12. On locally compact fields.

By Yosikazu OTOBE.

Mathematical Institute, Tokyo Imperial University.

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§ 1. Introduction. It is known that locally compact non-discrete fields are completely classified into the following five types:

a) $\mathbb{R}$ the field of real numbers;

b) $\mathbb{C}$ (i) the field of complex numbers ($i = \sqrt{-1}$);

c) $\mathbb{H}$ (j, k, l) the (non-commutative) field of (Hamilton's) quaternions;

d) $\mathbb{Q}_p$ the field of (Hensel's) $p$-adic numbers;

e) $K(p^n), \{ \xi \}$ the (in general non-commutative) field of formal power series $x = \sum_{i=0}^{\infty} a_i \xi^i (a_i \in K(p^n), h = \exp x = 0, \pm 1, \pm 2, \ldots)$ in one variable $\xi$ over a Galois field $K(p^n)$, where the transformation by $\xi$ of elements of $K(p^n)$ causes a fixed automorphism $\sigma$ of $K(p^n)$, and the topology of the field being defined by $\{x | \exp x \geq t\}$ for $t = 1, 2, \ldots$ as a complete system of neighborhoods of zero.

Clearly all of these fields satisfy the second countability axiom, and the fields of the first three types are connected and those of the second two totally disconnected.

This fact has been proved under the assumption that the field in question satisfies the first countability axiom and that the division in the field is continuous (except at 0). In the present paper the author proposes to prove the same classification without using the above-mentioned assumption, i.e. to prove that any locally compact non-discrete field in which the addition, subtraction and multiplication are continuous—that is what we call "locally compact field (= l.c. field)" in this paper—is one of the fields of the types a)—e), especially that necessarily every l.c. field satisfies the first (and second) countability axiom (Th. 1) and the division in the field becomes continuous (Th. 3). Theorem 4 that asserts a certain possibility of compactification of the field may also be interesting.

(1) L. Pontrjagin, Über stetige algebraische Körper, Ann. of Math. 33 (1932), proved this classification for the connected case (a), b) and c)). Also see his book "Topological groups", Princeton (1939), § 37, "Topologized algebraic fields".

N. Jacobson, Totally disconnected locally compact rings, Amer. Jour. Math. 58 (1936), Part II, proved it for the rest case (d), e) by reducing to the study of H. Hasse(14).

(2) We understand "bicompact" by the word "compact", i.e. a Hausdorff space for which Heine-Borel's covering axiom holds shall be called a compact space.
Since we already know(3) that every l.c. field is either connected or totally disconnected, and that connected l.c. fields are classified into the three types a), b) and c), so we have now only to consider the totally disconnected case.

Our method may be characterized as an alternating succession of clarifying the topologico-algebraic structure and normalizing the 'evaluation' of the field: By means of the 'sufficiently many quasi-evaluations'(4) of a compact open subring $R$ of a l.c. totally disconnected field $\mathfrak{F}$, we prove first that the right-ideals $b^rR$ generated by the powers of a certain element $b$ form a complete system of neighborhoods of zero (Lem. E, Th. 1), and further that the powers $\mathfrak{P}^r$ of a prime ideal $\mathfrak{P}$ in $R$ form another one (Th. 2). This enables us to prove the continuity of inverse (Th. 3). On the other hand, we can in a way 'evaluate' $\mathfrak{F}$ by $\{b^rR\}$ (Lem. G), by which the continuity of inverse leads to the compactification of $\mathfrak{F}$ (Th. 4). This proves the existence of the maximal compact open subring $R_0$, and of the ordinary field-evaluation ($\S$ 4). Many parts of our proof are originally due to N. Jacobson, loc. cit.(1), and the final determination of the structure of $\mathfrak{F}$ is due to H. Hasse.

In Appendix, we shall make clear the uniform structure of the multiplicative group $\mathfrak{F}^*$ of $\mathfrak{F}$ (cf. Th. 3, Cor.) and that of the compactification $\mathfrak{F}$ of $\mathfrak{F}$ (cf. Th. 4) by introducing suitable metrics.

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$\S$ 2. First we define exactly:

Definition. $\mathfrak{F}$ is called a "locally compact (= l. c.) totally disconnected (= t. d.) field", if $\mathfrak{F}$ satisfies the following three axioms:

1) $\mathfrak{F}$ is a locally compact totally disconnected topological space;
2) $\mathfrak{F}$ forms algebraically an associative field, where the commutativity of multiplication is not assumed;
3) The mappings $(x, y) \rightarrow x-y$ and $(x, y) \rightarrow xy$ of the topological product space $(\mathfrak{F}, \mathfrak{F})$ of $\mathfrak{F}$ into $\mathfrak{F}$ are both continuous.

Let, from now on, $\mathfrak{F}$ be a non-discrete l. c. t. d. field. We begin with some lemmas:

Lemma A. For any $a \in \mathfrak{F}$, $a \neq 0$, the mapping $x \rightarrow ax$ (or $x \rightarrow xa$) is a homeomorphism of an arbitrary compact set $M \subseteq \mathfrak{F}$ onto $aM$ (or $Ma$).

(4) Y. Otobe, On quasi-evaluations of compact rings, Proc. Imp. Acad. Tokyo, 20 (1944). We shall use the same notation as there.
For, since $x \rightarrow ax$ is one-to-one and continuous, it is homeomorphic on the compact space $M$.

**Lemma B.** If a subgroup $\mathcal{H}$ of a compact group $\mathcal{G}$ contains an open set, then $\mathcal{H}$ is compact and open (c.o.).

For, since $\mathcal{H}$ is a subgroup, $\mathcal{H}$ is open and hence closed in the compact group $\mathcal{G}$, therefore also it is compact.

