^\rho} = A_1$.

(a) If $A = 0$, then $A_1 = 0$ from (31).

(b) If $0 < A < \infty$, then $A \leq A_1 \leq 3.2^\rho A$ from (31), (32).

(c) If $A = \infty$, then $A_1 = \infty$. For, if $A_1 < \infty$, then $T(r) = O(r^{\rho + \varepsilon})$ by (i). Hence $A \leq A_1 < \infty$ from (32), which contradicts the hypothesis. Hence $A_1 = \infty$.

(iii) (a) If $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$, then $\int_{r_0}^{\infty} \frac{\log M(r)}{r^{\rho+1}} dr < \infty$ from (31).

(b) If $\int_{r_0}^{\infty} \frac{\log M(r)}{r^{\rho+1}} dr < \infty$, then $\log M(r) = O(r^\rho)$, so that $T(r) = O(r^{\rho + \varepsilon})$ by (i), hence $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$ from (32).

(5) NEVANLINNA: Eindeutige analytische Funktionen. p. 294.

From (a), (b), $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr$ and $\int_{r_0}^{\infty} \frac{\log M(r)}{r^{\rho+1}} dr$ are finite or infinite at the same time.

Theorem V. Let $U(w)$ be a linear transformation, which makes $[w, w_0] < \delta$ invariant. Then

$$(i) \quad T(r, U(w); \Delta) = T(r, w; \Delta) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right),$$

$$(ii) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, w; \Delta)}{\log r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, U(w); \Delta)}{\log r}.$$

Proof. (i) follows from Theorem II.

$$(ii) \quad \text{We put } \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, w; \Delta)}{\log r} = \rho, \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, U(w); \Delta)}{\log r} = \rho_1.$$

If $\rho < \infty$, then from (i) and (2),

$$T(r, U(w); \Delta) = T(r, w; \Delta) + O(r^{\rho'}) \quad (\rho' < \rho).$$

Hence $\rho = \rho_1$. Similarly $\rho = \rho_1$ if $\rho_1 < \infty$, q. e. d.

If $[w, w_0] < \delta$ contains a transcendental singularity a of the inverse function $\varphi(w)$ of $w = f(z)$, then it is proved by K. Noshiro⁽⁶⁾ and K. Kunugui⁽⁷⁾ that $\lim_{r \rightarrow \infty} A(r) = \infty$. We will give a simple proof for it. By the hypothesis, Δ contains a curve C extending to infinity, along which $f(z)$ tends to a . Let γ be the image of C on K , then γ ends at a . There are two cases.

(i) Δ contains a boundary curve which extends to infinity.

In this case, an arc α_r of θ_r meets C and the boundary of Δ , so that the image of α_r on K meets $[w, w_0] = \delta$ and γ . Hence the length of this image has a length $\geq d > 0$ for $r > r_0$, so that $L(r) \geq d > 0$ for $r > r_0$. From (29), we have

$$d^2 \log \frac{r}{r_0} \leq \int_{r_0}^r \frac{[L(r)]^2}{r} dr \leq 2\pi A(r),$$

hence $\lim_{r \rightarrow \infty} A(r) = \infty$.

(ii) Δ contains no boundary curve which extends to infinity.

Then the boundary of Δ consists of infinitely many closed curves

(6) K. NOSHIRO: On the singularities of analytic functions. Jap. Journ. Math. 17 (1940).

(7) K. KUNUGUI: l. c. (4).

clustering to infinity. For, if there are only finite number of closed boundary curves, then $|z| > r_0$ belongs to Δ , so that $f(z)$ is meromorphic or has an essential singularity at $z = \infty$. By the hypothesis, $f(z)$ is not meromorphic at $z = \infty$. If $f(z)$ has an essential singularity at $z = \infty$, then by Picard's theorem, $f(z)$ takes the values outside $[w, w_0] < \delta$, which contradicts the hypothesis. Hence there are infinitely many closed boundary curves clustering to infinity. In this case we have two cases.

Case (a). $\varphi(w)$ has no transcendental singularity other than a in $[w, w_0] < \delta$. Then starting from any element of $\varphi(w)$, we can continue $\varphi(w)$ analytically along any curve which does not pass a . Then we see easily that if $f(z) - b$ ($b \neq a$) has a finite number of zero points in Δ , n , say, then $f(z) - w$ has n zero points in Δ for any $w (\neq a)$ in $[w, w_0] < \delta$. Consequently there are at most n closed boundary curves, which is a contradiction. Hence $f(z) - w$ has infinitely zero points for any $w (\neq a)$ in $[w, w_0] < \delta$. Hence $\lim_{r \rightarrow \infty} A(r) = \infty$.

Case (b). $\varphi(w)$ has a transcendental singularity $b (\neq a)$ in $[w, w_0] < \delta$. Then there exists a curve C' extending to infinity along which $f(z)$ tends to b , so that its image γ' on K ends at b . Then an arc a_r of θ_r which meets C , meets the boundary of Δ or C' . In each case, the image of a_r on K has a length $\geq d > 0$ for $r > r_0$, so that $L(r) \geq d > 0$ for $r > r_0$. Whence as before we conclude that $\lim_{r \rightarrow \infty} A(r) = \infty$.

Since $\lim_{r \rightarrow \infty} S(r) = \lim_{r \rightarrow \infty} \frac{A(r)}{\pi \delta^2} = \infty$, we have

Theorem VI. *If $[w, w_0] < \delta$ contains a transcendental singularity of the inverse function $\varphi(w)$ of $w = f(z)$, then*

$$\lim_{r \rightarrow \infty} \frac{T(r)}{\log r} = \infty.$$

5. Second Fundamental Theorem.

Let $D_1, \dots, D_q$ ($q \geq 2$) be q simply connected domains in $[w, w_0] < \delta$, which have no common points with each other and D be any one of D_i .

Let F_r be the image of Δ_r on K by $w = f(z)$. The part of F_r which lies above D consists of a certain number of connected pieces which are of two types. Those pieces which have no relative boundaries in D are called islands and the other pieces are called peninsulas. We apply Ahlfors' theory of covering surfaces⁽⁸⁾ to F_r .

We denote the circular disc $[w, w_0] \leq \delta$ by F_0 and the Euler's characteristic of a domain D by $\rho(D)$ and $p(D) = -\rho(D)$ is called the simple

(8) AHLFORS; l. c. (3).

multiplicity of D , which is $+1$, if D is simply connected, otherwise $p(D) \leq 0$.

We denote the sum of the simple multiplicities of all islands D^i of F_r above D_k by $\bar{p}(D_k)$.

First we take off from F_r all the peninsulas above $D_1 + \dots + D_q$, then there remains a certain number of connected pieces $\sum F'_r$.

From $\sum F'_r$ we take off all the islands D^i above $D_1 + \dots + D_q$, then there remains a certain number of connected pieces $\sum \bar{F}_r$, so that

$$\sum F'_r = \sum D^i + \sum \bar{F}_r. \quad (33)$$

Hence

$$\sum \rho(F'_r) = \sum \rho(D^i) + \sum \rho(\bar{F}_r).$$

Considering on the z -plane, we have easily

$$\begin{aligned} \sum_{k=1}^q \bar{p}(D_k) &= \sum \rho(\bar{F}_r) - \sum \rho(F'_r) = \sum \rho^+(\bar{F}_r) - (\sum \rho(F'_r) + N_1(\bar{F}_r)) \\ &\geq \sum \rho^+(\bar{F}_r) - \sum \rho^+(F'_r), \end{aligned} \quad (34)$$

where $N_1(\bar{F}_r)$ is the number of simply connected pieces in $\sum \bar{F}_r$.

