26. Theory of conformal mapping of a multiply connected domain, II.

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Let F be the Riemann surface of an algebraic function of genus ρ (≥ 2) spread over the w -plane and $C_1, \dots, C_p$ be p Jordan curves on F , which have no common points with each other and do not separate F into two pieces.

We cut F along $C_1, \dots, C_p$ and the resulting surface be denoted by $\bar{F}_0$. We take infinitely many same samples as $\bar{F}_0$ and connect them along the opposite shores of C_i in the well known way and obtain a covering surface $F^{(\infty)}$ of F . $F^{(\infty)}$ is of planar character, hence we can map $F^{(\infty)}$ on a schlicht domain D on the z -plane by $w = f(z)$. D is the outside of a certain closed set E . In my former paper⁽¹⁾, which we refer as (I) in the following, I have proved:

Theorem IV. E is of capacity zero.

I have found that the proof is incomplete so far as I have used an unproved theorem in the proof⁽²⁾, so that I will here give another proof of Theorem IV.

I will first prove some lemmas.

1. Some Lemmas.

Lemma 1. Let D be a domain in $|\zeta| < 1$, such that every point ζ in D has a distance $\geq k$ from a point a on $|\zeta| = 1$. We map $|\zeta| < 1$ on the upper half of the $v = x + iy$ -plane by $\zeta = a \frac{v-i}{v+i}$. Then for any point ζ in D ,

$$\frac{k^2}{2}y \leq 1 - |\zeta| \leq 4y.$$

Proof. From $\zeta = a \frac{v-i}{v+i}$, we have $1 - |\zeta|^2 = \frac{4y}{|v+i|^2}$. Hence

(1) M. Tsuji: Theory of conformal mapping of a multiply connected domain. This Journ., **18** (1943), 759-775. Mr. K. Arima has given another proof of Theorem IV in Zengoku Sizyô Danwakai, **257** (1943).

(2) I could prove the theorem, which will appear in this Journal, **19** (1944). (Added during the correction).

$$1 - |\zeta| = \frac{4y}{(1 + |\zeta|)|v + i|^2} \leq 4y,$$

and by the hypothesis, $|\zeta - \alpha| = \frac{2}{|v + i|} \geq k$, so that

$$1 - |\zeta| = \frac{4y}{(1 + |\zeta|)|v + i|^2} \geq \frac{4y}{2} \left(\frac{k}{2}\right)^2 = \frac{k^2}{2}y.$$

Lemma 2. Let G be a group of linear transformations, which make the upper half of the $v = x + iy$ -plane invariant and v_0, v'_0 be two points in $\Im v > 0$ and v_n, v'_n be the transforms of v_0, v'_0 by G respectively. Then $\frac{\Im v'_n}{\Im v_n}$ lies between two positive constants independent of the transformations of G .

Proof. Let $\frac{|dv|}{y}$ be the hyperbolic line element in $\Im v > 0$ and s_0 be the hyperbolic distance between v_0, v'_0 , then $s_0 = \int_C \frac{|dv|}{y}$, where C is a part bounded by v_0 and v'_0 of an orthogonal circle to the real axis through v_0 and v'_0 . Let $\Im v_n = y_n \leq \Im v'_n = y'_n$ and C_n be the transform of C by G , then

$$s_0 = \int_{C_n} \frac{|dv|}{y} \geq \int_{y_n}^{y'_n} \frac{dy}{y} = \log \frac{y'_n}{y_n},$$

so that $1 \leq \frac{y'_n}{y_n} \leq e^{s_0}$.

Lemma 3. Let G be a group of linear transformations, which make $|z| < 1$ invariant, such that its fundamental domain D_0 with its boundary lies entirely in $|z| < 1$. Let z_0 be a point in D_0 and z_n be its transforms by G . Then $\sum_{n=0}^{\infty} (1 - |z_n|) = \infty$.