Since $\mathcal{G}$ is a l.c.t.d. additive group, there exist arbitrarily small c.o. subgroups $\mathcal{S}$ of $\mathcal{G}$ (5). For any such $\mathcal{S}$ let us put

$$\mathcal{R}(\mathcal{S}) = \{a | a\mathcal{S} \subseteq \mathcal{S}, a \in \mathcal{G}\}.$$  

Then

**Lemma C.** $\mathcal{R}(\mathcal{S})$ is a c.o. subring of $\mathcal{G}$ and contains the unit 1 of $\mathcal{G}$.

**Proof.** $\mathcal{R}(\mathcal{S})$ is a subring of $\mathcal{G}$, since $x, y \in \mathcal{R}(\mathcal{S})$ implies $(x-y) \mathcal{S} \subseteq x\mathcal{S} - y\mathcal{S} \subseteq \mathcal{S} - \mathcal{S} = \mathcal{S}$, $xy\mathcal{S} \subseteq x\mathcal{S} \subseteq \mathcal{S}$ and $1\mathcal{S} = \mathcal{S}$. Since $\mathcal{S}$ is c.o. and the product in $\mathcal{G}$ is continuous as function of two variables, so there is a neighborhood $U$ of 0 such that $U\mathcal{S} \subseteq \mathcal{S}$, i.e. $U \subseteq \mathcal{R}(\mathcal{S})$. Take a $g \in \mathcal{S}$ with $g \neq 0$, then $\mathcal{R}(\mathcal{S}) \subseteq \mathcal{S} \cdot g^{-1}$. Since $\mathcal{S} \cdot g^{-1}$ is a compact group (Lem. A), $\mathcal{R}(\mathcal{S})$ is compact and open by Lem. B.

In §§2, 3 let $\mathcal{N}$ be an arbitrary fixed c.o. subring of $\mathcal{G}$ containing 1, the existence of which is just proved.

**Lemma D.** $b\mathcal{R} = \mathcal{R}$ is equivalent to $\mathcal{R}b = \mathcal{R}$.

**Proof.** Suppose $b\mathcal{R} = \mathcal{R}$, then $b \in \mathcal{R}$ and $b^{-1} \in \mathcal{R}$ follow from $1 \in \mathcal{R}$. Hence $\mathcal{R} \subseteq \mathcal{R}$ and $\mathcal{R}b^{-1} \subseteq \mathcal{R}$, i.e. $\mathcal{R} \subseteq \mathcal{R}b$, therefore $\mathcal{R}b = \mathcal{R}$.

$\mathcal{R}$ is a compact topological ring and evidently contains no null-divisor. Therefore, there exist sufficiently many quasi-evaluations $|x|_a$ ($a < \Omega_0$) of $\mathcal{R}$, the totality of which determines the topology of $\mathcal{R}$ (7).

Now define

$$\mathcal{B} = \mathcal{B}(\mathcal{R}) = \{b | b\mathcal{R} \subseteq \mathcal{R}, b \in \mathcal{G}\},$$  

then $\mathcal{B}(\mathcal{R}) \subseteq \mathcal{B}$ by $1 \in \mathcal{R}$. And holds the principal lemma:

**Lemma E.** If $b \in \mathcal{B}(\mathcal{R})$ and $b \neq 0$, then $b^{\nu} \rightarrow 0$ ($\nu \rightarrow +\infty$), i.e. for any neighborhood $U$ of zero there is a $\nu(U)$ such that

$$b^{\nu} \mathcal{R} \subseteq U \text{ for all } \nu \geq \nu(U).$$

**Proof.** From $b\mathcal{R} \subseteq \mathcal{R}$ and $b \neq 0$ we have

$$\mathcal{R} \supseteq b\mathcal{R} \supseteq b^2\mathcal{R} \supseteq \cdots, b \in \mathcal{R}.$$  


(7) Cf. Th. 2 in the paper cited(4).
Now we shall prove
\[ (*) \quad \bigwedge_{\nu=1}^{\nu=2} b^\nu \mathbb{R} = \{0\}. \]

If it were not the case, there would exist an element \( c \) such that
\[ c \neq 0, \quad b^{-\nu} c \in \mathbb{R} \quad (\nu = 1, 2, \ldots). \]

As \( b^\nu \in \mathbb{R} \) and \( \mathbb{R} \) is compact, \( \{b^\nu | \nu = 1, 2, \ldots\} \) has a limit point \( l \) in \( \mathbb{R} \). Then for any fixed finite set \( a_1, a_2, \ldots, a_s \) of \( a < \mathcal{O}_0 \) we can choose a subsequence \( \{\nu_k | k = 1, 2, \ldots\} \) of \( \{\nu\} \), which in general depends upon \( \{a_1, a_2, \ldots, a_s\} \), such that
\[ |b^{\nu_k} - l|_{aj} \to 0 \quad (k \to \infty) \quad \text{for} \quad j = 1, 2, \ldots, s. \]

Put \( \mu_k = \nu_{k+1} - \nu_k \geq 1 \), then for each fixed \( x \in \mathbb{R} \) and \( j \) we have
\[
|xb^{\mu_k}c - xc|_{aj} = |x(b^{\nu_k+1} - b^{\nu_k})b^{-\nu_k}c|_{aj}
\leq \min \left\{ |x|_{aj}, |b^{\nu_k+1} - b^{\nu_k}|_{aj}, |b^{-\nu_k}c|_{aj} \right\} \to 0 \quad (k \to \infty),
\]
for \( x, b^{\nu_k+1} - b^{\nu_k} \) and \( b^{-\nu_k}c \in \mathbb{R} \) and \( |b^{\nu_k+1} - b^{\nu_k}|_{aj} \leq \max \left\{ |b^{\nu_k+1} - l|_{aj}, |b^{\nu_k} - l|_{aj} \right\} \to 0 \quad (k \to \infty) \). Hence, for any \( \epsilon > 0 \) holds for some sufficiently large \( k_0 \)
\[ \rho_{aj}(xb^{\mu_k}c, xc) = |xb^{\mu_k}c - xc|_{aj} < \epsilon \quad (j = 1, 2, \ldots, s). \]

As \( xb^{\mu_k}c = xb^{\mu_k-1} \cdot bc \in \mathbb{R}bc \), and \( \epsilon, \{a_1, a_2, \ldots, a_s\} \) are arbitrary, we have from the compactness of \( \mathbb{R}bc \) (Lem. A)
\[ xc \in \mathbb{R}bc, \quad \text{i.e.} \quad x \in \mathbb{R}b. \]

Since \( x \) was an arbitrary element of \( \mathbb{R} \), we would have \( \mathbb{R}b \supseteq \mathbb{R} \), hence \( \mathbb{R}b = \mathbb{R} \), which contradicts the assumption \( b\mathbb{R} \supseteq \mathbb{R} \) by Lem. D. Thus \((*)\) has been proved.