We put $\bar{F}_0 = F_0 - (D_1 + \dots + D_q)$. By Ahlfors' fundamental theorem,

$$\sum \rho^+(\bar{F}_r) \geq (q-1)S(\bar{F}_0) - hL(r),$$

where h is a constant and $S(\bar{F}_0) = \frac{\text{area of } \sum \bar{F}_r}{\text{area of } \bar{F}_0}$ and by Ahlfors' first covering theorem, $S(\bar{F}_0) \geq S(r) - hL(r)$, so that

$$\sum \rho^+(\bar{F}_r) \geq (q-1)S(r) - hL(r),$$

where $S(r)$, $L(r)$ have the same meanings as in §1.

Considering on the z -plane, we see easily

$$\sum \rho^+(F'_r) \leq \lambda(r),$$

where

$$\lambda(r) = \text{the number of holes in } \Delta_r. \quad (35)$$

Hence

$$\sum_{k=1}^q \bar{p}(D_k) \geq (q-1)S(r) - \lambda(r) - hL(r). \quad (36)$$

We take a_k in D_k , then, since $\bar{p}(D_k) \leq n(r, a_k)$, we have

$$(q-1)S(r) \leq \sum_{k=1}^q n(r, a_k) + \lambda(r) + hL(r). \quad (37)$$

Next we will evaluate $\lambda(r)$. Now F_r consists of a certain number of sheets and let Ω_i be such a sheet whose boundary consists of the whole circle $[w, w_0] = \delta$ and one part of the image of θ_r , which form holes in Ω_i . Let the sum of the lengths of the boundaries of the holes of Ω_i be denoted by L_i , then $|\text{area of } \Omega_i - \pi\delta^2| \leq hL_i$, where h is a constant. Hence summing up for such Ω_i , we have

$$S(r) \geq \frac{\sum \text{area of } \Omega_i}{\pi\delta^2} \geq \lambda(r) - hL(r). \quad (38)$$

Hence

$$(q-1)S(r) \leq \sum_{k=1}^q n(r, a_k) + \lambda(r) + hL(r), \quad (39)$$

where

$$\lambda(r) \leq S(r) + hL(r). \quad (40)$$

From (39), (40), we have the following theorem, which corresponds to Nevanlinna's second fundamental theorem.

Theorem VII (SECOND FUNDAMENTAL THEOREM). *For any q points, $a_1, \dots, a_q$ in $[w, w_0] < \delta$,*

$$(q-1)T(r) \leq \sum_{k=1}^q N(r, a_k) + A(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right), \quad (41)$$

where

$$A(r) = \int_{r_0}^r \frac{\lambda(r)}{r} dr \leq T(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right), \quad (42)$$

$\lambda(r)$ being the number of holes in A_r .

6. Some consequences of the Second Fundamental Theorem.

(A) Let $f(z)$ take q values, $a_1, \dots, a_q$ in $[w, w_0] < \delta$ finite times in A , then from (37), (40),

$$(q-1)S(r) \leq \lambda(r) + hL(r) + O(1) \leq S(r) + hL(r) + O(1). \quad (43)$$

We suppose that $[w, w_0] < \delta$ contains a transcendental singularity of the inverse function $\varphi(w)$ of $w = f(z)$, then by Theorem VI, $\lim_{r \rightarrow \infty} S(r) = \infty$. Let $L(r) > [A(r)]^{\frac{1}{2} + \frac{\varepsilon}{2}}$ on a sequence of intervals I_n , then since by (29), $[L(r)]^2 \leq 2\pi r \frac{dA(r)}{dr}$, we have

$$\sum_n \int_{I_n} d \log r = \sum_n \int_{I_n} \frac{dr}{r} < 2\pi \sum_n \int_{I_n} \frac{dA(r)}{[A(r)]^{1+\varepsilon}} < 2\pi \int_{r_0}^{\infty} \frac{dA}{A^{1+\varepsilon}} < \infty.$$

Since outside I_n , $L(r) \leq [A(r)]^{\frac{1}{2} + \frac{1}{2}\varepsilon} = [\pi\delta^2 S(r)]^{\frac{1}{2} + \frac{1}{2}\varepsilon}$, we have

$$\lim_{r \rightarrow \infty} \frac{L(r)}{S(r)} = 0, \quad (44)$$

where r tends to infinity by taking the values outside I_n . Hence we have from (43), $q \leq 1 + \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} \leq 2$.

Theorem VIII. *If $[w, w_0] < \delta$ contains a transcendental singularity of the inverse function $\varphi(w)$ of $w = f(z)$, then $f(z)$ takes any value a in $[w, w_0] < \delta$, except at most $q \leq 1 + \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} \leq 2$ values, infinitely many times in Δ , where r tends to infinity by taking the values outside certain intervals I_n , such that $\sum_n \int_{I_n} d \log r < \infty$.*

K. Noshiro⁽⁹⁾ proved that $q \leq 1$, if Δ is simply connected and K. Kunugui⁽¹⁰⁾ proved that $q \leq 2$. The general case is due to Y. Tumura.⁽¹¹⁾

(B) Defect relation.

By (1), (3) we have outside I_n , $N(r, a) \leq T(r) + O(\sqrt{T(r)} \log T(r))$, so that $\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a)}{T(r)} \leq 1$ and by (41), (42), (3), we have outside I_n ,

$$\begin{aligned} (q-1)T(r) &\leq \sum_{k=1}^q N(r, a_k) + A(r) + O(\sqrt{T(r)} \log T(r)) \\ &\leq \sum_{k=1}^q N(r, a_k) + T(r) + O(\sqrt{T(r)} \log T(r)), \end{aligned}$$

so that

$$(q-1) \leq \sum_{k=1}^q \overline{\lim}_{r \rightarrow \infty} \frac{N(r, a_k)}{T(r)} + \lim_{r \rightarrow \infty} \frac{A(r)}{T(r)} \leq \sum_{k=1}^q \overline{\lim}_{r \rightarrow \infty} \frac{N(r, a_k)}{T(r)} + 1. \quad (45)$$

We will call

$$\delta(a) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N(r, a)}{T(r)} \geq 0 \quad (46)$$

the defect of a . Then we have from (45),

$$\textbf{Theorem IX.} \quad \sum \delta(a) \leq 1 + \lim_{r \rightarrow \infty} \frac{A(r)}{T(r)} \leq 2,$$

⁽⁹⁾ K. NOSHIRO: l. c. (5).

⁽¹⁰⁾ K. KUNUGUI: Une généralisation des théorèmes de MM. Picard-Nevanlinna sur les fonctions méromorphes. Proc. Imp. Acad. **17** (1914).

⁽¹¹⁾ Y. TUMURA: l. c. (1) also M. TSUJI: On the behaviour of an inverse function of a meromorphic function at its transcendental singular point, I. Proc. Imp. Acad. **17** (1941).

where r tends to infinity by taking the values outside certain intervals I_n , such that $\sum_n \int_{I_n} d \log \log r < \infty$.

(C) Picard-Borel's theorem.

Theorem X. If $\int_{r_0}^{\infty} \frac{N(r, a)}{r^{\rho+1}} dr < \infty$ ($\rho > 0$) for three (two, if $\int_{r_0}^{\infty} \frac{\Delta(r)}{r^{\rho+1}} dr < \infty$) values a in $[w, w_0] < \delta$, then $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$ and hence $\int_{r_0}^{\infty} \frac{N(r, a)}{r^{\rho+1}} dr < \infty$ for any value a in $[w, w_0] < \delta$.

Proof. Let $\int_{r_0}^{\infty} \frac{N(r, a_k)}{r^{\rho+1}} dr < \infty$ ($k = 1, 2, 3$). Then from (41), (42), (3), we have outside $I_n = (r_1^{(n)}, r_2^{(n)})$ ($n = 1, 2, \dots$), $T(r) \leq \sum_{k=1}^3 N(r, a_k) + O(\sqrt{T(r)} \log T(r))$, so that for a large r , $T(r) \leq \frac{1}{1-\varepsilon} \sum_{k=1}^3 N(r, a_k)$.