Proof. Let $\frac{|dz|}{1 - |z|^2}$ be the hyperbolic line element in $|z| < 1$ and $\sigma_0, |D_0|$ be the hyperbolic diameter and area of D_0 respectively. We denote the number of z_n in $|z| \leq r$ by $n(r)$. Let $r < 1$ and we determine r_0 ($< r < 1$) such that

$$\sigma_0 = \int_{r_0}^r \frac{dr}{1 - r^2} = \frac{1}{2} \log \frac{1 + r}{1 - r} \cdot \frac{1 - r_0}{1 + r_0}.$$

Then, since the transform D_n of D_0 by G has the same hyperbolic diameter σ_0 as D_0 , any D_n , which meets $|z| = r_0$, is contained in $|z| \leq r$. Hence, since the hyperbolic area S_0 of a disc: $|z| \leq r_0$ is given by

$$S_0 = \int_0^{2\pi} \int_0^{r_0} \frac{r dr d\theta}{(1-r^2)^2} = \frac{\pi r_0^2}{1-r_0^2},$$

we have

$$n(r) |D_0| \geq S_0 = \frac{\pi r_0^2}{1-r_0^2} = \frac{\pi(1+r-\lambda(1-r))^2}{4\lambda(1-r)(1+r)} \geq \frac{\text{const.}}{1-r},$$

where $\lambda = e^{2\sigma_0}$.

Hence $N(1) = \int_0^1 \frac{n(r)}{r} dr = \infty$, so that $\sum_{n=0}^{\infty} (1-|z_n|) = \infty$, q. e. d.

2. Proof of Theorem IV.

We assume $p = 2$. Let $w = f(z)$ map the outside of E on $F^{(\infty)}$, then $f(z)$ is automorphic with respect to a group G of transformations $U: z' = U(z)$, where z, z' correspond to the same point on F . U transforms the outside of E into itself. Let $\bar{F}_0$ be the starting sheet of $F^{(\infty)}$ and let it correspond to $\bar{S}_0$ on the z -plane. $\bar{S}_0$ is bounded by four Jordan curves: I_1, I_2, I_3, I_4 , where $I_1, I_2; I_3, I_4$ correspond to the both shores of C_1, C_2 respectively. Let z_0 be a point in $\bar{S}_0$ and we connect z_0 to a point z_i on I_i by a Jordan arc A_i ($i = 1, 2, 3, 4$), where $z_1, z_2; z_3, z_4$ correspond to the same point on C_1, C_2 respectively and A_i have no common points with each other except z_0 . We cut $\bar{S}_0$ along A_1, A_2, A_3, A_4 and the resulting domain be denoted by S_0 . If we apply all transformations of G to S_0 and I_i , we obtain domains: $S_0; S_1, S_2, S_3, S_4; S_{11}, S_{12}, S_{13}; S_{21}, S_{22}, S_{23}; S_{31}, S_{32}, S_{33}; S_{41}, S_{42}, S_{43}, \dots, S_{ik_1 \dots k_n}, \dots$ and curves: $I_1, I_2, I_3, I_4; I_{11}, I_{12}, I_{13}; I_{21}, I_{22}, I_{23}; I_{31}, I_{32}, I_{33}; I_{41}, I_{42}, I_{43}, \dots, I_{ik_1 \dots k_n}, \dots$ ($i = 1, 2, 3, 4, k_1, \dots, k_n = 1, 2, 3$), where $S_{ik_1 \dots k_n}$ is contained in $S_{ik_1 \dots k_{n-1}}$ and $I_{ik_1 \dots k_n}$ is the outer boundary of $S_{ik_1 \dots k_n}$ as shown in Fig. 1 in (I).

We put

$$S = S_0 + S_1 + S_2 + S_3 + S_4 + \dots + S_{ik_1 \dots k_n} + \dots \quad (1)$$

Let $\mathfrak{F}^{(\infty)}$ be the simply connected universal covering Riemann surface of the outside of E and G' be the group of transformations, which interchange one sheet of $\mathfrak{F}^{(\infty)}$ into another sheet. Then S is the fundamental domain of G' and S_0 is the fundamental domain of a group $\bar{G} = G + G'$.

Next we map $\mathfrak{F}^{(\infty)}$ on $|\zeta| < 1$ by $z = \varphi(\zeta)$ and let $S, S_0, S_i, S_{ik_1 \dots k_n}$ ($i = 1, 2, 3, 4, k_1, \dots, k_n = 1, 2, 3$) correspond to $D^{(k)}, D_0^{(k)}, D_i^{(k)}, D_{ik_1 \dots k_n}^{(k)}$, ($k = 0, 1, 2, 3, \dots$) in $|\zeta| < 1$ respectively, where

$$D^{(0)} = D_0^{(0)} + D_1^{(0)} + D_2^{(0)} + D_3^{(0)} + D_4^{(0)} + \dots + D_{ik_1 \dots k_n}^{(0)} + \dots \quad (2)$$

$$D^{(k)} = D_0^{(k)} + D_1^{(k)} + D_2^{(k)} + D_3^{(k)} + D_4^{(k)} + \dots + D_{ik_1 \dots k_n}^{(k)} + \dots \quad (3)$$