Since \( b^\nu \mathbb{R} \) are all compact (Lem. A) and monotonously decreasing, for any neighborhood \( U \) of 0 there is \( \nu(U) \) such that
\[ b^\nu \mathbb{R} \subseteq U \quad (\nu \geq \nu(U)). \]

Now we can prove\((8)\)

Theorem 1. Every l.c.t.d. field \( \mathcal{F} \) satisfies necessarily the (Hausdorff's) first countability axiom.

Proof. As \( \mathcal{F} \) is c.o., by the same reason as in Proof of Lem. C,

\( (8) \) T. Nakayama has proved independently Lemma E and Theorem 1 under the essetinal assumption of the continuity of division. Cf. Isö-sṳgaku, 4 (1944).
for a neighborhood $V (\subseteq \mathfrak{A})$ of $0$ there is $b \neq 0$ sufficiently near by $0$ such that

$$b\mathfrak{A} \subseteq V \subseteq \mathfrak{A}.$$ 

$b\mathfrak{A}$ are open in $\mathfrak{A}$ (Lem. A), hence also in $\mathfrak{A}$ where $\mathfrak{A}$ is open. Therefore, Lem. E assures that $\{b\mathfrak{A}|\nu = 0, 1, 2, \ldots\}$ is a complete system of neighborhoods of zero in $\mathfrak{A}$, q.e.d.

Corollary. The topology of $\mathfrak{A}$ can be described in terms of the (countably) sequential convergence (Notation: $a_{\nu} \to a$) or by means of an additively invariant metric.$^9$

Now $\mathfrak{B}$ can be characterized topologically as follows:

Lemma F. $\mathfrak{B} = \mathfrak{B}(\mathfrak{A}) = \{b|b\nu \to 0 (\nu \to +\infty), b \in \mathfrak{A}\}$.

Hence symmetrically

$$\mathfrak{B}(\mathfrak{A}) = \{b|\mathfrak{A}b \subseteq \mathfrak{A}, b \in \mathfrak{A}\}.$$ 

Proof. If $b \in \mathfrak{B}$, $b\mathfrak{A} \to 0$ (Lem. E), hence $b^\nu \to 0$ (from $\mathfrak{A} \epsilon 1$). Conversely, if $b^\nu \to 0$, $b^\nu \mathfrak{A} \subseteq \mathfrak{A}$ for some $\nu_0$ as $\mathfrak{A}$ is c.o. Hence $b\mathfrak{A} \subseteq \mathfrak{A}$, i.e. $b \in \mathfrak{B}$.

Lemma G. Let $b_0$ be an arbitrary fixed element $\neq 0$ of $\mathfrak{B}(\mathfrak{A})$. Then to each element $x \neq 0$ of $\mathfrak{A}$ corresponds uniquely an integer $\exp_{\mathfrak{B}, b_0}(x)$ such that

$$b_0 \nu x \in \mathfrak{A} \quad \text{for} \quad \nu \leq \exp_{\mathfrak{B}, b_0}(x),$$

and

$$b_0 \nu x \notin \mathfrak{A} \quad \text{for} \quad \nu > \exp_{\mathfrak{B}, b_0}(x).$$

Proof. From $b_0 \mathfrak{A} \subseteq \mathfrak{A}$ we have

$$b_0^{-1} \mathfrak{A} \subseteq b_0 \mathfrak{A} \quad (\nu = 0, \pm 1, \pm 2, \ldots),$$  

hence $x \in b_0 \mathfrak{A}$ and $\nu' > \nu''$ implies $x \in b_0 \nu'' \mathfrak{A}$. From (*) and $x \neq 0$ follows $x \in b_0 \nu \mathfrak{A}$ for a sufficiently large $\nu_1$. On the other hand, there is a $\nu_2$ with $b_0^\nu x \in \mathfrak{A}$ i.e. $x \in b_0 \nu \nu_2 \mathfrak{A}$, as $b^\nu \to 0 (\nu \to +\infty)$ (Lem. F) and $\mathfrak{A}$ is a neighborhood of $0$. Therefore, there exists a unique integer $\nu_0 (\nu_1 > \nu_0 \geq -\nu_2)$ such that

$$x \in b_0 \nu \mathfrak{A} \quad \text{for} \quad \nu > \nu_0 \quad \text{and} \quad x \in b_0 \nu \mathfrak{A} \quad \text{for} \quad \nu \leq \nu_0.$$  

This $\nu_0$ is the desired $\exp_{\mathfrak{B}, b_0}(x)$.

§ 3. In this section we shall consider the algebraic properties of $\mathfrak{B}$ and then prove the continuity of inverse and the possibility of compactification of $\mathfrak{A}$. $\mathfrak{A}$ and $\mathfrak{B} = \mathfrak{B}(\mathfrak{A})$ are as before.

$^9$ The latter-half of the assertion is due to Kakutani-Birkhoff's theorem. However we do not use them explicitly in the following.
Lemma H. \( \mathfrak{B} \) contains arbitrarily small two-sided c. o. ideals \( \mathfrak{I} \).

Proof. For an arbitrarily small compact group-neighborhood \( U \) of \( 0 \) with \( UR \subseteq R \) (Cf. the beginning of §2) there is an element \( a \neq 0 \) sufficiently near by \( 0 \) such that \( RaR \subseteq U \) by the same reason as in Proof of Lem. C. As \( RaR = \bigcap_{x,y \neq 0} Ra \cdot x \) is open (Lem. A), the subgroup \( \mathfrak{I} \) generated by \( RaR \) is, as it is contained in the c. o. group \( U \), a c. o. ideal in \( R \) (Lem. B), and \( \mathfrak{I} \subseteq \mathfrak{B} \) because \( \mathfrak{B} \) is a c. o. group with respect to multiplication.