From $\int_{r_0}^{\infty} \frac{N(r, a_k)}{r^{\rho+1}} dr < \infty$, we have $N(r, a_k) = O(r^{\rho})$, so that outside I_n ,

$$T(r) \leq \text{const. } r^{\rho}. \quad (47)$$

From $\sum_n \int_{I_n} d \log \log r < \infty$, we have $r_2^{(n)} \leq [r_1^{(n)}]^{1+\varepsilon}$, so that for $r_1^{(n)} \leq r \leq r_2^{(n)}$,

$$\begin{aligned} T(r) &\leq T(r_2^{(n)}) \leq \text{const. } [r_2^{(n)}]^{\rho} \leq \text{const. } [r_1^{(n)}]^{\rho(1+\varepsilon)} \\ &\leq \text{const. } r^{\rho(1+\varepsilon)}. \end{aligned} \quad (48)$$

From (47), (48), we see that (48) holds for all sufficiently large r . Since ε may be taken arbitrarily small, we have from (41), (42), (2), (1*),

$$T(r) \leq \sum_{k=1}^3 N(r, a_k) + O(r^{\rho'}) \quad (\rho' < \rho), \quad (49)$$

so that

$$\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty.$$

If $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$, then $\int_{r_0}^{\infty} \frac{T(r, a)}{r^{\rho+1}} dr < \infty$ from Theorem III. Hence $\int_{r_0}^{\infty} \frac{N(r, a)}{r^{\rho+1}} dr < \infty$ for any value a in $[w, w_0] < \delta$.

By using $T(r) \leq \sum_{k=1}^2 N(r, a_k) + A(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right)$, we can prove that, if $\int_{r_0}^{\infty} \frac{A(r)}{r^{\rho+1}} dr < \infty$, then $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$ follows from $\int_{r_0}^{\infty} \frac{N(r, a_k)}{r^{\rho+1}} dr < \infty$ ($k = 1, 2$).

The following theorem corresponds to Picard-Borel's theorem.

Theorem XI. Let $f(z)$ be meromorphic and of finite order ρ in Δ and a be any point in $[w, w_0] < \delta$. Then $N(r, a)$ is of the same order (class, type) as $T(r)$, except for at most two (one, if $\overline{\lim}_{r \rightarrow \infty} \frac{\log A(r)}{\log r} < \rho$) values of a in $[w, w_0] < \delta$, for which $N(r, a)$ may be of lower order (class, type) than $T(r)$.

Proof.

(i) We put $\rho(a) = \overline{\lim}_{r \rightarrow \infty} \frac{\log N(r, a)}{\log r}$, then by Theorem III,

$$\rho(a) \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, a)}{\log r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \rho.$$

If $\rho(a_k) < \rho$ ($k = 1, 2, 3$), then from (49), the order of $T(r)$ would be lower than ρ , contradictory to the hypothesis. Hence $\rho(a) = \rho$, except for at most two values a in $[w, w_0] < \delta$.

(ii) From Theorem III, if $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr < \infty$, then $\int_{r_0}^{\infty} \frac{N(r, a)}{r^{\rho+1}} dr < \infty$ for every value a in $[w, w_0] < \delta$ and if $\int_{r_0}^{\infty} \frac{T(r)}{r^{\rho+1}} dr = \infty$, then from (49) $\int_{r_0}^{\infty} \frac{N(r, a)}{r^{\rho+1}} dr = \infty$, except for at most two values a in $[w, w_0] < \delta$.

(iii) We put $\overline{\lim}_{r \rightarrow \infty} \frac{T(r)}{r^{\rho}} = A$, $\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho}} = \nu(a)$, then by Theorem III,

$$\nu(a) \leq \overline{\lim}_{r \rightarrow \infty} \frac{T(r, a)}{r^{\rho}} = \overline{\lim}_{r \rightarrow \infty} \frac{T(r)}{r^{\rho}} = A, \quad (50)$$

and from (49),

$$A \leq \sum_{k=1}^2 \nu(a_k). \quad (51)$$

From (50), (51), we have

(a) If $A = 0$, then $\nu(a) = 0$ for any value a in $[w, w_0] < \delta$.

(b) If $0 < A < \infty$, then $0 < \nu(a) < \infty$, except for at most two values a in $[w, w_0] < \delta$.

(c) If $A = \infty$, then $\nu(a) = \infty$, except for at most two values a in $[w, w_0] < \delta$.

We can prove that, if $\lim_{r \rightarrow \infty} \frac{\log \lambda(r)}{\log r} < \rho$, then the number of exceptional values is reduced to one.

(D) **Discs-theorem.**

The Riemann surface F of $\varphi(w)$ is called μ -ply ramified above D , if all the simply connected islands above D have at least μ sheets. If F ramifies μ_k -ply above D_k , then from (36),

$$\sum_{k=1}^q \frac{1}{\mu_k} S(D_k) \geq \sum_{k=1}^q \bar{p}(D_k) \geq (q-1)S(r) - \lambda(r) - hL(r). \quad (52)$$

Since by Ahlfors' first covering theorem, $S(D_k) \leq S(r) + hL(r)$, we have

$$\sum_{k=1}^q \left(1 - \frac{1}{\mu_k}\right) S(r) \leq S(r) + \lambda(r) + hL(r),$$

so that

$$\sum_{k=1}^q \left(1 - \frac{1}{\mu_k}\right) \leq 1 + \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} \leq 2.$$

Hence we have the following discs-theorem:

Theorem XII. *If $[w, w_0] < \delta$ contains a transcendental singularity of the inverse function $\varphi(w)$ of $w = f(z)$ and the Riemann surface F of $\varphi(w)$ ramifies μ_k -ply above D_k ($k = 1, 2, \dots, q$), then*

$$\sum_{k=1}^q \left(1 - \frac{1}{\mu_k}\right) \leq 1 + \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} \leq 2,$$

where D_k are simply connected domains in $[w, w_0] < \delta$ and r tends to infinity by taking the values outside certain intervals I_n , such that $\sum_n \int_{I_n} d \log r < \infty$.

Hence one of any three, four or five simply connected domains in $[w, w_0] < \delta$ is covered schlicht by the Riemann surface F , according as

$$\lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} < \frac{1}{2}, \quad \frac{1}{2} \leq \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} < 1, \quad \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} = 1.$$

K. Noshiro⁽⁹⁾ proved that $\sum_{k=1}^q \left(1 - \frac{1}{\mu_k}\right) \leq 1$, if λ is simply connected and K. Kunugui⁽¹⁰⁾ proved that $\sum_{k=1}^q \left(1 - \frac{1}{\mu_k}\right) \leq 2$. The general case is due to Y. Tumura⁽¹²⁾.

(12) Y. TUMURA: l. c. (11) also M. TSUJI: l. c. (11).

Remark. I. In Theorem VIII and XII, we can replace $1 + \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} \leq 2$ by $1 + \lim_{r \rightarrow \infty} \frac{A(r)}{T(r)} \leq 2$, where r tends to infinity by taking the values outside certain intervals I_n , such that $\sum_n \int_{I_n} d \log \log r < \infty$.

II. Similarly as Ahlfors, we can prove the defect relation:

$$\sum_{k=1}^q \delta(D_k) + \sum_{k=1}^q \vartheta(D_k) \leq 1 + \lim_{r \rightarrow \infty} \frac{\lambda(r)}{S(r)} \leq 2. \quad (53)$$

(E) Uniqueness theorem.

Let $\bar{n}(r, a)$ be the number of zero points of $f(z) - a$ in Δ_r , where the multiple zero point is counted only once and put $\bar{N}(r, a) = \int_{r_0}^r \frac{\bar{n}(r, a)}{r} dr$.