$G, G', \bar{G}$ become groups of transformations in $|\zeta| < 1$, which we denote by the same letters. Since any transformation of $G, G', \bar{G}$ makes $|\zeta| < 1$ invariant, $G, G', \bar{G}$ are groups of linear transformations in $|\zeta| < 1$.

$D^{(0)}$ is the fundamental domain of G' and $D_0^{(0)}$ is the fundamental domain of $\bar{G}$. We take a point ζ_0 in $D_0^{(0)}$ and let $\zeta_0^{(k)}$ be its transform in $D_0^{(k)}$ by G' and $\zeta_i^{(k)}, \zeta_{ik_1 \dots k_n}^{(k)}$ be its transform in $D_i^{(k)}, D_{ik_1 \dots k_n}^{(k)}$ by G respectively, so that

$$\zeta_0^{(k)} \in D_0^{(k)}, \quad \zeta_i^{(k)} \in D_i^{(k)}, \quad \zeta_{ik_1 \dots k_n}^{(k)} \in D_{ik_1 \dots k_n}^{(k)}. \quad (4)$$

We will prove that $\sum_{k=0}^{\infty} (1 - |\zeta_0^{(k)}|) = \infty$, which is equivalent to our Theorem.

Now $D_1^{(k)}$ is separated from $D_0^{(k)}$ by a part of a Jordan arc L_1 , which corresponds to Γ_1 on the z -plane.

Fig. 1.

L_1 separates $|\zeta| < 1$ into two domains: Δ, Δ' , where Δ contains $D_1^{(k)}$ in it. Since Γ_1 contains $S_{1k_1 \dots k_n}$ ($k_1, \dots, k_n = 1, 2, 3$) in it, we see that Δ contains $D_{1k_1 \dots k_n}^{(k)}$.

Since the diameter of $D^{(k)}$ diminishes to zero for $k \rightarrow \infty$, we can find a point a on $|\zeta| = 1$, such that for any point ζ in $\Delta + D_0^{(k)}$, $|\zeta - a| \geq k$, where k is a positive constant independent of the transformations of G .

By $\zeta = a \frac{v-i}{v+i}$, we map $|\zeta| < 1$ on the upper half of the $v = x + iy$ -plane and let $G, G', \bar{G}$ become groups of linear transformations in $\Im v > 0$, which we denote by the same letters. Let

$$D_0^{(k)}, \quad \zeta_0^{(k)}; \quad D_i^{(k)}, \quad \zeta_i^{(k)}; \quad D_{ik_1 \dots k_n}^{(k)}, \quad \zeta_{ik_1 \dots k_n}^{(k)} \quad (5)$$

correspond to

$$\sigma_0^{(k)}, \quad v_0^{(k)} = x_0^{(k)} + iy_0^{(k)}; \quad \sigma_i^{(k)}, \quad v_i^{(k)} = x_i^{(k)} + iy_i^{(k)}; \\ \sigma_{1k_1 \dots k_n}^{(k)}, \quad v_{1k_1 \dots k_n}^{(k)} = x_{1k_1 \dots k_n}^{(k)} + iy_{1k_1 \dots k_n}^{(k)} \quad (6)$$

in $\Im v > 0$ respectively, so that

$$v_0^{(k)} \in \sigma_0^{(k)}, \quad v_i^{(k)} \in \sigma_i^{(k)}, \quad v_{1k_1 \dots k_n}^{(k)} \in \sigma_{1k_1 \dots k_n}^{(k)}. \quad (7)$$

Since $D_{1k_1 \dots k_n}^{(k)}$ is contained in J , we have by Lemma 1,

$$\frac{k^2}{2} y_0^{(k)} \leq 1 - |\zeta_0^{(k)}| \leq 4 y_0^{(k)}, \quad \frac{k^2}{2} y_1^{(k)} \leq 1 - |\zeta_1^{(k)}| \leq 4 y_1^{(k)}, \\ \frac{k^2}{2} y_{1k_1 \dots k_n}^{(k)} \leq 1 - |\zeta_{1k_1 \dots k_n}^{(k)}| \leq 4 y_{1k_1 \dots k_n}^{(k)}. \quad (8)$$

We consider only $\sigma_{1k_1 \dots k_n}^{(k)}$ in the following.

