15. On Hopf's ergodic theorem.⁽¹⁾

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(Received October 30, 1944.)

1. Hopf's theorem.

1. Let E be a measurable set of points in $|z| < 1$. We define its hyperbolic measure $\sigma(E)$ by

$$\sigma(E) = \iint_E \frac{dx dy}{(1-|z|^2)^2} \quad (z = x + iy),$$

so that for a disc $K_r: |z| \leq r$, we have $\sigma(K_r) = \frac{\pi r^2}{1-r^2}$, hence

$$\frac{\pi}{6(1-r)} < \sigma(K_r) < \frac{\pi}{2(1-r)} \quad \left(r \geq \frac{1}{2}\right). \quad (1)$$

Similarly the hyperbolic linear measure $\lambda(e)$ of a set e on a rectifiable curve C in $|z| < 1$ is defined by

$$\lambda(e) = \int_e \frac{|dz|}{1-|z|^2}.$$

Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and D_0 be its fundamental domain, which contains $z_0 = 0$ and is bounded by at most enumerably infinite number of orthogonal circles to $|z| = 1$, z_n be equivalents of $z_0 = 0$ and $n(r)$ be the number of z_n in $|z| \leq r$. For any point z in $|z| < 1$, we denote its equivalent in D_0 by (z) .

Let $E(\theta)$ be the set of points $(re^{i\theta})$ in D_0 , which are equivalent to points on a radius $z = re^{i\theta} (0 \leq r < 1)$. Then I have proved in my former paper⁽²⁾:

Theorem 1. (i) If $\sum_{n=0}^{\infty} (1-|z_n|) = \infty$, then $E(\theta)$ is everywhere dense in D_0 for almost all $e^{i\theta}$ on $|z| = 1$.

(ii) If $\sum_{n=0}^{\infty} (1-|z_n|) < \infty$, then $\lim_{r \rightarrow 1} |(re^{i\theta})| = 1$ for almost all $e^{i\theta}$ on $|z| = 1$.

(1) I have reported the results of the present paper in a paper, M. Tsuji: On Hopf's ergodic theorem. Proc. Imp. Acad. **20** (1944).

(2) M. Tsuji: Theory of conformal mapping of a multiply connected domain, III. This Journ. **19** (1944).

Theorem 2. *The necessary and sufficient condition, that there exists a set e on $|z|=1$, which is invariant by G and $0 < m_e < 2\pi$, is that $\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$.*

Theorem 1 (i) is an extension of Myrberg's theorem⁽³⁾, who assumed that D_0 lies with its boundary entirely in $|z| < 1$, in which case, it is easily proved that $\sum_{n=0}^{\infty} (1 - |z_n|) = \infty$.

2. Let $\eta_1 = e^{i\theta}$, $\eta_2 = e^{i\varphi}$ be two points on $|z|=1$, $|w|=1$ respectively. Then the pair (η_1, η_2) can be considered as a point on a torus Θ ($0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$). For any measurable set E on Θ , we define its measure mE by

$$mE = \iint_E d\theta d\varphi, \quad \text{so that} \quad m\Theta = 4\pi^2.$$

Let S be any substitution of G and

$$T: \quad \eta'_1 = S(\eta_1), \quad \eta'_2 = S(\eta_2),$$

then the totality of T constitutes a group $\mathfrak{G}$, which is isomorphic to G . Hopf⁽⁴⁾ proved the theorem:

Theorem 3. (Hopf.) *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and D_0 be its fundamental domain. If the hyperbolic area $\sigma(D_0)$ of D_0 is finite, then there does not exist a set E on Θ , which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$.*

To prove this theorem, Hopf proved a lemma:

Lemma. (Hopf.) *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and the hyperbolic area $\sigma(D_0)$ of its fundamental domain D_0 be finite. Let B be a set of points in $|z| < 1$, which is invariant by G and $K_r: |z| \leq r$ be a disc. Then*

$$\sigma(K_r \cdot B) \leq \text{const. } \sigma(K_r) \sigma(D_0 \cdot B) \leq \frac{\text{const. } \sigma(D_0 \cdot B)}{1-r}. \quad (2)$$

Let $K_{r_0}: |z| \leq r_0$ be a disc and $D'_0 = D_0 \cdot K_{r_0}$, $B_0 = D_0 - D'_0$. Since $\sigma(D_0) < \infty$, we take r_0 , so that $\sigma(B_0) < \varepsilon$.

Let B_n be equivalents of B_0 and $B = \sum_{n=0}^{\infty} B_n$, then B is invariant by G and $\sigma(D_0 \cdot B) = \sigma(B_0) < \varepsilon$. Hence by (2), for a small ε ,

$$\sigma(K_r \cdot B) \leq \frac{\text{const. } \varepsilon}{1-r} \leq \frac{\pi}{12(1-r)}. \quad (3)$$

⁽³⁾ Myrberg: Ein Satz über die Fuchsschen Gruppen und seine Anwendungen in der Funktionentheorie. *Annales Académie Sci. Fennicae*. **32** (1929).

⁽⁴⁾ E. Hopf: Fuchsian groups and ergodic theory. *Trans. Amer. Math. Soc.* **39** (1936). *Ergodentheorie*. Berlin (1937).

Then by (1) for $r \geq \frac{1}{2}$,

$$n(r)\sigma(D'_0) + \sigma(K_r \cdot B) \geq \sigma(K_r) > \frac{\pi}{6(1-r)},$$

so that by (3),

$$n(r)\sigma(D'_0) > \frac{\pi}{6(1-r)} - \frac{\pi}{12(1-r)} = \frac{\pi}{12(1-r)}, \quad \text{or} \quad n(r) > \frac{\text{const.}}{1-r}.$$

Hence we have :

Theorem 4. *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant. If the hyperbolic area $\sigma(D_0)$ of its fundamental domain D_0 is finite, then*

$$n(r) > \frac{\text{const.}}{1-r}. \quad (4)$$

We will prove the following theorem, which, in virtue of (4), is an extension of Hopf's theorem.

Theorem 5. (Main Theorem.) *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant. If*

$$\overline{\lim}_{r \rightarrow 1} n(r)(1-r) > 0, \quad (5)$$

then there does not exist a set E on Θ , which is invariant by $\mathbb{G}$ and $0 < mE < 4\pi^2$.

To prove this theorem, we will first prove some theorems on bounded harmonic functions $u(z, w)$ in $|z| < 1$, $|w| < 1$. Hopf used in his proof the mean convergence of $u(z, w)$ to its boundary value $u(e^{i\theta}, e^{i\varphi})$, but we will use its Abel's limit to $u(e^{i\theta}, e^{i\varphi})$, whose existence will be proved in § 2.

2. Bounded harmonic functions $u(z, w)$.

First we will prove a lemma.

Lemma 1. *Let $f(x, y)$ be a bounded measurable function defined in $\Delta: 0 < x < 1, 0 < y < 1$ and $Q: |x-x_0| < \delta, |y-y_0| < \delta'$ be an interval contained in Δ .*

Then

$$\lim_{Q \rightarrow (x_0, y_0)} \frac{1}{mQ} \iint_Q |f(x, y) - f(x_0, y_0)| dx dy = 0 \quad \text{almost everywhere in } \Delta.$$

Proof. By means of the strong density theorem, Busemann and Feller⁽⁵⁾ proved that almost everywhere in Δ ,

$$\lim_{Q \rightarrow (x_0, y_0)} \frac{1}{mQ} \iint_Q f(x, y) dx dy = f(x_0, y_0).$$

Let $\{r\}$ be the set of all rational numbers and N_r be the set of points (x_0, y_0) , such that

$$\lim_{Q \rightarrow (x_0, y_0)} \frac{1}{mQ} \iint_Q |f(x, y) - r| dx dy = |f(x_0, y_0) - r|$$

does not hold, then $mN_r = 0$, so that $N = \sum_r N_r$ is of measure zero.

Let $(x_0, y_0) \in \Delta - N$ and a be any real number, then since

$$|f(x, y) - r| - |r - a| \leq |f(x, y) - a| \leq |f(x, y) - r| + |r - a|,$$

we have

$$\begin{aligned} |f(x_0, y_0) - r| - |r - a| &\leq \overline{\lim}_{Q \rightarrow (x_0, y_0)} \frac{1}{mQ} \iint_Q |f(x, y) - a| dx dy \\ &\leq |f(x_0, y_0) - r| + |r - a|. \end{aligned}$$

Making $r \rightarrow a$,

$$\lim_{Q \rightarrow (x_0, y_0)} \frac{1}{mQ} \iint_Q |f(x, y) - a| dx dy = |f(x_0, y_0) - a|, \quad (x_0, y_0) \in \Delta - N.$$

If we take $a = f(x_0, y_0)$, then

$$\lim_{Q \rightarrow (x_0, y_0)} \frac{1}{mQ} \iint_Q |f(x, y) - f(x_0, y_0)| dx dy = 0$$

almost everywhere in Δ , q.e.d.

Theorem 6. Let $f(\theta, \varphi)$ be a bounded measurable function defined on a torus Θ ($0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$) and

$$\begin{aligned} u(z, w) &= u(re^{i\theta}, \rho e^{i\varphi}) = \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta' - \theta) + r^2)(1-2\rho \cos(\varphi' - \varphi) + \rho^2)} \\ &\quad (0 \leq r < 1, 0 \leq \rho < 1). \end{aligned}$$

(5) H. Busemann und W. Feller: Zur Differentiation der Lebesgueschen Integrale. Fund. Math. **22** (1934).

Then $u(z, w) \rightarrow f(\theta, \varphi)$ almost everywhere on Θ , when $z \rightarrow e^{i\theta}$, $w \rightarrow e^{i\varphi}$ non-tangentially to $|z| = 1$, $|w| = 1$ respectively.

Proof. By Lemma 1,

$$\frac{1}{\delta\delta'} \int_{\theta_0-\delta}^{\theta_0+\delta} \int_{\varphi_0-\delta'}^{\varphi_0+\delta'} |f(\theta, \varphi) - f(\theta_0, \varphi_0)| d\theta d\varphi \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \delta' \rightarrow 0 \quad (6)$$

almost everywhere on Θ . Suppose that (6) holds at $\theta_0 = 0$, $\varphi_0 = 0$.