Since we have instead of (37), $(q-1)S(r) \leq \sum_{k=1}^q \bar{n}(r, a_k) + \lambda(r) + hL(r)$, we have the second fundamental theorem in the more precise form:

$$(q-1)T(r) \leq \sum_{k=1}^q \bar{N}(r, a_k) + A(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right). \quad (54)$$

From (54), similarly as Nevanlinna⁽¹³⁾, we can prove the uniqueness theorem:

Theorem XIII. Let $f_1(z)$ and $f_2(z)$ ($f_1(z) \not\equiv f_2(z)$) be meromorphic in Δ , where $[f_1(z), w_0] < \delta$, $[f_2(z), w] < \delta$ in Δ and $[f_1(z), w_0] = \delta$, $[f_2(z), w_0] = \delta$ on the boundary of Δ . If $[w, w_0] < \delta$ contains a transcendental singularity of one of the inverse functions of $f_1(z)$, $f_2(z)$ and $f_1(z) - a_k$, $f_2(z) - a_k$ ($k = 1, 2, \dots, q$) have the same zero points in Δ , then $q \leq 3 + \lim_{r \rightarrow \infty} \frac{A(r)}{T(r)} \leq 4$. Hence if $q > 3 + \lim_{r \rightarrow \infty} \frac{A(r)}{T(r)}$, then $f_1(z) \equiv f_2(z)$, where r tends to infinity by taking the values outside certain intervals I_n , such that $\sum_n \int_{I_n} d \log \log r < \infty$.

PART II. AHLFORS' THEOREM ON THE NUMBER OF ASYMPTOTIC VALUES.

7. Quasi-direct transcendental singularity.

With the same notation as in §1, we assume that w_0 is a transcendental singularity of the inverse function $\varphi(w)$ of $w = f(z)$. Then Δ

(13) NEVANLINNA: Le théorème de Picard-Borel et la théorie des fonctions méromorphes. p. 109.

contains a curve C extending to infinity along which $[f(z), w_0]$ tends to zero. w_0 is called by Iversen a direct transcendental singularity of $\varphi(w)$, if $f(z) - w_0$ has no zero point in Δ . Ahlfors⁽¹⁴⁾ proved that if $f(z)$ is meromorphic for $|z| < \infty$ and of finite order ρ , then the number n of direct transcendental singularities of $\varphi(w)$ is $\leq 2\rho$, if $n \geq 2$. We will show that if the number of zero points of $f(z) - w_0$ in Δ is not so large, then the number n of such singularities is $\leq 2\rho$, if $n \geq 2$. For this purpose, we introduce a new notion "quasi-direct transcendental singularity" as follows.

Let $n(r)$ be the number of zero points of $f(z) - w_0$ in Δ_r . We will call w_0 a *quasi-direct transcendental singularity*, if

$$n(r) \leq K \int_{r_0}^r \frac{dr}{r\bar{\theta}(r)}$$

for any choice of z_0 and U , where K is independent of r , but may depend on z_0 and U . Then we will prove the following theorem.

Theorem XIV. *Let $f(z)$ be meromorphic for $|z| < \infty$ and of finite order ρ , then the number n of quasi-direct transcendental singularities of the inverse function $\varphi(w)$ of $w = f(z)$ is $\leq 2\rho$, if $n \geq 2$.*

8. Some lemmas.

Lemma I. *If $|z| < 1$, $|a| < 1$, then*

$$\left| \frac{z-a}{1-\bar{a}z} \right| \geq \left| \frac{|z| - |a|}{1 - |a| \cdot |z|} \right|.$$

Proof. We put $z = re^{i\theta}$, $a = \rho e^{i\varphi}$, then

$$\left| \frac{z-a}{1-\bar{a}z} \right|^2 - \left| \frac{|z| - |a|}{1 - |a| \cdot |z|} \right|^2 = \frac{2r\rho(1-r^2)(1-\rho^2)(1-\cos(\theta-\varphi))}{(1-2r\rho\cos(\theta-\varphi)+r^2\rho^2)(1-r\rho)^2} \geq 0,$$

so that

$$\left| \frac{z-a}{1-\bar{a}z} \right| \geq \left| \frac{|z| - |a|}{1 - |a| \cdot |z|} \right|.$$

Lemma II. *Let $0 < a_n < 1$ and $n(r)$ = the number of a_n such that $a_n \leq r$ satisfy the condition:*

$$n(r) \leq K \log \frac{1}{1-r} \quad (K = \text{const.})$$

(14) L. AHLFORS: Über die asymptotischen Wert der meromorphen Funktionen endlicher Ordnung. Acta Acad. Aboensis. Math. et Phys. **6** (1932) or R. NEVANLINNA's book p. 294. l.c. (4).

and let

$$F(r) = \prod_{n=1}^{\infty} \left| \frac{r-a_n}{1-a_n r} \right| \quad (0 < r < 1).$$

Then there exists a sequence $\rho_n \rightarrow 1$, such that

$$F(\rho_n) \geq \delta > 0, \quad (\delta = \text{const.}).$$

Proof. We put $q = e^{-\frac{1}{K}}$ and $\frac{1-a_n}{1-r} = \theta_n = \theta_n(r)$, then

$$\left| \frac{r-a_n}{1-a_n r} \right| = \left| \frac{1-\theta_n}{1+r\theta_n} \right| \geq \left| \frac{1-\theta_n}{1+\theta_n} \right| = \left| 1 - \frac{2\theta_n}{1+\theta_n} \right| = \left| 1 - \frac{\frac{2}{\theta_n}}{1+\frac{1}{\theta_n}} \right|.$$

If $\theta_n \leq q$, then $\frac{2\theta_n}{1+\theta_n} \leq \frac{2q}{1+q} < 1$. Hence $\left| 1 - \frac{2\theta_n}{1+\theta_n} \right| \geq e^{-K_1 \theta_n}$, if $\theta_n \leq q$ and $\left| 1 - \frac{\frac{2}{\theta_n}}{1+\frac{1}{\theta_n}} \right| \geq e^{-K_1 \frac{1}{\theta_n}}$, if $\frac{1}{\theta_n} \leq q$ ($K_1 = \text{const.}$), so that

$$\left| \frac{r-a_n}{1-a_n r} \right| \geq e^{-K_1 \theta_n(r)}, \quad \text{if } \theta_n(r) \leq q, \quad (55)$$

$$\geq e^{-K_1 \frac{1}{\theta_n(r)}}, \quad \text{if } \frac{1}{\theta_n(r)} \leq q. \quad (56)$$

We put $r_k = 1 - q^k$ and let

I_k : $r_{k-1} < r \leq r_k$ and $n_k = n(r_k) - n(r_{k-1})$ = the number of a_n in I_k , then since $n(r_k) \leq K \log \frac{1}{q^k} = Kk \log \frac{1}{q} = k$, we have

$$n_1 + n_2 + \dots + n_k = n(r_k) \leq k, \quad n_k \leq k. \quad (57)$$

Let $N \geq 5$ and we divide I_k into two classes I'_k and I''_k , such that we denote I_k by I'_k , if $n_k \geq N$ and by I''_k , if $n_k < N$.

We take $\left[\frac{n_k}{N} \right]$ consecutive intervals I_s on the left and on the right of I'_k and denote this intervals-group by $I_{(k)}$:

$$I_{(k)}: I_k - \left[\frac{n_k}{N} \right], \dots, I_{k-1}, I_k, I_{k+1}, \dots, I_k + \left[\frac{n_k}{N} \right].$$

Then s_k = the number of intervals in $I_{(k)} = 2 \left[\frac{n_k}{N} \right] + 1 \leq 3 \frac{n_k}{N}$.