Theorem 2. \( \mathfrak{B}^\nu, \nu = 1, 2, \ldots \) are c. o. two-sided ideals in \( R \), and form a complete system of neighborhoods of zero in \( \mathfrak{B} \). \( \mathfrak{B} \) is a maximal prime ideal in \( R \), and

\[
\Gamma = \Gamma(R) = R \cap c(\mathfrak{B}) = \{ a | aR = R, a \in R \},
\]

is a c. o. group with respect to multiplication.

Proof. That \( \mathfrak{I} \) mentioned in Lem. H is c. o. and \( R \) is compact concludes that the factor ring \( R/\mathfrak{I} \) is finite. Otherhands, Lem. F and the invariance of \( \mathfrak{B} \) (i.e. \( b \in \mathfrak{B} \) and \( a \in R \) implies \( ab, ba \in \mathfrak{B} \)) shows that the image in \( R/\mathfrak{I} \) of elements of \( \mathfrak{B} \) are properly nilpotent. Therefore, \( \mathfrak{B} = \bigcap \mathfrak{I}_3 \), where \( \mathfrak{I}_3 \) is the inverse image of the radical \( R_3/\mathfrak{I}_3 \) of \( R/\mathfrak{I} \), hence \( \mathfrak{B} \) is a two-sided ideal and \( \mathfrak{B}^\nu = 0 \) (\( \mathfrak{I} \)), i.e. \( \mathfrak{B}^\nu \subseteq \mathfrak{I} \) for some \( \nu_0 = \nu(\mathfrak{I}) \). In order to prove that \( \mathfrak{B}^\nu \) are c. o., it is sufficient (by Lem. B) to show that \( \mathfrak{B}^\nu \) contain open sets. \( \mathfrak{B} \) contains \( \mathfrak{I} \) which is open. Suppose \( \mathfrak{B}^\nu \) is open, then for \( b \in \mathfrak{B}^\nu \) \( (b \neq 0) \) \( \mathfrak{B}^\nu \) is open (Lem. A) and contained in \( \mathfrak{B}^{\nu+1} \). Thus \( \mathfrak{B}^\nu \) are all c. o. As \( \mathfrak{I} \) was arbitrarily small, \( \{ \mathfrak{B}^\nu | \nu = 1, 2, \ldots \} \) forms a complete system of neighborhoods of zero in \( \mathfrak{B} \). We can easily see that \( \Gamma \) forms a group with resp. to multiplication and hence that \( \mathfrak{B} \) is maximal.

Theorem 3. The mapping of elements of a l. c. t. d. field \( \mathfrak{F} \) into their inverses:

\[
x \rightarrow x^{-1}
\]

is (locally uniformly) continuous at every point of \( \mathfrak{F} \) except 0.

Proof. Suppose \( y \in \mathfrak{F} \). Then it holds

\[
(-1)^\nu(y^\nu - y^{\nu+1} + \cdots \pm y^\mu) \in \mathfrak{B}^\nu, \quad \mu > \nu \geq 1.
\]

Therefore, \( 1-y+y^2-\ldots+(-1)^\nu y^\nu, \nu = 1, 2, \ldots \) form a Cauchy sequence by Th. 2, and there exists a unique \( z \in \mathfrak{F} \) such that

\[
z = \lim\limits_{\nu \to \infty} (1-y+y^2-\ldots+(-1)^\nu y^\nu) = 1-y+y^2-\ldots.
\]

\( (10) \) \( \mathfrak{B} \) is the two-sided ideal generated by \( \{ b_1b_2\ldots b_\nu | b_i \in \mathfrak{B}(i = 1, \ldots, \nu) \} \). \( c(A) \) denotes the complement of a subset \( A \) of \( \mathfrak{F} \).
because the sequential completeness of \( \mathcal{F} \) follows from the local compactness of \( \mathcal{F} \) and Th. 1. We have further (by the continuity of product and Lem. F.) \( z(1+y) = \lim_{y \to \infty} (1 - (-1)^{y+1}) = 1 \). Hence
\[
(1+y)^{-1} = 1 - y + y^2 - \cdots \quad \text{for} \quad y \in \mathcal{F}.
\]
Applying this formula to \( y \in \mathcal{F} \), we obtain, as \( \mathcal{F} \) is closed,
\[
(1+\mathcal{F})^{-1} \subseteq 1 + \mathcal{F} \quad (\nu = 1, 2, \ldots).
\]
Now for any fixed element \( a \neq 0 \) of \( \mathcal{F} \)
\[
(a+a\mathcal{F})^{-1} = (1+\mathcal{F})^{-1}a^{-1} \subseteq (1+\mathcal{F})a^{-1} = a^{-1} + \mathcal{F}a^{-1} \quad (\nu = 1, 2, \ldots).
\]
This proves the theorem, because \( \{\mathcal{F}\}, \{a\mathcal{F}\} \) or \( \{a^{-1}\mathcal{F}\} \) forms respectively a complete system of neighborhoods of zero (Th. 2, Lem. A).

**Corollary.** The multiplicative group \( \mathcal{F}^* \) of \( \mathcal{F} \) forms a topological group.

(see Appendix I.)

After establishing the continuity of inverse, we can prove by means of the evaluation in Lem. G the fundamental lemma for l. c. (t. d.) fields:

**Lemma 1.** \( \{a_\nu | \nu = 1, 2, \ldots \} \ (a_\nu \in \mathcal{F}) \) is absolutely divergent, i.e. has no limit point, if and only if \( a_\nu^{-1} \to 0 \ (\nu \to \infty) \).