We will prove that $u(z, w) \rightarrow f(0, 0)$, when $z \rightarrow 1$, $w \rightarrow 1$ non-tangentially to $|z| = 1$, $|w| = 1$ respectively.

We put

$$F(\theta, \varphi) = |f(\theta, \varphi) - f(0, 0)|,$$

$$\int_{-\delta}^{\delta} \int_{-\delta'}^{\delta'} F(\theta, \varphi) d\theta d\varphi = \delta\delta' \varepsilon(\delta, \delta'), \quad (7)$$

then

$$\varepsilon(\delta, \delta') \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \delta' \rightarrow 0. \quad (8)$$

It is easily proved that if $z = re^{i\theta}$, $w = \rho e^{i\varphi}$ lie in angular domains:

$$|\arg(1-z)| \leq \frac{\pi}{2} - \varepsilon, \quad |\arg(1-w)| \leq \frac{\pi}{2} - \varepsilon,$$

then

$$|\theta| \leq K(1-r), \quad |\varphi| \leq K(1-\rho) \quad (K = \text{const.}), \quad (9)$$

so that if

$$1-r = \delta[\varepsilon(\delta, \delta')]^{\frac{1}{3}}, \quad 1-\rho = \delta'[\varepsilon(\delta, \delta')]^{\frac{1}{3}}, \quad (10)$$

we have

$$|\theta| \leq K\delta[\varepsilon(\delta, \delta')]^{\frac{1}{3}}, \quad |\varphi| \leq K\delta'[\varepsilon(\delta, \delta')]^{\frac{1}{3}}. \quad (11)$$

Since

$$f(0, 0) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{f(0, 0)(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta'-\theta) + r^2)(1-2\rho \cos(\varphi'-\varphi) + \rho^2)},$$

$$|u(z, w) - f(0, 0)|$$

$$\begin{aligned} &\leq \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{F(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta'-\theta) + r^2)(1-2\rho \cos(\varphi'-\varphi) + \rho^2)} \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{4\pi^2} \int_{|\theta'| \leq \delta} \int_{|\varphi'| \leq \delta'}, & I_2 &= \frac{1}{4\pi^2} \int_{|\theta'| \leq \delta} \int_{\delta' \leq |\varphi'| \leq \pi}, \\ I_3 &= \frac{1}{4\pi^2} \int_{\delta \leq |\theta'| \leq \pi} \int_{|\varphi'| \leq \delta'}, & I_4 &= \frac{1}{4\pi^2} \int_{\delta \leq |\theta'| \leq \pi} \int_{\delta' \leq |\varphi'| \leq \pi}. \end{aligned}$$

Since

$$1 - 2r \cos \theta + r^2 = (1-r)^2 + 4r \sin^2 \frac{\theta}{2} \geq (1-r)^2 + a^2 \theta^2 \quad (a = \text{const.}), \quad (12)$$

$$1 - r^2 \leq 2(1-r),$$

we have

$$\begin{aligned} I_1 &\leq \frac{(1-r)(1-\rho)}{\pi^2} \int_{|\theta'| \leq \delta} \int_{|\varphi'| \leq \delta'} \frac{F(\theta', \varphi') d\theta' d\varphi'}{(1-r)^2(1-\rho)^2} = \frac{\delta \delta' \varepsilon(\delta, \delta')}{\pi^2(1-r)(1-\rho)} \\ &= \frac{[\varepsilon(\delta, \delta')]^{\frac{1}{3}}}{\pi^2} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0, \quad \delta' \rightarrow 0. \end{aligned} \quad (13)$$

By (11), we take δ, δ' , so small that $|\theta| \leq \frac{\delta}{2}, |\varphi| \leq \frac{\delta'}{2}$, then

$$\text{if } \delta \leq |\theta'| \leq \pi, \quad |\theta' - \theta| \geq |\theta'| - |\theta| \geq |\theta'| - \frac{\delta}{2} \geq |\theta'| - \frac{|\theta'|}{2} = \frac{|\theta'|}{2}, \quad (14)$$

$$\text{and if } \delta' \leq |\varphi'| \leq \pi, \quad |\varphi' - \varphi| \geq \frac{|\varphi'|}{2}. \quad (15)$$

Let $0 \leq F(\theta, \varphi) \leq M$ on Θ , then by (12), (15),

$$\begin{aligned} I_2 &\leq \frac{M(1-r)(1-\rho)}{\pi^2} \int_{|\theta'| \leq \delta} \frac{d\theta'}{(1-r)^2 + a^2(\theta' - \theta)^2} \int_{\delta' \leq |\varphi'| \leq \pi} \frac{d\varphi'}{a^2(\varphi' - \varphi)^2} \\ &\leq \frac{2M(1-r)(1-\rho)}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{(1-r)^2 + a^2 t^2} \int_{\delta'}^{\infty} \frac{d\varphi'}{a^2 \left(\frac{\varphi'}{2}\right)^2} \\ &= \frac{8M(1-\rho)}{a^3 \pi \delta'} = \frac{8M[\varepsilon(\delta, \delta')]^{\frac{1}{3}}}{a^3 \pi} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0, \quad \delta' \rightarrow 0. \end{aligned} \quad (16)$$

Similarly

$$I_3 \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0, \quad \delta' \rightarrow 0. \quad (17)$$

From (12), (14), (15),

$$\begin{aligned} I_4 &\leq \frac{M(1-r)(1-\rho)}{\pi^2} \int_{\delta \leq |\theta'| \leq \pi} \frac{d\theta'}{a^2(\theta' - \theta)^2} \int_{\delta' \leq |\varphi'| \leq \pi} \frac{d\varphi'}{a^2(\varphi' - \varphi)^2} \\ &\leq \frac{4M(1-r)(1-\rho)}{\pi^2} \int_{\delta}^{\infty} \frac{d\theta'}{a^2 \left(\frac{\theta'}{2}\right)^2} \int_{\delta'}^{\infty} \frac{d\varphi'}{a^2 \left(\frac{\varphi'}{2}\right)^2} = \frac{64M(1-r)(1-\rho)}{a^4 \pi^2 \delta \delta'} \\ &= \frac{64M[\varepsilon(\delta, \delta')]^{\frac{2}{3}}}{a^4 \pi^2} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0, \quad \delta' \rightarrow 0. \end{aligned} \quad (18)$$

Hence $u(z, w) \rightarrow f(0, 0)$, when $z \rightarrow 1, w \rightarrow 1$, q.e.d.

Theorem 7. Let $u(z, w)$ be a harmonic function of each one of z, w in $|z| < 1, |w| < 1$. If $u(z, w)$ be bounded in $|z| < 1, |w| < 1$, then $\lim u(z, w) = f(\theta, \varphi)$ exists almost everywhere on a torus Θ ($0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq 2\pi$), when $z \rightarrow e^{i\theta}, w \rightarrow e^{i\varphi}$ non-tangentially to $|z| = 1, |w| = 1$ respectively and $u(z, w)$ can be expressed by

$$u(z, w) = u(re^{i\theta}, \rho e^{i\varphi}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta'-\theta)+r^2)(1-2\rho \cos(\varphi'-\varphi)+\rho^2)} \\ (0 \leq r < 1, 0 \leq \rho < 1).$$

Proof. For a fixed $w = \rho e^{i\varphi} (\rho < 1)$, $u(z, \rho e^{i\varphi})$ is a harmonic function of z in $|z| < 1$, so that for $r (< R < 1)$,

$$u(z, w) = u(re^{i\theta}, \rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta'}, \rho e^{i\varphi})(R^2-r^2)d\theta'}{R^2-2Rr \cos(\theta'-\theta)+r^2}.$$

Since $u(Re^{i\theta'}, w)$ is a harmonic function of w , we have for $\rho < R$,

$$u(Re^{i\theta'}, \rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta'}, Re^{i\varphi'})(R^2-\rho^2)d\varphi'}{R^2-2R\rho \cos(\varphi'-\varphi)+\rho^2}.$$

Hence

$$u(z, w) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta'}, Re^{i\varphi'})(R^2-r^2)(R^2-\rho^2)d\theta'd\varphi'}{(R^2-2Rr \cos(\theta'-\theta)+r^2)(R^2-2R\rho \cos(\varphi'-\varphi)+\rho^2)}.$$

Now we define an additive set function $\mu_R(e)$ ($0 < R < 1$) for Borel sets e on Θ by

$$\mu_R(e) = \iint_e u(Re^{i\theta}, Re^{i\varphi})d\theta d\varphi. \quad (19)$$

Then

$$u(z, w) = \frac{1}{4\pi^2} \iint_{\Theta} \frac{(R^2-r^2)(R^2-\rho^2)d\mu_R(e)}{(R^2-2Rr \cos(\theta'-\theta)+r^2)(R^2-2R\rho \cos(\varphi'-\varphi)+\rho^2)}. \quad (20)$$

Let $|u(z, w)| \leq M$ in $|z| < 1, |w| < 1$, then

$$|\mu_R(e)| \leq M.me. \quad (21)$$

Hence $\mu_R(e)$ ($0 < R < 1$) are uniformly absolutely continuous additive set functions on Θ , so that by Radon's theorem, we can select $R_n \rightarrow 1$, such that $\mu_{R_n}(e) \rightarrow \mu(e)$, where $\mu(e)$ is an absolutely continuous additive set function on Θ , such that, by (21),

$$|\mu(e)| \leq M.me. \quad (22)$$

Taking $R = R_n \rightarrow 1$ in (20), we have

$$u(z, w) = \frac{1}{4\pi^2} \iint_{\Theta} \frac{(1-r^2)(1-\rho^2)d\mu(e)}{(1-2r \cos(\theta' - \theta) + r^2)(1-2\rho \cos(\varphi' - \varphi) + \rho^2)}. \quad (23)$$

Since $\mu(e)$ is absolutely continuous, we can write (23) as a Lebesgue integral:

$$u(z, w) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta' - \theta) + r^2)(1-2\rho \cos(\varphi' - \varphi) + \rho^2)},$$

where $f(\theta, \varphi)$ is the derivative of $\mu(e)$ at (θ, φ) , so that by (22), $|f(\theta, \varphi)| \leq M$ on Θ . Hence by Theorem 4, $u(z, w) \rightarrow f(\theta, \varphi)$ almost everywhere on Θ , when $z \rightarrow e^{i\theta}$, $w \rightarrow e^{i\varphi}$ non-tangentially to $|z| = 1$, $|w| = 1$ respectively, q.e.d.