Since

$$\frac{3}{N}(n_1 + n_2 + \cdots + n_k) \leq \frac{3k}{N}, \quad (58)$$

we see that

at most $\frac{3k}{N}$ intervals among $I_1, \dots, I_k$ are covered by the intervals-groups $I_{(\nu)}$ ($\nu \leq k$). (59)

Next we consider $I_{(k+\nu)}$ ($\nu \geq 1$). Its left end interval is $I_{k+\nu} - \left[\frac{n_{k+\nu}}{N} \right]$.

Now

$$\begin{aligned} k + \nu - \left[\frac{n_{k+\nu}}{N} \right] &\geq k + \nu - \frac{n_{k+\nu}}{N} \geq k + \nu - \frac{k + \nu}{N} = (k + \nu) \left(1 - \frac{1}{N} \right) \\ &\geq k \left(1 - \frac{1}{N} \right). \end{aligned}$$

Hence

at most $\frac{k}{N}$ intervals among $I_1, \dots, I_k$ are covered by $I_{(k+\nu)}$ ($\nu \geq 1$). (60)

Consequently from (59) and (60), we see that at least $\left(1 - \frac{4}{N} \right)k$ intervals among $I_1, \dots, I_k$ are uncovered by $I_{(\nu)}$ ($\nu = 1, 2, \dots$).

Let I_i be an uncovered interval. There are $s = j - i - 1$ intervals between I_i and I'_j ($j \geq i + 2$), so that $s \geq \left[\frac{n_j}{N} \right] \geq \frac{n_j}{N} - 1$, hence $n_j \leq N(s + 1)$.

If $r \in I_i$, $a_n \in I'_j$, then $\theta_n(r) = \frac{1 - a_n}{1 - r} \leq q^s$, so that for $r \in I_i$,

$$\sum_{a_n \in I'_j} \theta_n(r) \leq n_j q^s \leq N(s + 1) q^s,$$

so that

$$\sum_{j=i+2}^{\infty} \sum_{a_n \in I_j'} \theta_n(r) \leq N \sum_{s=0}^{\infty} (s+1)q^s = A.N \text{ (say)}. \quad (61)$$

Since $n_j < N$ in $I_j'' (j \geq i+2)$, we have for $r \in I_i$,

$$\sum_{a_n \in I_j''} \theta_n(r) \leq n_j q^s \leq N q^s,$$

so that

$$\sum_{j=i+2}^{\infty} \sum_{a_n \in I_j''} \theta_n(r) \leq N \sum_{s=0}^{\infty} q^s < A.N. \quad (62)$$

Hence from (61), (62),

$$\sum_{j=i+2}^{\infty} \sum_{a_n \in I_j} \theta_n(r) \leq 2A.N. \quad (63)$$

Similarly

$$\sum_{j=1}^{i-2} \sum_{a_n \in I_j} \frac{1}{\theta_n(r)} \leq 2A.N. \quad (64)$$

We put $I_i^0 = I_{i-1} + I_i + I_{i+1}$, then from (55), (56), (63), (64), we have for $r \in I_i$,

$$\prod_{a_n \notin I_i^0} \left| \frac{r - a_n}{1 - a_n r} \right| \geq e^{-4AK_1 N}, \quad (65)$$

where $a_n \notin I_i^0$ means that a_n does not belong to I_i^0 .

Since I_i^0 contains at most $3N$ of a_n 's, we can find ρ_i in I_i , such that

$$\prod_{a_n \in I_i^0} \left| \frac{\rho_i - a_n}{1 - a_n \rho_i} \right| \geq K_0 = K_0(N). \quad (66)$$

From (65), (66), we have

$$F(\rho_i) \geq K_0 e^{-\alpha N} \quad (\alpha = 4AK_1). \quad (67)$$

Since the number of uncovered intervals increases indefinitely with k , we have a sequence $\rho_i \rightarrow 1$ for which (67) holds, which proves our lemma.

Remark. If we apply H. Cartan's theorem on $\prod_{a_n \in I_i^0} \left| \frac{r - a_n}{1 - a_n r} \right|$, we can prove that $F(\rho) \geq e^{-\alpha N \log N}$ holds in $0 \leq \rho \leq r$, except certain

intervals I_n , such that $\sum_n \int_{I_n} d \log \frac{1}{1-\rho} \leq \frac{1}{N} \log \frac{1}{1-r}$, where a is a constant independent of N .

Theorem XV. Let $0 < |a_n| < 1$ and $n(r)$ = number of a_n in $|z| \leq r$ satisfy the condition:

$$n(r) \leq K \log \frac{1}{1-r} \quad (K = \text{const.}),$$

and let

$$F(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \bar{a}_n z} \quad (|z| < 1).$$

Then there exists a sequence $\rho_n \rightarrow 1$, such that

$$\min_{|z|=\rho_n} |F(z)| \geq \delta > 0 \quad (\delta = \text{const.}).$$

Proof. By Lemma I, $|F(z)| \geq \prod_{n=1}^{\infty} \left| \frac{|z| - |a_n|}{1 - |a_n| \cdot |z|} \right|$, so that the theorem follows from Lemma II.

9. Proof of Theorem XIV.

We assume that $w_0 = \infty$ is a quasi-direct transcendental singularity of $\varphi(w)$ and a δ -neighbourhood U of w_0 correspond to Δ on the z -plane, so that $|f(z)| > \lambda$ in Δ and $|f(z)| = \lambda$ on the boundary $\bar{\Delta}$ of Δ . We suppose that there exists a boundary curve of Δ , which extends to infinity. Let z_0 be a point on the outermost boundary of Δ and we take r_0 so small that $|z - z_0| = r_0$ meets Δ at two points z_1, z_2 , so that the common part of $|z - z_0| \leq r_0$ and Δ is a simply connected domain and let θ_{r_0} be the common part of $|z - z_0| = r_0$ and Δ and $\bar{\Delta}$ be the boundary of $\bar{\Delta}$. We map $\bar{\Delta}$ on $|\zeta| < 1$, such that z_0, z_1, z_2 become $\zeta = -1, i, -i$. We denote the image of θ_{r_0} in $|\zeta| < 1$ by σ_{r_0} .

Let z' be point on $\bar{\Delta}$, such that $|z' - z_0| = r' \geq 2r_0$ and let z' correspond to $\zeta' = e^{i\varphi}$ ($|\varphi| < \frac{\pi}{2}$) on $|\zeta| = 1$. Next we map $|\zeta| < 1$ on $|x| < 1$, such that $\zeta = -1, i, \zeta'$ become $x = -1, i, 1$, so that $\frac{1+x}{1-x} = i \frac{\zeta+1}{\zeta-\zeta'} \cdot \frac{i-\zeta'}{i+1}$.

Hence

$$\left| \frac{1+x}{1-x} \right| \leq \frac{4}{|\zeta - \zeta'|}. \quad (68)$$

If we denote the image of σ_{r_0} in $|x| < 1$ by γ_{r_0} , then on γ_{r_0} ,

$$\left| \frac{1+x}{1-x} \right| \geq \eta > 0, \quad (69)$$

where η is a constant independent of z' . Next by $w = u + iv = \log \frac{1+x}{1-x}$, we map $|x| < 1$ on an infinite strip $S: -\infty < u < \infty, |v| < \frac{\pi}{2}$, such that $x = -1, 1$ correspond to $w = -\infty, \infty$ and let γ_{r_0} correspond to λ_{r_0} in S .