**Proof.** The "if" part is an immediate sequel of the continuity of product. The proof of the "only if" part is as follows: If it were \( a_\nu^{-1} \to l \neq 0 \ (k \to \infty) \), then \( a_\nu \to l^{-1} \) by the continuity of inverse, which contradicts the assumption. Therefore, it suffices to prove that \( \{a_\nu^{-1}\} \) has really \( 0 \) as a limit point. Let \( b_0 \) be, as in Lem. G, a fixed element \( \neq 0 \) of \( \mathcal{F} \), and put \( \mu(\nu) = -\exp_{b_0}(a_\nu) \), then
\[
b_0^{-\mu(\nu)}a_\nu \in \mathcal{F} \quad \text{for} \quad \mu \geq \mu(\nu) \quad \text{and} \quad b_0^{-\mu(\nu)}a_\nu \in \mathcal{F} \; ;
\]
hence
\[
b_0^{-\mu(\nu)}a_\nu \in J = \mathcal{F} \cap c(b_0\mathcal{F}) ,
\]
i.e.
\[
a_\nu^{-1} \in J^{-1}b_0^{-\mu(\nu)} \quad (\nu = 1, 2, \ldots).
\]
If \( \mu(\nu) \) were bounded upwards \( (\leq \mu_0) \), then \( a_\nu \in b_0^{-\mu_0} \mathcal{F} \ (\nu = 1, 2, \ldots) \) and \( \{a_\nu\} \) would have a limit point (Lem. A). Therefore, by choosing a suitable subsequence, we may assume \( \mu(\nu) \to + \infty (\nu \to \infty) \) without the loss of generality. As \( J \) is compact (\( \mathcal{F} \) and \( b_0\mathcal{F} \) are c. o.) and does not contain \( 0, J^{-1} \) is also compact as a continuous image of \( J \). Hence by Lem. F
\[
J^{-1}b_0^{-\mu(\nu)} \to 0 \ (\nu \to \infty) \quad \text{and} \quad a_\nu^{-1} \to 0 , \ q.e.d.
\]
We obtain by combining this lemma with Th. 3.
Theorem 4. Denote by $\tilde{\mathfrak{F}}$ the union of a l.c.t.d. field $\mathfrak{F}$ and the "point at infinity" $\infty$, and define $a_\to \to \infty$ if and only if $\{a_\to\}$ is absolutely divergent in $\mathfrak{F}$. Then $\tilde{\mathfrak{F}}$ is a totally disconnected compact space which contains $\mathfrak{F}$ as a dense subspace. The mapping $x \to x^{-1}$, where $0^{-1} = \infty$, $\infty^{-1} = 0$, is a homeomorphism (and a reciprocal isomorphism with respect to multiplication) of $\tilde{\mathfrak{F}}$ onto itself.

The compactness of $\tilde{\mathfrak{F}}$ is evident from Th. 1, Cor. The above lemma and Th. 3 shows that the one-to-one mapping $x \to x^{-1}$ is continuous on the compact space $\tilde{\mathfrak{F}}$, hence it is topological.

Remark. Every locally compact Hausdorff space $S$ can be compactified by adding the point at infinity $\infty$ in the same way as above. But for general topological algebras this topological process has no corresponding algebraic meaning. This theorem means that every l. c. field is, in a sense, topologico-algebraically compactifiable, and this fact facilitates considerably the treatment of such fields (see the next section)(11). See Appendix II.

§ 4. In this section we shall prove by Th. 4 the existence of the maximal c. o. subring $\mathfrak{R}_0$ and accomplish the final normalization of evaluation.

Denote by $\mathfrak{R}_1$ an arbitrary fixed c. o. subring $\exists 1$, and by $\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{R}_1)$ its maximal ideal, the existence of which was shown at the beginning of § 2. Suppose $\mathfrak{R}$ is any c. o. subring in $\tilde{\mathfrak{F}}$. Then $\mathfrak{R} \cap \mathfrak{P}_1^{-1} = \emptyset$, where $\mathfrak{P}_1^{-1} = \{a^{-1} | a \in \mathfrak{P}_1\}$. For, from $a \in \mathfrak{R}$ and $a^{-1} \in \mathfrak{P}_1$ would follow $a^2 \in \mathfrak{R}$ and $a^{-2} \to 0$ (Lem. F), in contradiction with the compactness of $\mathfrak{R}$ (Lem. I). Thus we have

$$\mathfrak{R} \subseteq c(\mathfrak{P}_1^{-1}),$$

where $c(\mathfrak{P}_1^{-1})$ is the complement of $\mathfrak{P}_1^{-1}$ in $\tilde{\mathfrak{F}}$. Otherhands, Th. 4 shows that $\mathfrak{P}_1^{-1}$ is c. o. in $\tilde{\mathfrak{F}}$, hence $c(\mathfrak{P}_1^{-1})$ is c. o. in $\tilde{\mathfrak{F}}$. Therefore, every strictly increasing sequence of c. o. subrings breaks off in finite steps (Heine-Borel's covering axiom), hence there exists a maximal c. o. subring $\mathfrak{R}_0 (\exists 1)$ in $\tilde{\mathfrak{F}}$. Put $\mathfrak{P}_0 = \mathfrak{P}(\mathfrak{R}_0)$, then it holds conversely

$$\mathfrak{R}_0 = \mathfrak{R}(\mathfrak{P}_0),$$

because $\mathfrak{R}(\mathfrak{P}_0)$ is a c. o. subring (Lem. C) and contains evidently the maximal c. o. subring $\mathfrak{R}_0$. Now, following Jacobson, we have

Theorem 5.

$$\mathfrak{R}_0 = c(\mathfrak{P}_0^{-1}).$$

(11) Theorem 4 holds also in the connected case. Indeed, when $\mathfrak{F}$ is the field of complex numbers, $\tilde{\mathfrak{F}}$ is nothing other than the Gauss-Riemann sphere. Cf. My paper cited in (23).
Proof. It is sufficient to prove that $b \not\in R_0$ implies $b \in \mathbb{P}_0^{-1}$. From $b \not\in R_0 = R(\mathbb{P}_0)$ follows that there is $c_1 \in \mathbb{P}_0$ with $b_1 = bc_1 \not\in \mathbb{P}_0$. If $b_1 \not\in R_0$, there is similarly a $c_2 \in \mathbb{P}_0$ with $b_2 = b_1c_2 = b_1c_2 \not\in \mathbb{P}_0$, and so on. If this process could be continued infinitely, there would be $b_r = bc_1c_2...c_r \in \mathbb{P}_0^{-1}$ in contradiction with $b_r \not\in \mathbb{P}_0$. Hence for some $r$ must hold $b_r = bc_1c_2...c_r \in \mathbb{P}_0^{-1}$ (Th. 2)

This gives a topological characterization of $R_0$ (by Th. 2, Lem. 1):

Corollary. $a \in R_0$ if and only if $\{a^r \mid r = 1, 2, ...\}$ has no divergent (in $\mathfrak{R}$) subsequence. Hence $R_0$ is the unique maximal compact open subring of $\mathfrak{R}$.