From Theorem 7, we have:

Theorem 8. *Let $f(z, w)$ be an analytic function of two complex variables z, w in $|z| < 1$, $|w| < 1$. If $f(z, w)$ be bounded in $|z| < 1$, $|w| < 1$, then $\lim f(z, w)$ exists almost everywhere on a torus Θ , when $z \rightarrow e^{i\theta}$, $w \rightarrow e^{i\varphi}$ non-tangentially to $|z| = 1$, $|w| = 1$ respectively.*

3. Some lemmas.

We will prove some lemmas.

Lemma 2. *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and z_n be equivalents of $z_0 = 0$ by G .*

If $\sum_{n=0}^{\infty} (1 - |z_n|) = \infty$, then there does not exist a set e on $|z| = 1$, which is invariant by G and $0 < me < 2\pi$.

Though this follows from Theorem 2, I will give a proof for the sake of completeness.

Proof. Suppose that there exists an invariant set e on $|z| = 1$, such that $0 < me < 2\pi$. Let

$$u(z) = u(r^{i\theta}) = \frac{1}{2\pi} \int_e \frac{1-r^2}{1-2r \cos(\varphi - \theta) + r^2} d\varphi \quad (0 \leq r < 1).$$

Then $u(z)$ is a non-constant bounded harmonic function in $|z| < 1$, which is invariant by G . Let $u(0) = k$, then $u(z) = k$ on an arc L_0 through $z_0 = 0$. Then $u(z) = k$ on arcs L_n , which are equivalent to L_0 passing through z_n . Let $v(z)$ be the conjugate harmonic function of $u(z)$ and let $v(z)$ change by an amount a , when z moves on L_0 from one

end point to another end point. Then $v(z)$ changes by the same amount a , when z moves on L_n . Let $f(z) = e^{\frac{2\pi}{a}(u+iv)}$, then $f(z)$ makes one turn on a circle $|w| = e^{\frac{2\pi k}{a}}$, when z moves on L_n from one end point to another end point. Hence there exists a point z'_n on L_n , such that $f(z'_n) = e^{\frac{2\pi k}{a}}$. Since $f(z)$ is bounded in $|z| < 1$, by Blaschke's theorem, $\sum_{n=0}^{\infty} (1 - |z'_n|) < \infty$.

Let σ be the hyperbolic length of L_0 and $|z_n| = r_n < |z'_n| = r'_n$, then

$$\sigma = \int_{L_n} \frac{|dz|}{1 - |z|^2} > \frac{1}{2} \int_{r_n}^{r'_n} \frac{dr}{1 - r} = \frac{1}{2} \log \frac{1 - r_n}{1 - r'_n}, \quad \text{or}$$

$1 - |z_n| < e^{2\sigma}(1 - |z'_n|)$, so that $\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$, which contradicts the hypothesis, which proves the lemma.

Lemma 3. Under the same condition as Theorem 5, there does not exist a set e on $|z| = 1$, which is invariant by G and $0 < me < 2\pi$.

Proof. If $\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$, then $\int_0^1 n(r) dr < \infty$, so that $\lim_{r \rightarrow 1} n(r)(1 - r) = 0$. Hence if $\lim_{r \rightarrow 1} n(r)(1 - r) > 0$, then $\sum_{n=0}^{\infty} (1 - |z_n|) = \infty$, so that Lemma 3 follows from Lemma 2.

Lemma 4. Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and D_0 be its fundamental domain, which contains $z_0 = 0$. Let $K_0: |z| \leq r_0$ be a disc contained in D_0 and K_n, z_n be equivalents of $K_0, z_0 = 0$ respectively, $n(r)$ be the number of z_n in $|z| \leq r$ and $rL(r)$ be the linear measure of the part of $|z| = r$ contained in $\sum_{n=0}^{\infty} K_n$.

If $\lim_{r \rightarrow 1} n(r)(1 - r) > 0$, then there exists $\rho_\nu \rightarrow 1$, such that

$$L(\rho_\nu) \geq a > 0 \quad (\nu = 1, 2, \dots).$$

Proof. Let σ be the hyperbolic diameter of K_0 . For $r' (< 1)$, we determine $r (> r')$ by $\sigma = \int_{r'}^r \frac{dr}{1 - r^2}$, then

$$\sigma > \frac{1}{2} \int_{r'}^r \frac{dr}{1 - r} = \frac{1}{2} \log \frac{1 - r'}{1 - r}, \quad \text{so that}$$

$$1 - r' < e^{2\sigma}(1 - r). \quad (24)$$

Let $K(r)$ be the part of $|z| \leq r$ contained in $\sum_{n=0}^{\infty} K_n$. Since any K_n , which has common points with $|z| \leq r'$ is contained in $|z| \leq r$,

$$\sigma(K(r)) > n(r')\sigma(K_0).$$

By the hypothesis, there exists $r'_v \rightarrow 1$, such that

$$n(r'_v) > \frac{\text{const.}}{1-r'_v}.$$

Hence by (24),

$$\sigma(K(r_v)) > \frac{\text{const.}}{1-r'_v} > \frac{2a}{1-r_v} \quad (a = \text{const.}). \quad (25)$$

Let $\kappa > 1$ and we define τ_v by $1-\tau_v = \kappa(1-r_v)$, then $\tau_v < r_v$.

Let $K(\tau_v, r_v)$ be the part of $\sum_{n=0}^{\infty} K_n$ contained in $\tau_v \leq |z| \leq r_v$, then by (1),

$$\begin{aligned} \sigma(K(\tau_v, r_v)) &> \sigma(K(r_v)) - \int_0^{\tau_v} \int_0^{2\pi} \frac{r dr d\theta}{(1-r^2)^2} > \frac{2a}{1-r_v} - \frac{\pi}{2(1-\tau_v)} \\ &= \frac{2a}{1-r_v} - \frac{\pi}{2\kappa(1-r_v)} \geq \frac{a}{1-r_v}, \end{aligned} \quad (26)$$

where we take κ , such that $\frac{\pi}{2\kappa} \leq a$.

Let $L_v = \text{Max. } L(r)$, then

$$\frac{a}{1-r_v} < \sigma(K(\tau_v, r_v)) = \int_{\tau_v}^{r_v} \frac{r L(r) dr}{(1-r^2)^2} < L_v \int_0^{r_v} \frac{r dr}{(1-r^2)^2} < \frac{L_v}{1-r_v},$$

or $L_v > a$. Hence there exists ρ_v ($\tau_v \leq \rho_v \leq r_v$), such that $L(\rho_v) > a$, where $\rho_v \rightarrow 1$, q.e.d.

Lemma 5. Let $K_0: |z| = r_0$ and $K: \left| \frac{z-a}{1-\bar{a}z} \right| = r_0$ ($|a| < 1$) be two circles in $|z| < 1$. We transform K into K_0 by S :

$$S: z' = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z}, \quad \text{such that } S(K) = K_0, \quad S(0) = -ae^{i\theta} = \rho_0 e^{i\varphi_0}.$$

Then $S(K_0) = \bar{K}$ is obtained from K by a rotation about $z = 0$.

Let e be a set on $|z| = 1$ contained in an arc C :

$$C: \pi \geq |\arg z - \arg a| \geq \eta > 0.$$

Then $S(e)$ is contained in an arc $\bar{C}$ on $|z| = 1$, whose center is at $e^{i\varphi_0}$, such that

$$m\bar{C} = \kappa R \quad (R = \text{radius of } K), \quad (27)$$

$$\frac{1}{2\pi} m e > \frac{m S(e)}{m \bar{C}} > \lambda m e, \quad (28)$$

where
$$\kappa = \frac{2\pi}{r_0 \sin^2 \eta}, \quad \lambda = \frac{(1-r_0^2) \sin^2 \eta}{8\pi}.$$

Proof. From $z' = e^{i\theta} \frac{z-a}{1-\bar{a}z}$, we have $z = \frac{z'+ae^{i\theta}}{e^{i\theta}+\bar{a}z'}$. Hence $S(K_0): \left| \frac{z'+ae^{i\theta}}{1+\bar{a}e^{-i\theta}z'} \right| = r_0$ is obtained from K by a rotation of an angle $\theta+\pi$ about $z=0$.