Fig. 2.

Let the boundary of $\sigma_{1k_1 \dots k_n}^{(k)}$ contain Jordan arcs:

$$\gamma_{1k_1 \dots k_n}^{(k)}, \quad \gamma_{1k_1 \dots k_{n1}}^{(k)}, \quad \gamma_{1k_1 \dots k_{n2}}^{(k)}, \quad \gamma_{1k_1 \dots k_{n3}}^{(k)}, \quad (9)$$

which are parts of Jordan arcs:

$$\bar{\gamma}_{1k_1 \dots k_n}^{(k)}, \quad \bar{\gamma}_{1k_1 \dots k_{n1}}^{(k)}, \quad \bar{\gamma}_{1k_1 \dots k_{n2}}^{(k)}, \quad \bar{\gamma}_{1k_1 \dots k_{n3}}^{(k)}, \quad (10)$$

which correspond to

$$\Gamma_{1k_1 \dots k_n}, \quad \Gamma_{1k_1 \dots k_{n1}}, \quad \Gamma_{1k_1 \dots k_{n2}}, \quad \Gamma_{1k_1 \dots k_{n3}} \quad (11)$$

on the z -plane. Each curve of (10) ends at two points on the real axis and let the distance between these two points be denoted by

$$J_{1k_1 \dots k_n}^{(k)}, \quad J_{1k_1 \dots k_n 1}^{(k)}, \quad J_{1k_1 \dots k_n 2}^{(k)}, \quad J_{1k_1 \dots k_n 3}^{(k)} \quad (12)$$

respectively. Then I have proved in (I) :

$$\sum_{k_1, \dots, k_n}^{1, 2, 3} J_{1k_1 \dots k_n}^{(k)} \leq \frac{1}{(1+\epsilon)^{n-1}} (J_{11}^{(k)} + J_{12}^{(k)} + J_{13}^{(k)}), \quad (13)$$

where ϵ is a positive constant independent of the transformations of $\bar{G}$. Hence

$$\sum_{n=1}^{\infty} \sum_{k_1, \dots, k_n}^{1, 2, 3} J_{1k_1 \dots k_n}^{(k)} \leq \text{const.} (J_{11}^{(k)} + J_{12}^{(k)} + J_{13}^{(k)}). \quad (14)$$

Let $\bar{\gamma}_{1k_1 \dots k_n}^{(k)}$ end at two points a_0, b_0 on the real axis and c_0 be one end point of $\gamma_{1k_1 \dots k_n}^{(k)}$. Then since $c_0, v_{1k_1 \dots k_n}^{(k)}$ are equivalent by $\bar{G}$ to the corresponding points in $\sigma_0^{(0)}$, we have by Lemma 2,

$$\Im v_{1k_1 \dots k_n}^{(k)} = y_{1k_1 \dots k_n}^{(k)} \leq \text{const.} \Im c_0. \quad (15)$$

Since a_0, c_0, b_0 are equivalent by $\bar{G}$ to the corresponding points similarly obtained from the boundary of $\sigma_0^{(0)}$, we see that a_0, c_0, b_0 lie on a circular arc, which makes a fixed angle with the real axis, so that

$$\Im c_0 \leq \text{const.} (b_0 - a_0) = \text{const.} J_{1k_1 \dots k_n}^{(k)}. \quad (16)$$

Hence we have from (14), (15), (16),

$$\sum_{n=1}^{\infty} \sum_{k_1, \dots, k_n}^{1, 2, 3} y_{1k_1 \dots k_n}^{(k)} \leq \text{const.} (J_{11}^{(k)} + J_{12}^{(k)} + J_{13}^{(k)}). \quad (17)$$

Fig. 3.