Hence by (69), we have on λ_{r_0} ,

$$\Re(w) = \log \left| \frac{1+x}{1-x} \right| \geq \log \eta. \quad (70)$$

We connect z_0 and z' by a curve L in Δ and let $|z - z_0| = r$ ($r' > r > r_0$) intercept $\bar{\Delta}$ in a finite number of cross cuts, at least one of which separates z_0 and z' in $\bar{\Delta}$. Let $C(r)$ be such a cross cut, which meets L at first, when we proceed from z_0 to z' on L . If we denote the length of $C(r)$ by $r\bar{\theta}_1(r)$, then $r\bar{\theta}_1(r) \leq r\bar{\theta}(r)$. Now by Ahlfors' Verzerrungssatz we have on any point of the image of $C(r)$ in S ,

$$\Re(w) = \log \left| \frac{1+x}{1-x} \right| > \pi \int_{r_0}^r \frac{dr}{r\bar{\theta}(r)} + \log \eta - 4\pi, \text{ if } \int_{r_0}^r \frac{dr}{r\bar{\theta}(r)} > 2.$$

Hence by (68), we have on the image of $C(r)$ in $|\zeta| < 1$,

$$|\zeta - \zeta'| \leq K_0 e^{-\pi \int_{r_0}^r \frac{dr}{r\bar{\theta}(r)}} \quad (K_0 = \text{const.}). \quad (71)$$

Let a_n be the poles of $f(z)$ in Δ , then since $w_0 = \infty$ is a quasi-direct transcendental singularity of $\varphi(w)$,

$$n(r) \leq K \int_{r_0}^r \frac{dr}{r\bar{\theta}(r)} \quad (K = \text{const.}), \quad (72)$$

where $n(r)$ is the number of a_n in Δ_r .

We put $|a_\nu - z_0| = r_\nu$ and suppose that $r_m = r_{m+1} = \dots = r_n$. Let $C_1, \dots, C_k$ be the cross cuts of $\bar{\Delta}$, which lie on $|z - z_0| = r_n$ and let a_ν, C_i correspond to ζ_ν, Γ_i in $|\zeta| < 1$.

Now Γ_i separates $|\zeta| < 1$ into two domains D_i, D_i' , where D_i' contains -1 on its boundary. D_i, D_j ($i \neq j$) may have no common points or D_i is contained in D_j . We assume that D_1 is not contained in any D_i and contains $D_2, \dots, D_s$. Then we see that there exists a point ζ' on $|\zeta| = 1$, which lies on the boundary of D_1 and lies outside $D_2, \dots, D_s$.

Let ζ' correspond to z' on $\bar{A}$, then $|z' - z_0| > r_n$. Hence by putting $r = r_n$ in (71), (72), we have on I_1 and hence in D_1 ,

$$|\zeta - \zeta'| \leq e^{-K_1 n} \quad (K_1 = \text{const.}). \quad (73)$$

Hence if $\zeta_\nu (m \leq \nu \leq n)$ is contained in D_1 ,

$$1 - |\zeta_\nu| \leq |\zeta_\nu - \zeta'| \leq e^{-K_1 n} \leq e^{-K_1 \nu},$$

so that for any n ($n \geq n_0$)

$$1 - |\zeta_n| \leq e^{-K_1 n}. \quad (74)$$

From (74), we see easily that

$$n_1(r) \leq A \log \frac{1}{1-r} \quad (A = \text{const.}), \quad (75)$$

where $n_1(r)$ is the number of ζ_n in $|\zeta| \leq r$. Hence by Theorem XV, there exists a sequence $\rho_n \rightarrow 1$, such that

$$\text{Min.}_{|\zeta| = \rho_n} |g(\zeta)| \geq \delta > 0 \quad (\delta = \text{const.}), \quad (76)$$

where

$$g(\zeta) = \prod_{n=1}^{\infty} \frac{\bar{\zeta}_n}{|\zeta_n|} \cdot \frac{\zeta_n - \zeta}{1 - \bar{\zeta}_n \zeta}.$$

If we put $G(z) = g(\zeta)$, then $|G(z)| < 1$ in Δ and $G(a_n) = 0$, so that $F(z) = f(z)G(z)$ is regular in Δ and $|F(z)| \leq |f(z)|$ in and on the boundary of Δ . $F(z)$ is unbounded in Δ . For, if $F(z)$ is bounded and $|F(z)| \leq K$ in Δ , then $|G(z)| \leq \frac{K}{|f(z)|}$. By the hypothesis, $f(z) \rightarrow \infty$ along C , so that $G(z) \rightarrow 0$ along C , which contradicts (76). Hence $F(z)$ is unbounded in Δ . Next we map $\bar{\Delta}$ on the half-plane $\Im(\zeta) > 0$ and put $\log \frac{|w(z)|}{\lambda} = u(\zeta)$, where $w(z) = F(z) = f(z)G(z)$.

Since $u(\zeta)$ is unbounded in $\Im(\zeta) > 0$, there exists a point ζ_0 in $\Im(\zeta) > 0$, such that $u(\zeta_0) > 0$. This point established, we proceed similarly as Ahlfors' proof and we obtain

$$\pi \int_{r_0}^r \frac{dr}{r \theta(r)} < \log \log M(r) + \text{const.}, \quad (77)$$

where $M(r) = \text{Max.}_{\theta_r} |w(z)|$.

Let $U(z)$ be a harmonic function in $|z - z_0| < r' (r' > r)$, such that $U(z) = \log |f(z)|$ on $\theta_{r'}$ and $U(z) = \log \lambda$ on the remaining part of $|z - z_0| = r'$. Since $\log |w(z)| \leq \log \lambda \leq U(z)$ on the boundary of $A_{r'}$, we have $\log |w(z)| \leq U(z)$ in $A_{r'}$. Hence, if $M(r) = |w(z_0 + re^{i\varphi})|$, we have

$$\begin{aligned} \log M(r) &\leq \frac{1}{2\pi} \int_0^{2\pi} U(z_0 + r'e^{i\theta}) \frac{r'^2 - r^2}{r'^2 - 2rr' \cos(\theta - \varphi) + r^2} d\theta \\ &\leq \frac{r' + r}{r' - r} \int_{\theta_{r'}} \log |f(z_0 + r'e^{i\theta})| d\theta + \text{const.} \leq \frac{r' + r}{r' - r} m(r', \infty) + \text{const.} \end{aligned}$$

Hence for $r' = 2r$, $\log M(r) \leq 3 T(2r) + \text{const.}$, so that

$$\pi \int_{r_0}^r \frac{dr}{r\theta(r)} < \log T(2r) + \text{const.} \quad (78)$$

From this we have $n \leq 2\rho$ by the well known reasonings.

Remark. If $T(r) = O((\log r)^2)$, then the order of $f(z)$ is zero and let $n(r, w_0)$ be the number of zero points of $f(z) - w_0$ in $|z| \leq r$. Then since $n(r, w_0) \log r \leq \int_r^{r^2} \frac{n(r, w_0)}{r} dr \leq N(r^2, w_0) \leq T(r^2) = O((\log r)^2)$, $n(r, w_0) = O(\log r)$. Since $\log r - \log r_0 \leq 2\pi \int_{r_0}^r \frac{dr}{r\theta(r)}$, it follows that every transcendental singularity of $\varphi(w)$ is quasi-direct, hence we have the following Valiron's theorem.⁽¹⁵⁾

Theorem XVI. If $T(r) = O((\log r)^2)$, then there exists at most one asymptotic value.

Y. Tumura⁽¹⁶⁾ proved that $T(r) = O((\log r)^2)$ can be replaced by $\lim_{r \rightarrow \infty} \frac{T(r)}{(\log r)^2} < \infty$.

10. Extension of Theorem XIV.

Let $f(z)$ satisfy the condition in §1 and be of finite order ρ in A . We will investigate the number of quasi-direct transcendental singularities of the inverse function of $f(z)$ in $[w, w_0] < \delta$.