We remark that the radius R of $K: \left| \frac{z-a}{1-\bar{a}z} \right| = r_0$ is given by $R = \frac{r_0(1-|a|^2)}{1-|a|^2 r_0^2}$, so that

$$r_0(1-|a|^2) < R < \frac{r_0(1-|a|^2)}{1-r_0^2}. \quad (29)$$

To prove (28), we may assume that $\theta = 0$, $0 < a < 1$, then $S: z' = \frac{z-a}{1-az}$. Since $|dz'| = \frac{1-a^2}{|1-az|^2} |dz|$, we have

$$mS(C) = \int_C \frac{1-a^2}{|\bar{z}-a|^2} |dz|, \quad mS(e) = \int_e \frac{1-a^2}{|\bar{z}-a|^2} |dz|.$$

Since $|\bar{z}-a| \geq \sin \eta$ for any z on C , we have by (29),

$$mS(C) < \frac{1-a^2}{\sin^2 \eta} 2\pi < \frac{2\pi}{r_0 \sin^2 \eta} R = \kappa R, \quad \text{where } \kappa = \frac{2\pi}{r_0 \sin^2 \eta}, \quad (30)$$

$$mS(e) < \frac{1-a^2}{\sin^2 \eta} me < \frac{R}{r_0 \sin^2 \eta} me = \frac{\kappa R}{2\pi} me. \quad (31)$$

Since $|\bar{z}-a| \leq 2$, we have from (29),

$$\begin{aligned} mS(e) &> \frac{1-a^2}{4} me > \frac{R(1-r_0^2)}{4r_0} me = \kappa R \cdot \frac{(1-r_0^2) \sin^2 \eta}{8\pi} me \\ &= \kappa R \cdot \lambda me, \quad \text{where } \lambda = \frac{(1-r_0^2) \sin^2 \eta}{8\pi}. \end{aligned} \quad (32)$$

Let $\bar{C}$ be an arc on $|z|=1$, whose center is at -1 and $m\bar{C} = \kappa R$. Then from (30), $\bar{C}$ contains $S(C)$, so that $\bar{C}$ contains $S(e)$ and from (31), (32),

$$\frac{1}{2\pi} me > \frac{mS(e)}{m\bar{C}} > \lambda me, \quad \text{q.e.d.}$$

4. Proof of Theorem 5.

1. Suppose that there exists a set E on Θ , which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$.

Let $f(\theta, \varphi)$ be its characteristic function, which is $=1$ on E and $=0$ on its complementary set on Θ and

$$u(z, w) = u(re^{i\theta}, \rho e^{i\varphi}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta'-\theta)+r^2)(1-2\rho \cos(\varphi'-\varphi)+\rho^2)} \\ (0 \leq r < 1, \quad 0 \leq \rho < 1). \quad (33)$$

Then by Theorem 6, $u(z, w) \rightarrow f(\theta, \varphi)$ almost everywhere on Θ , when $z \rightarrow e^{i\theta}$, $w \rightarrow e^{i\varphi}$ non-tangentially to $|z|=1$, $|w|=1$ respectively.

Since $f(\theta, \varphi)$ is invariant by $\mathfrak{G}$, we have for any substitution S of G

$$u(S(z), S(w)) = u(z, w). \quad (34)$$

Let $E(\theta_0)$, $\bar{E}(\varphi_0)$ be the sub-sets of E , which lie on the line $\theta = \text{const.} = \theta_0$ and $\varphi = \text{const.} = \varphi_0$ respectively, then

$$mE = \int_0^{2\pi} mE(\theta)d\theta = \int_0^{2\pi} m\bar{E}(\varphi)d\varphi. \quad (35)$$

Now

$$u(0, w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(\varphi')(1-\rho^2)d\varphi'}{1-2\rho \cos(\varphi'-\varphi)+\rho^2}, \quad (36)$$

where

$$F(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta, \varphi)d\theta = \frac{1}{2\pi} m\bar{E}(\varphi). \quad (37)$$

2. If $mE(\theta) = 0$ on a set e of positive measure, then since such a set e is invariant by G , by Lemma 3, $me = 2\pi$, so that $mE = 0$ by (35), which contradicts the hypothesis, hence $mE(\theta) > 0$ for almost all θ . Hence if η is small, there exists a set e of measure $me > 2\pi - \varepsilon$ ($\varepsilon < \pi$), such that

$$mE(\theta) \geq 4\eta \quad \text{for any } \theta \in e. \quad (38)$$

Let E_1 be a sub-set of E consisting of points (θ, φ) , such that

$$E_1: \quad \theta \in e, \quad |\varphi - \theta| \geq \eta. \quad (39)$$

Let $E_1(\theta)$ be defined as $E(\theta)$ with respect to E_1 , then by (38), (39),

$$mE_1(\theta) \geq mE(\theta) - 2\eta \geq 4\eta - 2\eta = 2\eta, \quad (40)$$

so that

$$mE_1 = \int_e mE_1(\theta)d\theta \geq 2\eta me \geq 2\eta (2\pi - \varepsilon) \geq 2\pi\eta. \quad (41)$$

By Egoroff's theorem, there exists a closed sub-set E_0 of E_1 of positive measure consisting of points (θ, φ) , such that

$$E_0: \begin{cases} \text{(i)} & \theta \in e_0, |\varphi - \theta| \geq \eta, \text{ where } e_0 \text{ is a closed sub-set of} \\ & e \text{ of measure } me_0 > 2\pi - \varepsilon, \\ \text{(ii)} & mE_0(\theta) \geq \eta, \text{ where } E_0(\theta) \text{ is defined as } E(\theta) \text{ with} \\ & \text{respect to } E_0, \\ \text{(iii)} & u(z, w) \rightarrow 1 \text{ uniformly, when } z \rightarrow e^{i\theta}, w \rightarrow e^{i\varphi} ((\theta, \varphi) \in E_0) \text{ from} \\ & \text{the inside of angular domains:} \end{cases} \quad (42)$$

$$\Delta(\theta): |\arg(1 - ze^{-i\theta})| \leq \omega, \quad \Delta(\varphi): |\arg(1 - we^{-i\varphi})| \leq \omega,$$

where ω is so chosen that if an equivalent K_j of a disc $K_0: |z| \leq r_0$ intersects a radius $z = re^{i\theta} (0 \leq r < 1)$, then its non-euclidean center z_j is contained in $\Delta(\theta)$.

Hence if $(\theta, \varphi) \in E_0$, $|z_j - e^{i\theta}| < \delta(\varepsilon)$, $|w - e^{i\varphi}| < \delta(\varepsilon)$, $(z_j \in \Delta(\theta), w \in \Delta(\varphi))$, then

$$1 - \varepsilon \leq u(z_j, w) \leq 1, \quad (44)$$

where $\delta(\varepsilon)$ depends on ε only and is independent of (θ, φ) on E_0 .

3. Let $L(r)$ be defined as Lemma 4. Since $\lim_{r \rightarrow 1} n(r)(1-r) > 0$, by Lemma 4, there exists $\rho_\nu \rightarrow 1$, such that

$$L(\rho_\nu) \geq a > 0 \quad (\nu = 1, 2, \dots).$$

Now the part of $|z| = \rho_\nu$ contained in $\sum_{n=0}^{\infty} K_n$ consists of a set of arcs. If we project these arcs from $z = 0$ on $|z| = 1$, then we have a set of arcs on $|z| = 1$. We divide these arcs on $|z| = 1$ into two classes: $\sum_j \alpha_j(\rho_\nu) + \sum_j \beta_j(\rho_\nu)$, where $\alpha_j(\rho_\nu)$ contains at least one point $\theta_j \in e_0$ and $\beta_j(\rho_\nu)$ does not contain such points. If we denote the arc length of an arc α on $|z| = 1$ by $|\alpha|$, then

$$L(\rho_\nu) = \sum_{j=1}^{m=m(\nu)} |\alpha_j(\rho_\nu)| + \sum_{j=1}^{n=n(\nu)} |\beta_j(\rho_\nu)| \geq a.$$

Let e'_0 be the complementary set of e_0 , then, by (42), $me'_0 < \varepsilon \leq \frac{a}{2}$, if we take $\varepsilon \leq \frac{a}{2}$.

Since $\sum_j |\beta_j(\rho_\nu)| \leq me'_0 \leq \frac{a}{2}$, we have

$$L'(\rho_\nu) = \sum_{j=1}^{m=m(\nu)} |\alpha_j(\rho_\nu)| \geq \frac{a}{2}, \quad (45)$$

where $\alpha_j(\rho_\nu)$ is the projection from $z = 0$ on $|z| = 1$ of an arc on $|z| = \rho_\nu$ contained in K_j , which intersects a radius $z = re^{i\theta_j} (0 \leq r < 1)$, such that $\theta_j \in e_0$. Let $z_j = r_j e^{i\psi_j}$ be the non-euclidean center of K_j and put

$$U_j(w) = u(z_j, w), \quad (46)$$

then $U_j(w)$ is a bounded harmonic function in $|w| < 1$, so that by Fatou's theorem, $\lim U_j(w)$ exists almost everywhere on $|w| = 1$, when w tends to $|w| = 1$ non-tangentially. We write this limiting value by $u(z_j, e^{i\varphi})$. Hence there exists a sub-set $E'_0(\theta_j)$ of $E_0(\theta_j)$, such that $mE'_0(\theta_j) = mE_0(\theta_j)$ and for any $\varphi \in E'_0(\theta_j)$, the limiting value $u(z_j, e^{i\varphi})$ exists.

Let $\theta_j \in e_0$, $\varphi \in E'_0(\theta_j)$. Since z_j is contained in $\Delta(\theta_j)$, if we make $w \rightarrow e^{i\varphi}$ in (44), then

$$1 - \varepsilon \leq u(z_j, e^{i\varphi}) \leq 1, \quad (47)$$

if $|z_j - e^{i\theta_j}| < \delta(\varepsilon)$, where $\delta(\varepsilon)$ depends on ε only and is independent of θ_j on e_0 .

