2. On the double sequences of linear transformations on normed linear separable spaces.*)

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1. W. Sierpiński has proved recently the following theorem⁽¹⁾:

Let E be denumerable set of any elements and $\{f_n^m(x)\}$ be a double sequence of real valued functions defined on E , and if there exists the limit $\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} f_n^m(x)) = f(x)$ for all $x \in E$, then there exist two increasing sequences of indices $\{m_k\}$ and $\{n_k\}$ such that $\lim_{k \rightarrow \infty} f_{n_k}^{m_k}(x) = f(x)$ for all $x \in E$.

It is natural then to ask whether this theorem can be extended by replacing the denumerable set E by a separable set. In this note we will prove that this theorem is valid in the case where the double sequence $\{f_n^m(x)\}$ defined on a separable set is a sequence of equally continuous functions, that is, a sequence of real valued functions such that corresponding to each $\varepsilon > 0$ there exists an $\delta > 0$ and we have $|f_n^m(x) - f_n^m(x')| < \varepsilon$ for any m, n and for all x, x' where $|x - x'| < \delta$. And next by applying this theorem we will obtain some theorems concerning the double sequences of linear transformations defined on normed linear separable spaces.

2. Theorem 1.

Let $\{f_n^m(x)\}$ be a double sequence of equally continuous functions defined on a separable set E , and if $\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} f_n^m(x)) = f(x)$ for all $x \in E$, then there exists a subsequence $\{f_{n_k}^{m_k}(x)\}$ such that $\lim_{k \rightarrow \infty} f_{n_k}^{m_k}(x) = f(x)$ for all $x \in E$.

Before proving Theorem 1 we proceed to establish the following Lemma.

If $\lim_{n \rightarrow \infty} f_n^m(x) = f^m(x)$ for $m = 1, 2, \dots$, then the sequence $\{f^m(x)\}$ is a sequence of equally continuous functions.

Proof. Since $\{f_n^m(x)\}$ is a sequence of equally continuous functions, for any given number $\frac{\varepsilon}{2} > 0$ there exists an $\delta > 0$ such that for all m, n and all x, x' where $|x - x'| < \delta$, we have

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(1) W. Sierpiński, Un théorème concernant la convergence des fonctions sur les ensemble dénombrables, Fund. Math. Vol. 31 (1938).

$$|f_n^m(x) - f_n^m(x')| < \frac{\varepsilon}{2} \quad \dots\dots\dots (1)$$

If there exist an m and two elements x, x' such that $|x - x'| < \delta$, and $|f^m(x) - f^m(x')| > \varepsilon$, then it can be shown that this contradicts the hypothesis (1).

In fact, corresponding to $\frac{\varepsilon}{4}$, there exists an N such that for all $n > N$ we have

$$|f_n^m(x) - f^m(x)| < \frac{\varepsilon}{4},$$

$$|f_n^m(x') - f^m(x')| < \frac{\varepsilon}{4}.$$

Therefore

$$\begin{aligned} |f^m(x) - f^m(x')| &\leq |f^m(x) - f_n^m(x)| + |f_n^m(x) - f_n^m(x')| + |f_n^m(x') - f^m(x')| \\ &< \frac{\varepsilon}{2} + |f_n^m(x) - f_n^m(x')|. \end{aligned}$$

Hence we have $|f_n^m(x) - f_n^m(x')| > \frac{\varepsilon}{2}$, this contradicts the hypothesis (1) and completes the proof.

Proof of Theorem 1.

Since the set E is separable, there is a denumerable set $E_0 \subset E$ such that $\overline{E_0} = E$. From W. Sierpiński's theorem, there exists a subsequence $\{f_{n_k}^{m_k}(x)\}$ of $\{f_n^m(x)\}$ such that for all $x \in E_0$ we have

$$\lim_{k \rightarrow \infty} f_{n_k}^{m_k}(x) = f(x) \quad \dots\dots\dots (2)$$

Let x be any element of E and $\varepsilon > 0$ be any number, then there is a sequence $\{x_i\}$ of elements of E_0 such that $\lim_{i \rightarrow \infty} x_i = x$. By Lemma, it is easy to see that $f(x)$ is a continuous function. Therefore there exists an M such that for any $i > M$, we have

$$|f(x) - f(x_i)| < \frac{\varepsilon}{3} \quad \dots\dots\dots (3)$$

Since $\{f_n^m(x)\}$ is a sequence of equally continuous functions, we have for any m_k, n_k and for any $i > M$,

$$|f_{n_k}^{m_k}(x) - f_{n_k}^{m_k}(x_i)| < \frac{\varepsilon}{3} \quad \dots\dots\dots (4)$$

On the other hand, by (2), corresponding to i we can find K such that for any $k > K$ we have

$$|f_{n_k}^{m_k}(x_i) - f(x_i)| < \frac{\varepsilon}{3} \dots\dots\dots (5)$$

From (3), (4) and (5), we have

$$|f_{n_k}^{m_k}(x) - f(x)| \leq |f_{n_k}^{m_k}(x) - f_{n_k}^{m_k}(x_i)| + |f_{n_k}^{m_k}(x_i) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon$$

for any $k > K$. Therefore we have $\lim_{k \rightarrow \infty} f_{n_k}^{m_k}(x) = f(x)$ for all $x \in E$. Thus our theorem is completely proved.

3. From the proof of Theorem 1, we may easily see that Theorem 1 remains true in the case where the domain of definition of the functions and whose range are in metric spaces. In this article we will deal with the double sequences of linear transformations with domain in a normed linear separable space and range in the same or other normed linear space.

Theorem 2.