⁽¹⁵⁾ G. VALIRON: Sur les valeurs asymptotiques de quelque fonction méromorphes. Rendiconti mat. Palermo. **46** (1925). Sur le nombre des singularités transcendentes des fonctions inverse d'une classe d'algebroides. C.R. **200** (1935).

⁽¹⁶⁾ Y. TUMURA: Sur les théorèmes de M. VALIRON et les singularités transcendentes indirectement critiques. Proc. Imp. Acad. **17** (1941).

Theorem XVII. Let $f(z)$ be meromorphic and of finite order ρ in Δ , where $[f(z), w_0] < \delta$ in Δ and $[f(z), w_0] = \delta$ on the boundary Λ of Δ and n be the number of quasi-direct transcendental singularities of the inverse function $\varphi(w)$ of $w = f(z)$ in $[w, w_0] < \delta$. Then $n \leq 2\rho$, when Δ contains a curve extending to infinity and $n \leq 2\rho$, if $n \geq 2$, when Δ consists of only closed curves.

Proof. Let $[w, w_0] < \delta$ contain n quasi-direct transcendental singularities, $a_1, \dots, a_n$ of $\varphi(w)$ and a δ_i -neighbourhood U_i of a_i correspond to Δ_i on the z -plane.⁽¹⁷⁾ We define $\bar{\Delta}_i$, $r\bar{\theta}_i(r)$, $T(r, a_i, w; \Delta_i)$ etc as in §1. We take δ_i so small that $\bar{\Delta}_i$ do not overlap, so that $\bar{\theta}_1(r) + \dots + \bar{\theta}_n(r) \leq 2\pi$.

By (78), we have

$$\pi \int_{r_0}^r \frac{dr}{r\bar{\theta}_i(r)} < \log T(2r, a_i, w; \Delta_i) + \text{const.}$$

From (1*),

$$\begin{aligned} T(2r, a_i, w; \Delta_i) &\leq T(2r, a_i, w; \Delta) \\ &= T(2r, w; \Delta) + O(r^{\rho'}) < r^{\rho+\varepsilon}, \end{aligned}$$

so that

$$\pi \int_{r_0}^r \frac{dr}{r\bar{\theta}_i(r)} < (\rho + \varepsilon) \log r + \text{const.} \quad (79)$$

Since $\frac{1}{\bar{\theta}_1(r)} + \dots + \frac{1}{\bar{\theta}_n(r)} \geq \frac{n^2}{\bar{\theta}_1(r) + \dots + \bar{\theta}_n(r)} \geq \frac{n^2}{2\pi}$, we have

$$n \log r \leq 2(\rho + \varepsilon) \log r + \text{const.} \quad \text{Hence } n \leq 2\rho.$$

Theorem XVIII. Let $f(z)$ be meromorphic and of finite order ρ in Δ , where $[f(z), w_0] < \delta$ in Δ and $[f(z), w_0] = \delta$ on the boundary of Δ and n be the number of quasi-direct transcendental singularities of the inverse function $\varphi(w)$ of $w = f(z)$ in $[w, w_0] < \delta$. We assume that the boundary of Δ contains a curve extending to infinity. Then

$$n \leq \frac{\rho}{\pi} / \overline{\lim}_{r \rightarrow \infty} \frac{1}{\log r} \int_{r_0}^r \frac{dr}{r\bar{\theta}(r)} \leq \frac{\bar{\theta}\rho}{\pi}, \quad \text{where } \bar{\theta} = \overline{\lim}_{r \rightarrow \infty} \bar{\theta}(r).$$

Proof. Suppose that $f(z)$ is regular in Δ , where $|f(z)| > \lambda$ in Δ and $|f(z)| = \lambda$ on the boundary of Δ and Δ contains only one asymptotic path for the value ∞ , so that $f(z)$ tends to infinity along a curve C , which

⁽¹⁷⁾ Δ_i has boundary curves extending to infinity, if Δ has an infinite branch or when Δ consists of only closed curves, if $n \geq 2$.

connects a point z_0 on the boundary Δ of Δ with $z = \infty$ in Δ . We map $\bar{\Delta}$ on $\Im(\zeta) > 0$ by $\zeta = g(z)$ ($\infty = g(\infty)$) and let the image of Δ and C in $\Im(\zeta) > 0$ be denoted by D and γ respectively and put $F(\zeta) = f(z)$. Then $F(\zeta) \rightarrow \infty$, when ζ tends to ∞ along γ . Now by the mapping $\zeta = g(z)$, $z = \infty$ corresponds to a closed set E of measure zero on the real axis $\Im(\zeta) = 0$. At any point ($\neq \infty$) on $\Im(\zeta) = 0$, $F(\zeta)$ is bounded, since otherwise, there would exist another asymptotic path for the value ∞ , contradictory to the hypothesis. Let $\zeta_1 (\neq \infty)$ be a point on E and its neighbourhood in D be denoted by U . Then considering on the z -plane we can easily prove that $\lim_{\zeta \rightarrow \zeta_1} |F(\zeta)| \leq \lambda$, when ζ tends to ζ_1 in U . Hence, since $|F(\zeta)| > \lambda$ in U , we have

$$\lim |F(\zeta)| = \lambda, \quad (80)$$

when ζ tends to any point ($\neq \infty$) on the real axis.

From (80), similarly as we have deduced (78), we have

$$\pi \int_{r_0}^r \frac{dr}{r \bar{\theta}(r)} < \log T(2r, \infty, w; \Delta) + \text{const.} \quad (81)$$

Similarly as in § 9 we can prove that (81) holds if the number of poles of $f(z)$ in Δ_r is $\leq K \int_{r_0}^r \frac{dr}{r \bar{\theta}(r)}$.

Now by Theorem XVII, the number of quasi-direct transcendental singularities of $\varphi(w)$ is finite. Let $a_1, \dots, a_n$ be such singularities in $[w, w_0] < \delta$ and let a δ_i -neighbourhood U_i of a_i correspond to Δ_i on the z -plane, where we take δ_i so small that $\bar{\Delta}_i$ do not overlap. Hence $\bar{\theta}_1(r) + \dots + \bar{\theta}_n(r) \leq \bar{\theta}(r)$ and there is only one asymptotic path for the value a_i in Δ_i . Then from (81),

$$\pi \int_{r_0}^r \frac{dr}{r \bar{\theta}_i(r)} < \log T(2r, a_i, w; \Delta_i) + \text{const.} \quad (82)$$

Similarly as (79), we have from (82),

$$\pi \int_{r_0}^r \left(\frac{1}{\bar{\theta}_1(r)} + \dots + \frac{1}{\bar{\theta}_n(r)} \right) \frac{dr}{r} \leq n(\rho + \varepsilon) \log r + \text{const.},$$

Since

$$\frac{1}{\bar{\theta}_1(r)} + \dots + \frac{1}{\bar{\theta}_n(r)} \geq \frac{n^2}{\bar{\theta}_1(r) + \dots + \bar{\theta}_n(r)} \geq \frac{n^2}{\bar{\theta}(r)},$$

$$\pi n \int_{r_0}^r \frac{dr}{r \bar{\theta}(r)} \leq (\rho + \varepsilon) \log r + \text{const.}.$$

Hence

$$n \leq \frac{\rho}{\pi} \left/ \overline{\lim}_{r \rightarrow \infty} \frac{1}{\log r} \int_{r_0}^r \frac{dr}{r \bar{\theta}(r)} \right. \leq \frac{\bar{\theta} \rho}{\pi},$$

where

$$\bar{\theta} = \lim_{r \rightarrow \infty} \bar{\theta}(r), \quad \text{q. e. d.}$$

11. Number of asymptotic values of a regular function of finite order in Δ .

Theorem XIX. Let $f(z)$ be regular and of finite order ρ in Δ , where $|f(z)| > \lambda$ in Δ and $|f(z)| = \lambda$ on the boundary of Δ .