Since as I have proved in (I), the ratio of any two numbers of $J_{11}^{(k)}$, $J_{12}^{(k)}$, $J_{13}^{(k)}$, δ, δ' in Fig. 3 lies between two positive constants, we have

$$J_{11}^{(k)} \geq \text{const.} (J_{11}^{(k)} + J_{12}^{(k)} + J_{13}^{(k)}) . \quad (18)$$

Let λ be a boundary arc⁽³⁾ of $\sigma_1^{(k)}$, such that its one end point a on $\gamma_{11}^{(k)}$ is transformed into another end point b on $\gamma_{12}^{(k)}$ by a transformation U of $\bar{G}$. We apply all transformations U^n ($n = 0, \pm 1, \pm 2, \dots$) to λ , then we obtain a Jordan arc $\bar{\lambda}$ ending at two points a, β on the real axis. Since the ratio of any two numbers of $a - a_1, b_1 - a, \beta - b_1$ lie between two fixed positive constants and a_1, a, b_1 and a, a, β lie on circular arcs making fixed angles with the real axis, we have

$$\Im a \geq \text{const.} (b_1 - a_1) = \text{const.} J_{11}^{(k)} . \quad (19)$$

By Lemma 2,

$$\Im a \leq \text{const.} \Im y_1^{(k)} = \text{const.} y_1^{(k)} , \quad (20)$$

$$y_1^{(k)} \leq \text{const.} y_0^{(k)} . \quad (21)$$

Hence we have from (17), (18), (19), (20), (21),

$$y_1^{(k)} + \sum_{n=1}^{\infty} \sum_{k_1, \dots, k_n}^{1, 2, 3} y_{1k_1 \dots k_n}^{(k)} \leq \text{const.} y_0^{(k)} .$$

Hence by (8)

$$(1 - |\zeta_1^{(k)}|) + \sum_{n=1}^{\infty} \sum_{k_1, \dots, k_n}^{1, 2, 3} (1 - |\zeta_{1k_1 \dots k_n}^{(k)}|) \leq \text{const.} (1 - |\zeta_0^{(k)}|) .$$

Similarly we have for $i = 1, 2, 3, 4$,

$$(1 - |\zeta_i^{(k)}|) + \sum_{n=1}^{\infty} \sum_{k_1, \dots, k_n}^{1, 2, 3} (1 - |\zeta_{ik_1 \dots k_n}^{(k)}|) \leq \text{const.} (1 - |\zeta_0^{(k)}|) .$$

Summing up for $i = 1, 2, 3, 4$, we have

$$(1 - |\zeta_0^{(k)}|) + \sum_{i=1}^4 (1 - |\zeta_i^{(k)}|) + \sum_{i=1}^4 \sum_{n=1}^{\infty} \sum_{k_1, \dots, k_n}^{1, 2, 3} (1 - |\zeta_{ik_1 \dots k_n}^{(k)}|) \leq \text{const.} (1 - |\zeta_0^{(k)}|) . \quad (22)$$

Let ζ_n be the transform of ζ_0 by $\bar{G}$, then the left hand side of (22) is nothing but $\sum_{\zeta_n \in D^{(k)}} (1 - |\zeta_n|)$, hence we have an important relation:

$$\sum_{\zeta_n \in D^{(k)}} (1 - |\zeta_n|) \leq \text{const.} (1 - |\zeta_0^{(k)}|) . \quad (23)$$

(3) For example we take λ such that it corresponds to the boundary of S_0 on the z -plane, which belongs to one of both shores of A_1 and A_2 .

Hence

$$\sum_{n=0}^{\infty} (1 - |\zeta_n|) = \sum_{k=0}^{\infty} \sum_{\zeta_n \in D(k)} (1 - |\zeta_n|) \leq \text{const.} \sum_{k=0}^{\infty} (1 - |\zeta_0^{(k)}|).$$

Since the fundamental domain $D_0^{(0)}$ of $\bar{G}$ lies entirely in $|\zeta| < 1$, we have by Lemma 3, $\sum_{n=0}^{\infty} (1 - |\zeta_n|) = \infty$, so that

$$\sum_{k=0}^{\infty} (1 - |\zeta_0^{(k)}|) = \infty. \quad (24)$$

If E is of capacity positive, then by Frostman's theorem⁽⁴⁾ the characteristic function $T(r)$ of $\varphi(\zeta)$ is bounded in $|\zeta| < 1$, so that $\sum_{k=0}^{\infty} (1 - |\zeta_0^{(k)}|) < \infty$, which contradicts (24). Hence E is of capacity zero, q. e. d.

Similarly we can prove Theorem VI and VII in my former paper.

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(4) Frostman: Potentiel d'équilibre et capacité des ensembles. Lund. (1935).