Let $\{U_n^m(x)\}$ be a double sequence of linear transformations defined on a normed linear separable space E and whose range be in a normed linear space. And let $|U_n^m| < M$ for all m, n .

If $\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} U_n^m(x)) = U(x)$ for all $x \in E$, then there exists a subsequence $\{U_{n_k}^{m_k}(x)\}$ such that $\lim_{k \rightarrow \infty} U_{n_k}^{m_k}(x) = U(x)$ for all $x \in E$.

Proof. Let $\varepsilon > 0$ be any number. For all x, x' such that $|x - x'| < \frac{\varepsilon}{M}$, we have, for all m, n

$$|U_n^m(x - x')| = |U_n^m(x) - U_n^m(x')| \leq M|x - x'| < \varepsilon.$$

Hence $\{U_n^m(x)\}$ is a sequence of equally continuous transformations. Therefore, by Theorem 1, our theorem is obtained.

Instead of the normed linear space we take the Banach space. Then we have the following.

Theorem 3.

Let $\{U_n^m(x)\}$ be a double sequence of linear transformations defined on a separable Banach space E and whose range be in a Banach space and let $|U_n^m| < M$ for all m, n . And let E_0 denote the denumerable set whose closure is the whole space E .

If $\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} U_n^m(x)) = U(x)$ for all $x \in E_1$, where E_1 denotes the subset of E_0 such that E_1 is dense in a sphere, then there exists a subsequence $\{U_{n_k}^{m_k}(x)\}$ such that $\lim_{k \rightarrow \infty} U_{n_k}^{m_k}(x) = U(x)$ for all $x \in E$.

Proof. Since $\lim_{n \rightarrow \infty} U_n^m(x) = U^m(x)$ for all $x \in E_1$ and $|U_n^m| < M$, there-

(2) S. Banach, Théorie des opérations linéaires, 1932, p. 79.

fore it can be shown that $\lim_{n \rightarrow \infty} U_n^m(x) = U^m(x)$ for all $x \in E^{(2)}$. Therefore $U^m(x)$ is a linear transformation.⁽³⁾

On the other hand, since $|U_n^m(x)| < M \cdot |x|$ for all m, n and for all $x \in E$, we have $\lim_{n \rightarrow \infty} |U_n^m(x)| \leq M|x|$. Therefore $|U^m(x)| \leq M \cdot |x|$. Hence we have $|U^m| \leq M$ for all m .

In the same way, we have $\lim_{m \rightarrow \infty} U_n^m(x) = U(x)$ for all $x \in E$. Therefore by Theorem 2, there exists a subsequence $\{U_{n_k}^{m_k}(x)\}$ such that $\lim_{k \rightarrow \infty} U_{n_k}^{m_k}(x) = U(x)$ for all $x \in E$. Hence the theorem is completely proved.

Finally we give some definitions in a Banach space E . If there exists a denumerable subset E_0 of E such that to each $x \in E$ there corresponds a sequence of elements of E_0 which converges weakly to x , then E is called to be separable in the weak sense. If, for any $\epsilon > 0$, there is an I such that for all $i > I$ and for all $\{x_i\}$ which are sequences converging weakly to each $x \in E$, we have $|U_n(x) - U_n(x_i)| < \epsilon$ for all n , then the sequence $\{U_n(x)\}$ of linear transformations is said to be equally continuous in the weak sense.

A Banach space E is called to be regular if every linear functional $X_0(f)$ defined on $\bar{E}$, where $\bar{E}$ denotes the space of all linear functionals on E , may be represented in the form: $X_0(f) = f(x_0)$ for any $f \in \bar{E}$, where x_0 is an element from E .

Theorem 4. Let E be a regular Banach space which is separable in the weak sense and let $\{U_n^m(x)\}$ be a double sequence of completely continuous linear transformations on E with range in a Banach space. If $\{U_n^m(x)\}$ is a sequence of equally continuous transformations in the weak sense and $\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} U_n^m(x)) = U(x)$ for all $x \in E$, then $U(x)$ is a completely continuous linear transformation and there exists a subsequence $\{U_{n_k}^{m_k}(x)\}$ such that $\lim_{k \rightarrow \infty} U_{n_k}^{m_k}(x) = U(x)$ for all $x \in E$.

Proof. It is evident that $U(x)$ is a linear transformation. In the same way as the proofs of Theorem 1 and Lemma, in virtue of the separability of E in the weak sense, we can prove that there exists a subsequence $\{U_{n_k}^{m_k}(x)\}$ such that $\lim_{k \rightarrow \infty} U_{n_k}^{m_k}(x) = U(x)$ for all $x \in E$ and that $\lim_{i \rightarrow \infty} U(x_i) = U(x_0)$ for any sequence $\{x_i\}$ converging weakly to x_0 where x_0 is any given element of E . Since the space E is a regular Banach space, it may be shown that $U(x)$ is a completely continuous linear transformation.⁽⁴⁾ Hence the proof of our theorem is thus completed.

⁽³⁾ S. Banach, loc. cit., p. 23.

⁽⁴⁾ S. Hanai: On completely continuous linear transformations on regular Banach spaces, Tōhoku Math. Journ. Vol. 50, Part 1 (under press).

Remark. Let ϕ denote a family of transformations. The families ϕ^α where $\alpha = 0, 1, 2, \dots$, are defined as follows: $\phi^0 = \phi$ and $\phi^\alpha (\alpha > 0)$ is the family of all limits of all infinite sequences from $\phi^{\alpha-1}$.

If ϕ is defined as the family which satisfies the conditions of each of the above obtained theorems, then we have $\phi^1 = \phi^2$.

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