Let n and n' be the number of asymptotic pathes for the values a ($\lambda < |a| < \infty$) and the value ∞ respectively. Then

(A) when the boundary of Δ contains a curve extending to infinity,

$$(i) \quad n' \leq \frac{\rho}{\pi} \left/ \overline{\lim}_{r \rightarrow \infty} \frac{1}{\log r} \int_{r_0}^r \frac{dr}{r \bar{\theta}(r)} \right. \leq \frac{\bar{\theta} \rho}{\pi}, \quad \text{where } \bar{\theta} = \overline{\lim}_{r \rightarrow \infty} \bar{\theta}(r),$$

$$(ii) \quad n \leq n' - 1.$$

(B) when the boundary of Δ consists of only closed curves,

$$(i) \quad n' \leq 2\rho, \quad \text{if } n \geq 2,$$

$$(ii) \quad n \leq n'.$$

Proof. By the hypothesis, $w = \infty$ is a direct transcendental singularity of $\varphi(w)$, so that (i) follows from Theorem XVIII. It remains to prove (ii). We will first prove (A) (ii).

We map Δ on $|\zeta| < 1$ by $z = g(\zeta)$ and let $F(\zeta) = f(z)$. Then $F(\zeta)$ tends to infinity along curves $\gamma_1, \dots, \gamma_{n'}$ in $|\zeta| < 1$ ending at $\zeta_1, \dots, \zeta_{n'}$ on $|\zeta| = 1$, where some of ζ_i may coincide.

(i) Let a ($\lambda < |a| < \infty$) be an asymptotic value of $F(\zeta)$, then $F(\zeta)$ tends to a along a curve γ ending at a point ζ_0 on $|\zeta| = 1$. By (80), ζ_0 must coincide with one of ζ_i . Let $\zeta_0 = \zeta_1$ and $\mathcal{Q}$ be the neighbourhood of ζ_1 in D , which is the image of Δ in $|\zeta| < 1$ by $z = g(\zeta)$. Now γ divides $\mathcal{Q}$ into two parts $\mathcal{Q}_1$ and $\mathcal{Q}_2$. We will show that $F(\zeta)$ is unbounded in each $\mathcal{Q}_i$. For, suppose that $F(\zeta)$ is bounded in $\mathcal{Q}_1$. Then since ζ_1 is a regular point for the Dirichlet problem for $\mathcal{Q}_1$, if $\overline{\lim} |F(\zeta)| \leq m$, when ζ tends to ζ_1 on the boundary of $\mathcal{Q}_1$, then $\overline{\lim} |F(\zeta)| \leq m$, when ζ tends to ζ_1 in $\mathcal{Q}_1$. Hence if we denote the cluster set of $F(\zeta)$ by E_1 and E , when ζ tends to ζ_1 on the boundary of $\mathcal{Q}_1$ and in $\mathcal{Q}_1$ respectively, then by Hössjer's theorem,⁽¹⁸⁾ E is contained in E_1 . Now

(18) G. HÖSSJER: Bemerkung über einen Satz von E. LINDELÖF. Kungl. Fysiografiska Sällskapet I Lund. Förhandlingar 6 (1935).

E_1 consists of a and $|w| = 1$, so that E consists of a and $|w| = 1$. But we see easily that E contains a continuum connecting a and $|w| = 1$, which is a contradiction. Hence $F(\zeta)$ is unbounded in Ω_1 , so that Ω_1 contains a curve along which $F(\zeta)$ tends to infinity. Similarity for Ω_2 .

(ii) Next suppose that there are two asymptotic pathes γ, γ' for the values a and b ($\lambda < |a| < \infty, \lambda < |b| < \infty, a \neq b$) ending at ζ_1 , so that $F(\zeta) \rightarrow a$ along γ and $F(\zeta) \rightarrow b$ along γ' . Then $F(\zeta)$ is unbounded in the neighbourhood Ω of ζ_1 in D between γ and γ' . For, suppose that $F(\zeta)$ is bounded in Ω , then as before by Hössjer's theorem, E_1 and hence E consists of a and b and $|w| = 1$ (when infinitely many holes cluster at ζ_1). But we see easily that E contains a continuum connecting a and b , which is a contradiction. Hence $F(\zeta)$ is unbounded in Ω , so that Ω contains a curve along which $F(\zeta)$ tends to infinity.

From (i) and (ii) we conclude that $n \leq n' - 1$, q. e. d.

Similarly we can prove (B) (ii).

Remark. We assumed that $f(z)$ is regular in Δ , but if the number of poles of $f(z)$ in Δ is finite, then the same result holds, since, then $w = \infty$ is a quasi-direct transcendental singularity of $\varphi(w)$, so that n' is finite and $f(z)$ is regular in the neighbourhood of the boundary of Δ . From Theorem XIX we deduce the following theorem.

Theorem XX. Let $f(z)$ be meromorphic and of finite order ρ in Δ , where $[f(z), w_0] < \delta$ in Δ and $[f(z), w_0] = \delta$ on the boundary of Δ . We assume that the boundary of Δ contains a curve extending to infinity. Then $f(z)$ takes any value in $[w, w_0] < \delta$, except at most one value, infinitely many times in Δ .

Proof. We may assume that $[w, w_0] < \delta$ is of the form $|w| > \lambda$, since by the rotation of the Riemann sphere, we can bring $[w, w_0] < \delta$ into the form $|w| > \lambda$ and the order of $f(z)$ is invariant by the rotation. Suppose that $f(z) - a$ and $f(z) - b$ ($\lambda < |a|, \lambda < |b|, a \neq b$) have a finite number of zero points in Δ , then

$$F(z) = U(f(z)) = \frac{\bar{a}f(z) - \lambda^2}{f(z) - a}, \quad F_1(z) = U_1(f(z)) = \frac{\bar{b}f(z) - \lambda^2}{f(z) - b}$$

have a finite number of poles in Δ and $|F(z)| > \lambda, |F_1(z)| > \lambda$ in Δ and $|F(z)| = \lambda, |F_1(z)| = \lambda$ on the boundary of Δ . Since $U(w)$ and $U_1(w)$ make $|w| > \lambda$ invariant, $F(z)$ and $F_1(z)$ are of the same order as $f(z)$ by Theorem VIII.

Let n and n' be the number of asymptotic pathes for the values a ($\lambda < |a| < \infty$) and the value ∞ of $F(z)$ respectively and n_1 and n'_1 be the corresponding numbers for $F_1(z)$. Then by Theorem XIX,

$$n \leq n' - 1, \quad (83)$$

$$n_1 \leq n'_1 - 1. \quad (84)$$

As we see easily,

$$n' \leq n_1, \quad n'_1 \leq n,$$

so that

$$n' \leq n_1 \leq n'_1 - 1 \leq n - 1 \leq n' - 2 \quad \text{or} \quad 0 \leq -2,$$

which is absurd. Hence $f(z)$ takes any value in $|w| > \lambda$, except at most one value, infinitely many times in Δ .

We know from Theorem VIII, that $f(z)$ takes any value in $[w, w_0] < \delta$, except at most two values, infinitely many times in Δ . We see from the above theorem that the exceptional case may occur only when the niveau curve $[f(z), w_0] = \delta$ consists of infinitely many closed curves clustering to infinity.

From Theorem IV, we have the following special case.

Theorem XXI. *Let $f(z)$ be regular in Δ , where $|f(z)| > \lambda$ in Δ and $|f(z)| = \lambda$ on the boundary of Δ . We assume that the boundary of Δ contains a curve extending to infinity. If $\overline{\lim}_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} < \infty$, where $M(r) = \text{Max}_{|z|=r} |f(z)|$, then $f(z)$ takes any value a ($\lambda < |a| < \infty$) infinitely many times in Δ .*

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