Since $\theta_j \in e_0$, we have from (43),

$$mE'_0(\theta_j) = mE_0(\theta_j) \geq \eta. \quad (48)$$

Since $|z_j - e^{i\theta_j}| \rightarrow 0$, we see from (42) that if $\nu \geq \nu_0$, $E'_0(\theta_j)$ is contained in an arc C_j on $|z| = 1$, such that

$$C_j: \quad \pi \geq |\arg z - \psi_j| \geq \frac{\eta}{2}. \quad (49)$$

Let S_j be the substitution of G , such that $S_j(K_j) = K_0$, $S_j(z_j) = 0$ and let $S_j(K_0) = \bar{K}_j$. Then, by Lemma 5, $\bar{K}_j$ is obtained from K_j by a rotation about $z = 0$. Let $|z| = \rho_\nu$ intersect $\bar{K}_j$ in an arc, whose projection from $z = 0$ on $|z| = 1$ be $\bar{a}_j(\rho_\nu)$. We put

$$A_\nu = \bar{a}_1(\rho_\nu) + \bar{a}_2(\rho_\nu) + \cdots + \bar{a}_m(\rho_\nu) \quad (m = m(\nu)), \quad (50)$$

$$A = \lim_{\nu \rightarrow \infty} A_\nu = (A_1 + A_2 + \cdots)(A_2 + A_3 + \cdots) \dots \dots \dots \quad (51)$$

Then since $|\bar{a}_j(\rho_\nu)| = |\alpha_j(\rho_\nu)|$, we have by (45),

$$mA_\nu = L'(\rho_\nu) = |\alpha_1(\rho_\nu)| + |\alpha_2(\rho_\nu)| + \cdots + |\alpha_m(\rho_\nu)| \geq \frac{a}{2} \quad (m = m(\nu)),$$

so that

$$mA \geq \frac{a}{2}. \quad (52)$$

By (48), (49) and Lemma 5, $S_j(E'_0(\theta_j))$ is contained in an arc $\bar{C}_j$ on $|z| = 1$ which is concentric with $\bar{a}_j(\rho_\nu)$ and $m\bar{C}_j = \kappa R_j$, where R_j is the radius of K_j , such that

$$\frac{mS_j(E'_0(\theta_j))}{m\bar{C}_j} > \lambda mE'_0(\theta_j) \geq \lambda \eta \quad (j = 1, 2, \dots, m = m(\nu)), \quad (53)$$

where κ, λ depend on η only. Since $\kappa > 2\pi$, $\bar{C}_j$ contains $\bar{a}_j(\rho_\nu)$.

We put

$$M_\nu = S_1(E'_0(\theta_1)) + S_2(E'_0(\theta_2)) + \cdots + S_m(E'_0(\theta_m)), \quad (m = m(\nu)),$$

$$M^{(\nu)} = M_\nu + M_{\nu+1} + \cdots, \quad M = \varlimsup_{\nu \rightarrow \infty} M_\nu = (M_1 + M_2 + \cdots)(M_2 + M_3 + \cdots) \dots$$

Let φ be any point of A , then $\varphi \in A_{\nu_n} (n = 1, 2, \dots)$, so that

$$\varphi \in \bar{a}_{j_n}(\rho_{\nu_n}) \in \bar{C}_{j_n} \quad (1 \leq j_n \leq m(\nu_n)). \quad (54)$$

Hence if $\nu \leq \nu_n$, then by (53),

$$m(M^{(\nu)}, \bar{C}_{j_n}) \geq m(M_{\nu_n}, \bar{C}_{j_n}) \geq m(S_{j_n}(E'_0(\theta_{j_n})), \bar{C}_{j_n}) = m(S_{j_n}(E'_0(\theta_{j_n}))) > \lambda \eta m \bar{C}_{j_n}.$$

Since $\bar{C}_{j_n} \rightarrow \varphi$ for $n \rightarrow \infty$, the lower density of $M^{(\nu)}$ at φ is $\geq \lambda \eta$. Since φ is any point of A , we have $mM^{(\nu)} \geq mA \geq \frac{a}{2}$. Hence

$$mM = \lim_{\nu \rightarrow \infty} mM^{(\nu)} \geq \frac{a}{2}. \quad (55)$$

Let $\varphi \in M$, then $\varphi \in M_{\nu_n} (n = 1, 2, \dots)$, so that

$$\varphi \in S_{j_n}(E'_0(\theta_{j_n})) \text{ or } S_{j_n}^{-1}(\varphi) = \varphi_{j_n} \in E'_0(\theta_{j_n}) \quad (1 \leq j_n \leq m(\nu_n)).$$

Let K_{j_n} be the disc intersecting a radius: $z = re^{i\theta_{j_n}} (0 \leq r < 1)$, whose non-euclidean center is z_{j_n} , such that $S_{j_n}(K_{j_n}) = K_0$, $S_{j_n}(z_{j_n}) = 0$, then by (47) and (34) we have

$$1 - \varepsilon \leq u(z_{j_n}, e^{i\varphi_{j_n}}) = u(0, e^{i\varphi}) \leq 1,$$

if $|z_{j_n} - e^{i\varphi_{j_n}}| < \delta(\varepsilon)$. Since $|z_{j_n} - e^{i\varphi_{j_n}}| \rightarrow 0$ for $n \rightarrow \infty$, we have $u(0, e^{i\varphi}) = 1$. Hence $u(0, e^{i\varphi}) = 1$ at every point φ on M . Since the set of such $e^{i\varphi}$ is invariant by G and $mM > 0$, we have, by Lemma 3, $mM = 2\pi$, so that $u(0, e^{i\varphi}) = 1$ almost everywhere on $|w| = 1$. Since by (36), (37), $u(0, e^{i\varphi}) = \frac{1}{2\pi} m\bar{E}(\varphi)$ almost everywhere on $|w| = 1$, $m\bar{E}(\varphi) = 2\pi$ almost everywhere on $|w| = 1$. Hence by (35), $mE = 4\pi^2$, which contradicts the hypothesis, which proves the Theorem.

From Theorem 5, we can easily prove:

Theorem 9. Under the same condition as Theorem 5, there does not exist a non-constant bounded harmonic function $u(z, w)$ of z, w in $|z| < 1, |w| < 1$, such that $u(S(z), S(w)) = u(z, w)$ for any substitution S of G .

Proof. Suppose that there exists such a non-constant bounded harmonic function $u(z, w)$, then by Theorem 7, $\lim u(z, w) = f(\theta, \varphi)$ exists almost everywhere on Θ , when $z \rightarrow e^{i\theta}, w \rightarrow e^{i\varphi}$ non-tangentially to $|z| = 1, |w| = 1$ respectively and $u(z, w)$ can be expressed by

$$u(z, w) = u(re^{i\theta}, \rho e^{i\varphi}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta'd\varphi'}{(1-2r \cos(\theta'-\theta)+r^2)(1-2\rho \cos(\varphi'-\varphi)+\rho^2)} \\ (0 \leq r < 1, \quad 0 \leq \rho < 1),$$

where $f(\theta, \varphi)$ is invariant by $\mathfrak{G}$. Since $u(z, w) \not\equiv \text{const.}$, $f(\theta, \varphi) \not\equiv \text{const.}$ almost everywhere on Θ , so that for suitable α, β , the set $E_{\alpha\beta} = \Theta(\alpha \leq f \leq \beta)$ is such that $0 < mE_{\alpha\beta} < 4\pi^2$. Since $E_{\alpha\beta}$ is invariant by $\mathfrak{G}$, this contradicts Theorem 5, which proves the Theorem.

5. Some remarks.

1. We will prove:

Theorem 10. *Let G be a Fuchsian group of linear transformations, which make $|z| = 1$ invariant and z_n be equivalents of $z_0 = 0$ by G . If $\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$, then there exists a set E on a torus Θ , which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$.*

Proof. Since $\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$, by Theorem 2, there exists a set e on $|z| = 1$, which is invariant by G and $0 < me < 2\pi$. Then the product set $E = e \times e$ is invariant by $\mathfrak{G}$ and $0 < mE = (me)^2 < 4\pi^2$.

We will give a direct proof as follows.

Let $Q: \left| \theta - \frac{\pi}{2} \right| \leq \varepsilon, \left| \varphi - \pi \right| \leq \varepsilon$ ($\varepsilon < \frac{\pi}{16}$) be a square contained in $\Theta(0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq 2\pi)$ and $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ be arcs on $|z| = 1$, such that

$$\alpha: \left| \arg z - \frac{\pi}{2} \right| \leq \varepsilon, \quad \bar{\alpha}: \left| \arg z - \frac{\pi}{2} \right| \leq \frac{\pi}{8},$$

$$\beta: \left| \arg z - \pi \right| \leq \varepsilon, \quad \bar{\beta}: \left| \arg z - \pi \right| \leq \frac{\pi}{8}$$

and $\bar{\omega}$ be the complementary set of $\bar{\alpha} + \bar{\beta}$ on $|z| = 1$.

Let $K_0: |z| \leq r_0$ be a disc contained in the fundamental domain of G and $K_n, z_n = r_n e^{i\theta_n}$ be equivalents of $K_0, z_0 = 0$ respectively, such that $S_n(K_n) = K_0, S_n(z_n) = 0, z_n \in K_n$ and ρ_n be the radius of K_n .

(i) If $\theta_n \in \bar{\omega}$, then, since $\varepsilon < \frac{\pi}{16}$, for any z on $\alpha, \beta, \left| \arg z - \theta_n \right| \geq \frac{\pi}{16}$.

Hence by Lemma 5,

$$mS_n(\alpha) < \frac{\kappa \rho_n}{2\pi} m\alpha = \frac{\varepsilon \kappa}{\pi} \rho_n, \quad mS_n(\beta) < \frac{\varepsilon \kappa}{\pi} \rho_n,$$

so that

$$mS_n(Q) < \frac{\varepsilon^2 \kappa^2}{\pi^2} \rho_n^2 < 2\varepsilon \kappa^2 \rho_n. \quad (56)$$

(ii) If $\theta_n \in \bar{\alpha}$, then for any z on β , $|\arg z - \theta_n| \geq \frac{\pi}{16}$, so that $mS_n(\beta) < \frac{\varepsilon\kappa}{\pi} \rho_n$. Since $mS_n(\alpha) \leq 2\pi$, $\kappa > 1$, we have

$$mS_n(Q) < 2\varepsilon\kappa^2 \rho_n. \quad (57)$$

We have the same inequality, if $\theta_n \in \bar{\beta}$.

Since by (29),

$$\sum_{n=0}^{\infty} \rho_n < \sum_{n=0}^{\infty} \frac{r_0(1-|z_n|^2)}{1-r_0^2} < \frac{2r_0}{1-r_0^2} \sum_{n=0}^{\infty} (1-|z_n|) < \infty,$$

we take ε so small, that

$$\sum_{n=0}^{\infty} mS_n(Q) < 2\varepsilon\kappa^2 \sum_{n=0}^{\infty} \rho_n < 4\pi^2,$$

then $E = \sum_{n=0}^{\infty} S_n(Q)$ is invariant by $\mathfrak{G}$ and $0 < mE \leq \sum_{n=0}^{\infty} mS_n(Q) < 4\pi^2$.