Now we can introduce an ordinary field-evaluation $|x|$ of $\mathfrak{R}$: Choose an arbitrary fixed element $x_0$ with $x_0 \in \mathbb{P}_0$ but $a \not\in \mathbb{P}_0$.

Theorem 6. $\mathbb{P}_0 = x_0 R_0 = R_0 x_0$ (12).

Proof. If $x \in \mathbb{P}_0$ but $a \notin x_0 R_0$, then $x_0^{-1}x \not\in R_0 = c(\mathbb{P}_0^{-1})$, i.e. $x_0 \in x \mathbb{P}_0 \subseteq \mathbb{P}_0^2$, that is impossible. Hence $\mathbb{P}_0 = x_0 R_0$ and similarly $R_0 x_0$.

Corollary. $x_0^{-1} R_0 x_0 = R_0$, $x_0^{-1} \mathbb{P}_0 x_0 = \mathbb{P}_0$,

and $x_0^{-1} \Gamma_0 x_0 = \Gamma_0$. ($\Gamma_0 \Gamma_0 = \Gamma_0$, $\Gamma_0^{-1} = \Gamma_0$ (Th. 2).)

Define $\exp x = \exp_{x_0} x_0 (x)$ for $x \neq 0$. (cf. Lem. G). Then $\exp x = \mu$ if and only if $x_0^{-1}x \in R_0$ and $x_0^{-1}x \not\in x_0 R_0 = \mathbb{P}_0$. Hence

Theorem 7. Every element $x \neq 0$ of $\mathfrak{R}$ can be expressed uniquely in the form $x = x_0^\mu u$, where $\exp x = \mu$ and $u \in \Gamma_0$.

We can now easily verify (13)

$\exp(xy) = \exp x + \exp y$ and $\exp(x+y) \geq \min \{\exp x, \exp y\}$.

Put $|x| = \gamma^{-\exp x}$ ($x \neq 0$), $|0| = 0$, where $\gamma > 1$ is fixed, then clearly it holds

Theorem 8. $|x|$ is a continuous real-valued discrete non-archimedean evaluation of $\mathfrak{R}$. That is, it has the following properties:

1) $|x| = |-x| \geq 0$, $|x| = 0$ if and only if $x = 0$;

2) $|x+y| \leq \max \{|x|, |y|\}$;

12) Therefore, $\mathbb{P}_0^\nu = x_0^\nu R_0 = R_0 x_0^\nu$, $\nu = 0, \pm 1, \pm 2, ...$ are principal fractional ideals in $\mathfrak{R}$, and $\exp x = \nu$ if and only if $x \in \mathbb{P}_0^\nu$, but $a \not\in \mathbb{P}_0^\nu$.

13) For $u, v \in \Gamma_0$, $x_0^\nu u \cdot x_0^\nu v = x_0^\nu + x_0^\nu \nu$ where $w = (x_0^{\nu-\nu}) v e(x_0^{-\nu} \Gamma_0 x_0^\nu) \Gamma_0 = \Gamma_0 \Gamma_0 = \Gamma_0$; for $\mu \leq \nu$, $x_0^\nu u + x_0^\nu v = x_0^\mu t$ where $t = u + x_0^{\nu-\nu} v \in \mathbb{P}_0$ (and $\in \Gamma_0$ if $\mu < \nu$, since $\Gamma_0 + \mathbb{P}_0 = \Gamma_0$).
3) $|xy| = |x||y|$;

4) $\mathbb{F}_0 = \{x||x| \leq 1\}$ is the evaluation-ring of $|x|$, and $\mathbb{F}_0^* = \{x||x| \leq \gamma^{-1}\}$

5) the additively invariant metric $\rho(x, y) = |x-y|$ gives the topology of $\mathbb{F}$.

By means of Th. 8, the problem to determine the structure of $\mathbb{F}$ has been perfectly reduced to the classical case studied by H. Hasse(14). Hence, we can obtain the same classification as mentioned in §1 under our weaker assumption:

Main Theorem. Every locally compact totally disconnected field (cf. Definition in §2), which is not discrete, is either $F$ or $K(p^n)\{\xi\}$ ($p$ is a prime number, $n$ is a natural number; cf. §1). The former is of characteristic 0, while the latter of characteristic $p$.

By appealing to the result in the connected case(3), we have

Corollary. Any minimal non-discrete locally compact totally disconnected field (i.e. over the field $F$ of rational numbers) is either the field $F$ of real numbers or the field $K$ of $p$-adic numbers. The former is connected, while the latter is totally disconnected.

This abstract topologico-algebraic characterization of $F$ and $K$ seems interesting by reason that both are fundamental in analysis and arithmetic respectively.(15)

Appendix.

I. The uniform structure of the multiplicative group $\mathbb{F}^*$ of $\mathbb{F}$.

Let $\mathbb{F}$ be, as in the text, a l.c.t.d. field and $|x| = \gamma^{-\exp x}$ ($\gamma > 1$) be its evaluation in Th. 8. Theorem 3, Corollary shows that the multiplicative group $\mathbb{F}^*$ of $\mathbb{F}$ forms a topological group. Of course, the topology of $\mathbb{F}^*$ is, as a subspace of $\mathbb{F}$, induced by that of $\mathbb{F}$, indeed, by the "additive uniform structure" $U_a$ determined by the additively invariant metric $\rho(x, y) = |x-y|$.(16) But neither the mapping $x \to xa$ nor $x \to ax$ is uniformly continuous with respect to $U_a$, so that $\mathbb{F}^*$ with this uniform structure $U_a$—we shall denote it by $\mathbb{F}^*(U_a)$—does not form a topological group in stronger sense of the words.(17) Now we shall introduce the


(15) We notice that we assume now the classic Dedekind-Cantor's theory of natural, real and transfinite numbers. Cf. Note.