Remark. The condition $\sum_{n=0}^{\infty} (1-|z_n|) < \infty$ is equivalent to $\int_0^1 n(r)dr < \infty$.

Hence we have three cases:

$$\begin{aligned} \text{(I)} \quad & \int_0^1 n(r)dr = \infty, \quad \text{(a)} \quad \lim_{r \rightarrow 1} \overline{n(r)(1-r)} > 0, \quad \text{(b)} \quad \lim_{r \rightarrow 1} n(r)(1-r) = 0, \\ \text{(II)} \quad & \int_0^1 n(r)dr < \infty. \end{aligned}$$

In case (I)(a), by Theorem 5, there does not exist a set E on Θ , which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$. In case (II), by Theorem 10, there exists such invariant sets. In case (I)(b), we have no informations about the existence of such invariant sets. It seems that there exist groups of class (I)(b), for which such invariant sets exist and groups, for which such invariant sets do not exist, but I have no examples for it.

2. Next we consider the n -dimensional case. Let $\eta_1 = e^{i\theta_1}, \dots, \eta_n = e^{i\theta_n}$ be n points on $|z_1| = 1, \dots, |z_n| = 1$ respectively, then the pair $(\eta_1, \dots, \eta_n)$ can be considered as a point on an n -dimensional torus $\Theta_n (0 \leq \theta_j \leq 2\pi, j = 1, 2, \dots, n)$ and the measure of a measurable set E on Θ_n is defined by

$$mE = \int_E \dots \int d\theta_1 \dots d\theta_n, \quad \text{so that} \quad m\Theta_n = (2\pi)^n.$$

Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and S be any substitution of G and

$$T: \eta'_1 = S(\eta_1), \dots, \eta'_n = S(\eta_n),$$

then the totality of T constitutes a group $\mathfrak{G}_n$, which is isomorphic to G . We will prove:

Theorem 11. *If $n \geq 3$, then there exists always a set E on Θ_n , which is invariant by $\mathfrak{G}_n$ and $0 < mE < (2\pi)^n$.*

Proof. We assume that $n = 3$, the other case can be proved similarly. Let $Q: \left| \theta - \frac{\pi}{2} \right| \leq \varepsilon, |\varphi - \pi| \leq \varepsilon, \left| \psi - \frac{3\pi}{2} \right| \leq \varepsilon$ ($-\varepsilon < \frac{\pi}{16}$) be a cube contained in $\Theta_3 (0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq 2\pi, 0 \leq \psi \leq 2\pi)$ and $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}$ be arcs on $|z| = 1$, such that

$$\alpha: \left| \arg z - \frac{\pi}{2} \right| \leq \varepsilon, \quad \bar{\alpha}: \left| \arg z - \frac{\pi}{2} \right| \leq \frac{\pi}{8},$$

$$\beta: |\arg z - \pi| \leq \varepsilon, \quad \bar{\beta}: |\arg z - \pi| \leq \frac{\pi}{8},$$

$$\gamma: \left| \arg z - \frac{3\pi}{2} \right| \leq \varepsilon, \quad \bar{\gamma}: \left| \arg z - \frac{3\pi}{2} \right| \leq \frac{\pi}{8}$$

and $\bar{\omega}$ be the complementary set of $\bar{\alpha} + \bar{\beta} + \bar{\gamma}$ on $|z| = 1$.

Let $K_0: |z| \leq r_0$ be a disc contained in the fundamental domain of G and $K_n, z_n = r_n e^{i\theta_n}$ be equivalents of $K_0, z_0 = 0$ respectively, such that $S_n(K_n) = K_0, S_n(z_n) = 0, z_n \in K_n$ and ρ_n be the radius of K_n . Since K_n are non-overlapping, $\sum_{n=0}^{\infty} \pi \rho_n^2 < \pi$ (the total area of $|z| < 1$), or $\sum_{n=0}^{\infty} \rho_n^2 < 1$.

(i) If $\theta_n \in \bar{\omega}$, then, since $\varepsilon < \frac{\pi}{16}$, for any z on α, β, γ , $|\arg z - \theta_n| \geq \frac{\pi}{16}$. Hence by Lemma 5,

$$mS_n(\alpha) < \frac{\kappa \rho_n}{2\pi} m\alpha = \frac{\varepsilon \kappa}{\pi} \rho_n, \quad mS_n(\beta) < \frac{\varepsilon \kappa}{\pi} \rho_n, \quad mS_n(\gamma) < \frac{\varepsilon \kappa}{\pi} \rho_n,$$

so that

$$mS_n(Q) < \left(\frac{\varepsilon \kappa}{\pi} \right)^3 \rho_n^3 < \varepsilon^2 \kappa^3 \rho_n^2. \quad (58)$$

(ii) If $\theta_n \in \bar{\alpha}$, then for any z on β, γ , $|\arg z - \theta_n| \geq \frac{\pi}{16}$, so that $mS_n(\beta) < \frac{\varepsilon \kappa}{\pi} \rho_n, mS_n(\gamma) < \frac{\varepsilon \kappa}{\pi} \rho_n$. Since $mS_n(\alpha) \leq 2\pi, \kappa > 1$, we have

$$mS_n(Q) < \frac{2\varepsilon^2 \kappa^2}{\pi} \rho_n^2 < \varepsilon^2 \kappa^3 \rho_n^2. \quad (59)$$

We have the same inequality, if $\theta_n \in \bar{\beta}$ or $\theta_n \in \bar{\gamma}$.

If we take ε so small, that

$$\sum_{n=0}^{\infty} mS_n(Q) < \varepsilon^2 \kappa^3 \sum_{n=0}^{\infty} \rho_n^2 < \varepsilon^2 \kappa^3 < 8\pi^3,$$

then $E = \sum_{n=0}^{\infty} S_n(Q)$ is invariant by $\mathfrak{G}$ and $0 < mE \leq \sum_{n=0}^{\infty} mS_n(Q) < 8\pi^3$.

6. Geodesic flow on the Riemann manifold of constant negative curvature.

1. Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and D_0 be its fundamental domain containing $z = 0$. We consider D_0 as a Riemann manifold F of constant negative curvature with $ds = \frac{|dz|}{1-|z|^2}$ and equivalent points by G are considered as the same point of F . Let $z = x+iy$ be any point of D_0 . We associate a direction φ at z , which makes an angle φ with the real axis. Then the line elements (z, φ) ($z \in D_0$, $0 \leq \varphi \leq 2\pi$) constitute a phase space $\mathcal{Q}$, which is the product space of D_0 and a unit circle U : $\mathcal{Q} = D_0 \times U$ and the volume element $d\mu$ in $\mathcal{Q}$ is defined by

$$d\mu = d\sigma d\varphi = \frac{dx dy d\varphi}{(1-|z|^2)^2}, \text{ so that } \mu(\mathcal{Q}) = 2\pi\sigma(D_0). \quad (60)$$

Now the line element (z, φ) determines a unique geodesic $g = g(z, \varphi)$ of F , which is an arc on an orthogonal circle to $|z| = 1$, which touches the direction φ at z . Let $\eta_1 = e^{i\theta_1}$, $\eta_2 = e^{i\theta_2}$ be the two end points of g on $|z| = 1$, where η_1 is such that if we proceed on g in the direction φ , then we meet $|z| = 1$ at η_1 . We call η_1 the end point of g . Let z_0 be the middle point of the arc $\widehat{\eta_1 \eta_2}$ on g , z be any point on g and s be the hyperbolic length of the arc $\widehat{z_0 z}$ on g , where s is positive, if z lies on $\widehat{z_0 \eta_1}$ and negative, if z lies on $\widehat{z_0 \eta_2}$. Then we have a one- to- one correspondence between (z, φ) and (η_1, η_2, s) . Hopf⁽⁴⁾ proved that

$$d\mu = C \cdot \frac{|d\eta_1| |d\eta_2| ds}{|\eta_1 - \eta_2|^2} \quad (C = \text{const.}). \quad (61)$$

We consider a geodesic flow T_t ($-\infty < t < \infty$) in $\mathcal{Q}$:

$$T_t : P = (\eta_1, \eta_2, s) \rightarrow P_t = (\eta_1, \eta_2, s+t). \quad (62)$$

By (61), T_t is a mass-preserving transformation of $\mathcal{Q}$ into itself.

The flow T_t is said of metric transitive, if a set E in $\mathcal{Q}$ is invariant by T_t , then $\mu E = 0$ or $\mu(\mathcal{Q} - E) = 0$.

From Theorem 3, Hopf⁽⁴⁾ proved:

Theorem 12 (Hopf). *If $\sigma(D_0) < \infty$, then the geodesic flow T_t is metric transitive.*

Similarly from Theorem 5, we have:

Theorem 13. *If $\lim_{r \rightarrow 1} n(r)(1-r) > 0$, then the geodesic flow T_t is metric transitive.*

As an application of Theorem 12, we will prove the following theorem, which is a precision of Theorem 1 (i).

Theorem 14. *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and D_0 be its fundamental domain containing $z = 0$ and $\sigma(D_0) < \infty$. For any z in $|z| < 1$, we denote its equivalent in D_0 by (z) . Let Δ be a set in D_0 , which is measurable in Jordan's sense and $g : z = te^{i\theta} (0 \leq t < 1)$ be a radius of $|z| = 1$ and l be a segment $(0 \leq t \leq r)$ on g of length r , whose hyperbolic length be L . Let $L(\Delta)$ be the hyperbolic measure of the set of t -values on $(0, r)$, such that $(te^{i\theta}) \in \Delta$. Then there exists a certain set e_0 of measure zero on a unit circle $U : |z| = 1$, which does not depend on Δ , such that if $e^{i\theta} \in U - e_0$, then for any Δ ,*

$$\lim_{L \rightarrow \infty} \frac{L(\Delta)}{L} = \frac{\sigma(\Delta)}{\sigma(D_0)}. \quad (63)$$

Proof. Let M be any set in D_0 and $S_n(M) (n = 0, 1, 2, \dots)$ be its equivalents, we put

$$[M] = \sum_{n=0}^{\infty} S_n(M).$$

Then $L(\Delta)$ is equal to the hyperbolic measure of the part of l contained in $[\Delta]$.

Since by Theorem 12, T_t is metric transitive and $\mu(\Omega) = 2\pi\sigma(D_0) < \infty$, by Birkhoff's ergodic theorem,

$$\lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L f(P_t) dt = \frac{\int_{\Omega} f(P) d\mu}{\mu(\Omega)} \quad (64)$$

for almost all points $P = (z, \varphi)$ in Ω , where $f < L^2$ in Ω .

Let M be a set in D_0 . We associate at every point z of M directions $\varphi (0 \leq \varphi \leq 2\pi)$. Then such line elements $(z, \varphi) (z \in M, 0 \leq \varphi \leq 2\pi)$ constitute a set E in Ω , which is a product set of M and a unit circle $U : E = M \times U$, so that $\mu(E) = 2\pi\sigma(M)$.

Consider a geodesic $g = g(z, \varphi)$ and an arc $C = \widehat{z, z'}$ on g of hyperbolic length L_g . Let $L_g([M])$ be the hyperbolic measure of the part of C contained in $[M]$. If we take $f(P)$ in (64) as the characteristic function of E , then (64) becomes

$$\lim_{L_g \rightarrow \infty} \frac{L_g([M])}{L_g} = \frac{\mu(E)}{\mu(\Omega)} = \frac{\sigma(M)}{\sigma(D_0)} \quad (65)$$

for almost all points $P = (z, \varphi)$ in Ω .

Let Δ be a polygonal domain in $|z| < 1$, which has common points with D_0 and whose sides consist of segments lying on lines $x = \text{const.} = a$ or

$y = \text{const.} = \beta$, where α, β are rationals. If Δ has points outside D_0 , we replace such points by their equivalents in D_0 . We call the so modified domain in D_0 a rational polygonal domain. Since the totality of rational polygonal domains is enumerable, let Δ_i ($i = 1, 2, \dots$) be all rational polygonal domains, then by (65),

$$\lim_{L_g \rightarrow \infty} \frac{L_g([\Delta_i])}{L_g} = \frac{\sigma(\Delta_i)}{\sigma(D_0)}, \quad (66)$$

if $P \in \Omega - N_i$, where $\mu(N_i) = 0$.

If D_0 has boundary points on $|z| = 1$, let $D_0^{(r)}$ be the part of D_0 contained in $|z| \leq r < 1$. Let ρ_i ($i = 1, 2, \dots$) be all rational numbers in $(0, 1)$, then by (65),

$$\lim_{L_g \rightarrow \infty} \frac{L_g([D_0 - D_0^{(\rho_i)}])}{L_g} = \frac{\sigma(D_0 - D_0^{(\rho_i)})}{\sigma(D_0)}, \quad (67)$$

if $P \in \Omega - N'_i$, where $\mu(N'_i) = 0$.

We put $N = \sum_{i=1}^{\infty} N_i + \sum_{i=1}^{\infty} N'_i$, then $\mu(N) = 0$ and if $P = (z, \varphi) \in \Omega - N$, then (66), (67) ($i = 1, 2, \dots$) hold for the geodesic $g = g(z, \varphi)$ through P .

By Fubini's theorem, there exists a set M_0 in D_0 , such that $\sigma(M_0) = 0$ and if $z \in D_0 - M_0$ and φ does not belong to a certain set $N(z)$ of measure zero, then (66), (67) ($i = 1, 2, \dots$) hold for the geodesic $g = g(z, \varphi)$. Let $z_0 \in D_0 - M_0$ and e_0 be the set of points on a unit circle U , which are the end points of the exceptional geodesics $g = g(z_0, \varphi_0)$ ($\varphi_0 \in N(z_0)$).

Then $me_0 = 0$. Let $e^{i\theta} \in U - e_0$ and $\eta = e^{i\theta}$ be the end point of a geodesic $g = g(z_0, \varphi)$, then (66), (67) ($i = 1, 2, \dots$) hold for such a geodesic. Consider a radius $g_0: z = re^{i\theta}$ ($0 \leq r < 1$) of $|z| = 1$, which is a geodesic $g_0 = g(0, \theta)$ touching g at η . We will prove that (63) holds for such a radius g_0 .

Let z', z and ζ', ζ be points on g_0 and g respectively, such that $|z'| = |\zeta'|$, $|z| = |\zeta|$ ($|z'| < |z|$) and $L_{g_0}(z', z)$, $L_g(\zeta', \zeta)$ be the hyperbolic lengths of the arc $\widehat{z', z}$ on g_0 and the arc $\widehat{\zeta', \zeta}$ on g respectively, then

$$L_{g_0}(z', z) = \int_{z'}^z \frac{dr}{1-r^2}, \quad L_g(\zeta', \zeta) = \int_{\zeta'}^{\zeta} \frac{|dz|}{1-r^2} \quad (|z| = r).$$

Since g_0 touches g at η , we have $(1-\varepsilon) dr \leq |dz| \leq (1+\varepsilon) dr$ for $r_0 \leq r < 1$, so that

$$(1-\varepsilon)L_{g_0}(z', z) \leq L_g(\zeta', \zeta) \leq (1+\varepsilon)L_{g_0}(z', z) \quad (r_0 \leq r < 1) \quad (68)$$

Let z, ζ be points on g_0 and g respectively, such that $|z| = |\zeta| = r$ and $\sigma(z, \zeta)$ be the hyperbolic distance between z and ζ , then

$$\sigma(z, \zeta) \leq \frac{\widehat{z, \zeta}}{1-r^2},$$

where $\widehat{z, \zeta}$ is the arc length of the arc $\widehat{z, \zeta}$ on $|z| = r$.

Since g_0 touches g at η , we have

$$\sigma(z, \zeta) \rightarrow 0 \quad \text{for } r \rightarrow 1. \quad (69)$$

(i) First we suppose that A is contained in $|z| \leq r < 1$.

Since A is measurable in Jordan's sense, we can find two polygonal domains A_1, A_2 in $|z| > 1$, such that $A_1 \subset A \subset A_2$, $\sigma(A_2) - \sigma(A_1) < \varepsilon$, where A_1 consists of only inner points of A and the boundary of A_2 consists of only outer points of A and the sides of A_1, A_2 consists of segments lying on lines $x = \text{const.} = \alpha$ or $y = \text{const.} = \beta$, where α, β are rationals.

If A_2 contains points outside D_0 , we replace such points by their equivalents in D_0 . Let the so modified domain in D_0 be A_2 . Then we have two rational polygonal domains A_1, A_2 in D_0 , such that $A_1 \subset A \subset A_2$, $\sigma(A_2) - \sigma(A_1) < \varepsilon$.

Now by (66),

$$\lim_{L_g \rightarrow \infty} \frac{L_g([A_i])}{L_g} = \frac{\sigma(A_i)}{\sigma(D_0)}, \quad (i = 1, 2). \quad (70)$$

By (69), there exists $\rho < 1$, such that if a point z ($|z| = r \geq \rho$) on g_0 lies in $[A]$, then the corresponding ζ ($|\zeta| = |z|$) on g lies in $[A_2]$ and if ζ lies in $[A_1]$, then z lies in $[A]$, so that by (68),

$$-\text{const.} + \frac{1}{1+\varepsilon} L_g([A_1]) \leq L_{g_0}([A]) \leq \text{const.} + \frac{1}{1-\varepsilon} L_g([A_2]), \quad (71)$$

$$-\text{const.} + \frac{1}{1+\varepsilon} L_g \leq L_{g_0} \leq \text{const.} + \frac{1}{1-\varepsilon} L_g, \quad (72)$$

where L_{g_0}, L_g are hyperbolic lengths of the arc $\widehat{0, z}$ on g_0 and the arc $\widehat{z_0, \zeta}$ on g respectively, where $|z| = |\zeta|$.

Hence by (70), (71), (72),

$$\frac{1-\varepsilon}{1+\varepsilon} \cdot \frac{\sigma(A_1)}{\sigma(D_0)} \leq \lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} \leq \lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} \leq \frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{\sigma(A_2)}{\sigma(D_0)}.$$

Making $\varepsilon \rightarrow 0$, $\sigma(A_1) \rightarrow \sigma(A)$, $\sigma(A_2) \rightarrow \sigma(A)$, we have

$$\lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} = \frac{\sigma(A)}{\sigma(D_0)}. \quad (73)$$

(ii) Next suppose that A contains points tending to $|z| = 1$.

Let $A^{(r)}$ be the part of A contained in $|z| \leq r < 1$, then $A^{(r)}$ is measurable in Jordan's sense, hence by (73),

$$\lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A^{(r)}])}{L_{g_0}} = \frac{\sigma(A^{(r)})}{\sigma(D_0)}. \quad (74)$$

Since $L_{g_0}([A]) \geq L_{g_0}([A^{(r)}])$, we have for $r \rightarrow 1$,

$$\lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} \geq \lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A^{(r)}])}{L_{g_0}} = \frac{\sigma(A^{(r)})}{\sigma(D_0)} \rightarrow \frac{\sigma(A)}{\sigma(D_0)}. \quad (75)$$

Let $D_0^{(\rho)}$ be the part of D_0 contained in $|z| \leq \rho < 1$. By (69), there exists a rational ρ ($0 < \rho < 1$), such that if a point z ($|z| \geq r_0$) on g_0 lies in $[D_0 - D_0^{(\rho)}]$, then the corresponding ζ ($|\zeta| = |z|$) on g lies in $[D_0 - D_0^{(\rho)}]$, where $\rho < r$ and $\rho \rightarrow 1$ with $r \rightarrow 1$.