(17) Weil defined the so-called "uniform structure of a topological group" as that in which the right (or left) group-translation $x \to xa$ (or $x \to ax$) is uniformly continuous. Cf. Weil, loc. cit., §5. Also see §17, Th. 10 in Pontrjagin's book cited(1). We shall call the integrity of a topological group and such a uniform structure on it a "topological group in the stronger sense".
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"multiplicative uniform structure" $\mathfrak{u}_m$ on $\mathfrak{H}^*$, so that $\mathfrak{H}^*$ becomes a multiplicative topological group in the stronger sense. For this purpose we define the "multiplicative" norm $|x|^*$ of $\mathfrak{H}^*$:

$$|x|^* = \max \{|x-1|, \, |x^{-1}-1|\} = \gamma\min \{\exp(x-1), \exp(x^{-1}-1)\}.$$ 

Then we can easily verify (by the known structure of $\mathfrak{H}$)

$$|x|^* = \begin{cases} |x-1| & \text{for } \exp x = 0 \text{ and } \exp(x-1) > 0, \\ |x| & \text{for } \exp x = 0 \text{ and } \exp(x-1) = 0, \\ |x|^{-1} & \text{for } \exp x < 0, \\ >1 & \text{for } \exp x > 0. \end{cases}$$

Hence we have

1) $|x|^* \geq 0$, and $|x|^* = 0$ if and only if $x = 1$,

2) $|x^{-1}|^* = |x|^*$,

3) $|xx^{-1}|^* = |x|^*$,

4a) $|xy|^* \leq \max \{|x|^*, |y|^*, |x^*|, |y^*|\}$,

4b) $|xy|^* = |yx|^* = \max \{|x|^*, |y|^*\}$ for $|x|^* < 1$ and $|y|^* < 1$.

Proof of 3): $|xx^{-1}|^* = |xx^{-1}-\delta|^* = |x-\delta| \cdot |x^{-1}|^* = |x|^*$ ($\delta = 0$ or 1).

Proof of 4a): It follows from $|x| \leq \max \{|x-1|, |1|\} \leq \max \{|x|^*, |1|\}$ and $|y^{-1}| \leq \max \{|y^{-1}-1|, |1|\} \leq \max \{|y|^*, |1|\}$ that $|xy-1| = |x(y-1)+(y-1)| \leq \max \{|x|^*, |y|^*, |x^*|, |y^*|\}$ and $|y^{-1}x^{-1}-1| \leq \max \{|y^{-1}|, |x^{-1}-1|, |y^{-1}-1|\} \leq \max \{|x|^*|y|^*, |x^*|, |y^*|\}$, hence we have 4a).

Proof of 4b): $|x|^* < 1$, $|y|^* < 1$ implies $|xy|^* < 1$ (4a), hence

$$|xy|^* = |xy-1| = \max \{|x-1|, |y-1|\} = \max \{|x|^*, |y|^*\} = |yx|^*.$$ 

Define now

$$\rho^*(x, y) = |xy^{-1}|^*.$$ 

then we have the following properties of $\rho^*(x, y)$:

1') $\rho^*(x, y) \geq 0$, and $\rho^*(x, y) = 0$ if and only if $x = y$,

2') $\rho^*(x, y) = \rho^*(y, x)$,

3') $\rho^*(x, y) = \rho^*(xz, yz) = \rho^*(zx, zy)$,

4b') $\rho^*(x, y) = \max \{\rho^*(x, z), \rho^*(y, z)\}$ for $\rho^*(x, z) < 1$ and $\rho^*(y, z) < 1$,

4c') $\rho^*(x, y) = \rho^*(x^{-1}, y^{-1})$ for $\rho^*(x, z) < 1$ and $\rho^*(y, z) < 1$.

Proof of 4c'): By 3) and 4b) from the assumption $|x^{-1}|^* = |xx^{-1}|^* < 1$ and $|yz^{-1}|^* < 1$ follows $\rho^*(x^{-1}, y^{-1}) = |x^{-1}y|^* = |xx^{-1} \cdot yz^{-1}|^* = |yz^{-1} \cdot xx^{-1}|^* = \rho^*(y, x)$.
Thus $\rho^*(x, y)$ is a multiplicatively two-sided invariant (non-archimedean) local metric of $\mathfrak{F}^*$. ("local" means that $\rho^*(x, y)$ is a true metric in the neighborhood $E_a = \{x | \rho^*(x, a) < 1\}$ of each $a \in \mathfrak{F}^*$). Therefore, $\rho(x, y)$ determines a uniform structure $\mathcal{U}_m$ on $\mathfrak{F}^*$, and the multiplicative group $\mathfrak{F}^*$ with $\mathcal{U}_m : \mathfrak{F}^*(\mathcal{U}_m)$ forms a topological group in the stronger sense, in fact, the mappings $x \rightarrow xa$ and $x \rightarrow ax$ are both $\rho^*$-isometric (3').