By (67),

$$\lim_{L_g \rightarrow \infty} \frac{L_g([D_0 - D_0^{(\rho)}])}{L_g} = \frac{\sigma(D_0 - D_0^{(\rho)})}{\sigma(D_0)} < \delta, \quad \text{if } \rho_0 \leq \rho < 1. \quad (76)$$

Since $[A - A^{(r)}] \subset [D_0 - D_0^{(\rho)}]$, we have from (71), (72),

$$\begin{aligned} \overline{\lim}_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A - A^{(r)}])}{L_{g_0}} &\leq \overline{\lim}_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([D_0 - D_0^{(\rho)}])}{L_{g_0}} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \lim_{L_g \rightarrow \infty} \frac{L_g([D_0 - D_0^{(\rho)}])}{L_g} \\ &= \frac{1 + \varepsilon}{1 - \varepsilon} \frac{\sigma(D_0 - D_0^{(\rho)})}{\sigma(D_0)} < \frac{1 + \varepsilon}{1 - \varepsilon} \delta. \end{aligned}$$

Hence by (74),

$$\overline{\lim}_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} \leq \lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A^{(r)}])}{L_{g_0}} + \overline{\lim}_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A - A^{(r)}])}{L_{g_0}} < \frac{\sigma(A^{(r)})}{\sigma(D_0)} + \frac{1 + \varepsilon}{1 - \varepsilon} \delta.$$

Making $r \rightarrow 1$, $\delta \rightarrow 0$, $\varepsilon \rightarrow 0$, we have

$$\overline{\lim}_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} \leq \frac{\sigma(A)}{\sigma(D_0)}. \quad (77)$$

From (75), (77),

$$\lim_{L_{g_0} \rightarrow \infty} \frac{L_{g_0}([A])}{L_{g_0}} = \frac{\sigma(A)}{\sigma(D_0)}.$$

Since $L(A) = L_{g_0}([A])$, we have (63), q.e.d.

Remark. The same result holds, if G contains anti-analytic transformations: $z' = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $\bar{z}$ is the conjugate complex of z . As a special case, consider a domain D_0 in $|z| < 1$, bounded by n circles: $C_1, \dots, C_n$, which are orthogonal to $|z| = 1$ and touch each other externally as for a modular figure. Let G be the group generated by inversions on $C_i (i = 1, 2, \dots, n)$. Then D_0 is its fundamental domain and $\sigma(D_0) < \infty$.

The set in D_0 , which is equivalent to a radius $g: z = re^{i\theta} (0 \leq r < 1)$ of $|z| = 1$ is obtained as follows. We start from $z = 0$ and proceed on g till we meet the boundary of D_0 , say C_1 , at z_1 . Then reflect g on C_1 at z_1 and proceed on the reflected line till we meet the boundary of D_0 and so on.

Let $L(A)$ be the hyperbolic measure of the part of such a path contained in A and L be the total hyperbolic length of the path, then (63) holds for almost all starting directions.

From Theorem 14, we can prove easily the following theorem.

Theorem 15. Under the same condition as Theorem 14, let $F(z)$ be a function defined in D_0 , which is integrable in Riemann's sense, such that $\iint_{D_0} F(z) d\sigma(z)$ exists in Riemann's sense, where $d\sigma(z) = \frac{dx dy}{(1-|z|^2)^2}$ ($z = x + iy$). Then there exists a certain set e_0 of measure zero on a unit circle $U: |z| = 1$, which does not depend on $F(z)$, such that if $e^{i\theta} \in U - e_0$, then

$$\lim_{r \rightarrow 1} \frac{1}{\log \frac{1}{1-r}} \int_0^r F((te^{i\theta})) \frac{dt}{1-t} = \frac{\iint_{D_0} F(z) d\sigma(z)}{\sigma(D_0)}.$$

2. Let $\sigma(D_0) < \infty$ and Ω be defined as before. We consider a product space $\Omega^2 = \Omega \times \Omega$, where $\Pi = (P, Q)$ ($P \in \Omega, Q \in \Omega$) is considered as a point of Ω^2 and the volume element $d\bar{\mu}$ in Ω^2 is defined by

$$d\bar{\mu}(\Pi) = d\mu(P) d\mu(Q), \text{ so that } \bar{\mu}(\Omega^2) = (\mu(\Omega))^2 < \infty.$$

The geodesic flow $T_t: P \rightarrow P_t$ in Ω generates a product flow $\Pi = (P, Q) \rightarrow \Pi_t = (P_t, Q_t)$ in Ω^2 . We will prove that the flow $\Pi \rightarrow \Pi_t$ is metric transitive.

Let

$$P = (\eta_1, \eta_2, s) \quad (\eta_1 = e^{i\theta}, \eta_2 = e^{i\varphi}), \quad Q = (\zeta_1, \zeta_2, \sigma) \quad (\zeta_1 = e^{i\psi}, \zeta_2 = e^{i\chi}).$$

Then (η_1, η_2) can be considered as a point on a torus Θ_1 ($0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$) and (ζ_1, ζ_2) as a point on a torus Θ_2 ($0 \leq \psi \leq 2\pi$, $0 \leq \chi \leq 2\pi$) and $(\eta_1, \eta_2, \zeta_1, \zeta_2)$ as a point on a four dimensional torus Θ ($0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$, $0 \leq \psi \leq 2\pi$, $0 \leq \chi \leq 2\pi$). Let E_1 , E_2 , E be measurable sets on Θ_1 , Θ_2 , Θ respectively, then their measures are defined by

$$m_1 E_1 = \iint_{E_1} d\theta d\varphi, \quad m_2 E_2 = \iint_{E_2} d\psi d\chi, \quad m E = \iiint_E d\theta d\varphi d\psi d\chi,$$

so that $m_1(\Theta_1) = m_2(\Theta_2) = (2\pi)^2$, $m(\Theta) = (2\pi)^4$.

As Hopf's proof of metric transitivity of $T_t: P \rightarrow P_t$, to prove the metric transitivity of $\Pi \rightarrow \Pi_t$, it suffices to prove that there does not exist a set E on Θ , such that $0 < mE < (2\pi)^4$ and is invariant by the substitutions:

$$\eta'_1 = S(\eta_1), \quad \eta'_2 = S(\eta_2), \quad \zeta'_1 = S'(\zeta_1), \quad \zeta'_2 = S'(\zeta_2),$$

where S and S' are any substitutions of G .

We will prove that there does not exist such an invariant set.

Suppose that there exists an invariant set E on Θ , such that $mE > 0$ and we will prove that $mE = (2\pi)^4$.

Let $f(\theta, \varphi, \psi, \chi)$ be the characteristic function of E , then by Fubini's theorem,

$$mE = \iiint_{\Theta} f(\theta, \varphi, \psi, \chi) d\theta d\varphi d\psi d\chi = \iint_{\Theta_2} d\psi d\chi \iint_{\Theta_1} f(\theta, \varphi, \psi, \chi) d\theta d\varphi.$$

Let $E_1(\psi, \chi)$ be the set of points $(\theta, \varphi, \psi, \chi)$ on Θ , which belong to E , where ψ, χ have fixed values. Then

$$mE = \iint_{\Theta_2} m_1 E_1(\psi, \chi) d\psi d\chi.$$

Now $E_1(\psi, \chi)$ is invariant by the substitutions:

$$\eta'_1 = S(\eta_1), \quad \eta'_2 = S(\eta_2),$$

where S is any substitution of G . If $m_1 E_1(\psi, \chi) > 0$, then by Theorem 3, $m_1 E_1(\psi, \chi) = (2\pi)^2$. Hence if E_2 is the set of points (ψ, χ) on Θ_2 , such that $m_1 E_1(\psi, \chi) > 0$, then $mE = (2\pi)^2 m_2 E_2 > 0$.

E_2 is invariant by the substitutions:

$$\zeta'_1 = S(\zeta_1), \quad \zeta'_2 = S(\zeta_2),$$

where S is any substitution of G . Since $m_2 E_2 > 0$, by Theorem 3, we have $m_2 E_2 = (2\pi)^2$, so that $mE = (2\pi)^4$, q.e.d.

Hence the flow $\Pi \rightarrow \Pi_t$ is metric transitive.

Similarly we can prove that the flow $\Pi = (P^{(1)}, \dots, P^{(n)}) \rightarrow \Pi_t = (P_t^{(1)}, \dots, P_t^{(n)})$ in a product space $\Omega^n = \overbrace{\Omega \times \dots \times \Omega}^n$ is metric transitive, where $\bar{\mu}(\Omega^n) = (\mu(\Omega))^n < \infty$. Hence by Birkhoff's ergodic theorem,

$$\lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L f(\Pi_t) dt = \frac{\int_{\Omega^n} f(\Pi) d\bar{\mu}}{\mu(\Omega^n)}$$

for almost all points Π in Ω^n , where $f < L^2$ in Ω^n .

From this we proceed similarly as the proof of Theorem 14 and can prove the following extension of Theorem 14.

Theorem 16. *Let G be a Fuchsian group of linear transformations, which make $|z| < 1$ invariant and D_0 be its fundamental domain containing $z = 0$ and $\sigma(D_0) < \infty$. Let $A_1, \dots, A_n$ be n sets in D_0 , which are measurable in Jordan's sense. Let $g_k: z = te^{i\theta_k}$ ($0 \leq t < 1$) ($k = 1, 2, \dots, n$) be n radii of $|z| = 1$ and l_k be segments ($0 \leq t \leq r$) on g_k of the same length r , whose hyperbolic length be L . Let $L(A_1 \times \dots \times A_n)$ be the hyperbolic measure of the set of t -values on $(0, r)$, such that $(te^{i\theta_1}) \in A_1, \dots, (te^{i\theta_n}) \in A_n$. Then there exists a certain set e_0 of measure zero on an n -dimensional torus $\Theta(0 \leq \theta_k \leq 2\pi, k = 1, 2, \dots, n)$, which does not depend on $A_1, \dots, A_n$, such that if $(\theta_1, \dots, \theta_n)$ does not belong to e_0 , then for any $A_1, \dots, A_n$,*

$$\lim_{L \rightarrow \infty} \frac{L(A_1 \times \dots \times A_n)}{L} = \frac{\sigma(A_1) \dots \sigma(A_n)}{[\sigma(D_0)]^n}.$$

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