$\rho^*(x, y) < 1$ implies $|x| = |y| = a > 0$ and $\rho^*(x, y) = a^{-1}\rho(x, y)$. For, if $|xy^{-1}|^* < 1$, $\rho^*(x, y) = |xy^{-1}|^* = |xy^{-1}-1| = |x-y||y|^{-1}$ and $\rho^*(y, x) = |y-x||x|^{-1}$. Hence, the topology of $\mathfrak{F}^*(\mathcal{U}_m)$ is identical to the original topology of $\mathfrak{F}^* = \mathfrak{F}^*(\mathcal{U}_a)$, if we do not regard the uniform structures.

\[ E = E_1 = \{y | |y|^* < 1\} = 1 + \mathcal{B}_0 \quad (\text{cf. } \S 4) \]

is a compact open invariant subgroup of $\mathfrak{F}^*(\mathcal{U}_m)$ (by 2, 3) and 4b)). The cosets containing $x$ of $E$ are

\[ E_x = \{y | \rho^*(y, x) < 1\}, \]

and all of them form a discrete multiplicative factor group $\mathfrak{D} = \mathfrak{F}^*/E$ since $E$ is c. o. The mapping $x \rightarrow x^{-1}$ is a $\rho^*$-isometric transformation of $E_a$ onto $E_{a^{-1}}$ (4b')). Therefore, the mapping $(x, y) \rightarrow xy$, $(x, y) \rightarrow x^{-1}y$ or $(x, y) \rightarrow xy^{-1}$ is a uniformly continuous mapping of $(E_a(\mathcal{U}_m), E_0(\mathcal{U}_m))$ onto $E_{a0}(\mathcal{U}_m)$, $E_{a^{-1}b}(\mathcal{U}_m)$ or $E_{ab}(\mathcal{U}_m)$ respectively. Indeed, $\rho^*(x, x') < 1$ and $\rho^*(y, y') < 1$ implies (by 3) and 4b))

\[
\rho^*(xy, x'y') = |xy y'^{-1}x'^{-1}|^* = |xx'^{-1}x'y'y'^{-1}x'^{-1}|^* = \max \{|xx'^{-1}|^*, |yy'^{-1}|^*\} = \max \{\rho^*(x, x'), \rho^*(y, y')\}
\]

and

\[
\rho^*(x^{-1}y, x'^{-1}y') = \max \{\rho^*(x^{-1}, x'^{-1}), \rho^*(y, y')\} = \max \{\rho^*(x, x'), \rho^*(y, y')\} \quad \text{and similarly } \rho^*(xy^{-1}, x'y^{-1}).
\]

From the structure of $\mathfrak{F}$ we can immediately see that $\mathfrak{D}$ is a group-extension of the (cyclic) multiplicative group $\mathfrak{D}_0$ of the Galois field $K(p^n)$ ($= \mathfrak{N}_0/\mathcal{B}_0; n = 1$ in the case of type d)), by the infinite cyclic group $\mathfrak{D}^{(18)}$.

The two uniform structures $\mathcal{U}_a$ and $\mathcal{U}_m$ on $\mathfrak{F}^*$ are not equivalent. In fact, $\mathfrak{F}^*(\mathcal{U}_a)$ does not form a multiplicative topological group in the stronger sense as remarked above; and while $\mathfrak{F}^*(\mathcal{U}_a)$ is not complete (as $0 \not\in \mathfrak{F}^*$), otherhands $\mathfrak{F}^*(\mathcal{U}_m)$ is complete since $\mathfrak{F}^*(\mathcal{U}_m)$ is a locally compact topological group in the stronger sense.

(18) $E \ni x = x_0^\nu \mathfrak{N}$, where $\exp x = \nu$ and $x_0^{-\nu}x = u \equiv \mathfrak{N}(\mathcal{B}_0)$. $\mathfrak{N} \in \mathfrak{D}$ (cf. \S 4). The automorphism $\sigma : \mathfrak{N} \rightarrow x_0\mathfrak{N}x_0^{-1} = \mathfrak{N}$ ($\sigma = p^0$) is uniquely (mod $p^n$) determined by $\mathfrak{N}$. Cf. Hasse, loc. cit(14).
II. The uniform structure of the compactification $\tilde{\mathbb{F}}$ of $\mathbb{F}$.

In this section we shall consider the uniform structure of the compactification $\tilde{\mathbb{F}}$ of $\mathbb{F}$ defined in Theorem 4. For this purpose we transform the additive metric $\rho(x, y) = |x-y|$ into its "asymptotic form":

$$\tilde{\rho}(x, y) = \frac{\rho(x, y)}{1 + |x| + |y|} = \frac{|x-y|}{(1 + |x|)(1 + |y|)}.$$

We have then

$$(*) \quad \tilde{\rho}(x^{-1}, y^{-1}) = \frac{|x^{-1}-y^{-1}|}{(1 + |x^{-1}|)(1 + |y^{-1}|)} = \frac{|y-x|}{(1 + |x|)(1 + |y|)} = \tilde{\rho}(x, y),$$

and we can easily verify from $|x-y| \leq \max \{|x|, |y|\}$ and $|x-y| = |x|$ (for $|x| > |y|$)

$$(**) \quad \tilde{\rho}(x, z) \leq \tilde{\rho}(x, y) + \tilde{\rho}(y, z).$$

Define moreover

$$\tilde{\rho}(x, \infty) = \tilde{\rho}(\infty, x) = 1 - \tilde{\rho}(x, 0) = \frac{1}{1 + |x|}$$

then $0 \leq \tilde{\rho}(x, y) = \tilde{\rho}(y, x) \leq 1$, $(*)$ and $(**)$ hold for any $x, y, z \in \tilde{\mathbb{F}}$. Hence $\tilde{\rho}(x, y)$ is a metric on $\tilde{\mathbb{F}}$, and the mapping $x \rightarrow x^{-1}$ is a $\tilde{\rho}$-isometric transformation of $\tilde{\mathbb{F}}$ onto itself. $\tilde{\rho}(x, y)$ is equivalent to $\rho(x, y)$ on $\{x||x| \leq M\}$ for any $M < +\infty$. For, always $\tilde{\rho}(x, y) \leq \rho(x, y)$, and conversely $\rho(x, y) \leq (1 + M)^2 \tilde{\rho}(x, y)$ for $|x| \leq M$, $|y| \leq M$ ($0 < M < +\infty$). In the next, $\tilde{\rho}(x, \infty) = \tilde{\rho}(x^{-1}, 0) \rightarrow 0$ if and only if $x^{-1} \rightarrow 0$, i.e. $x \rightarrow \infty$.

Thus, the topology of $\tilde{\mathbb{F}}$ is given by $\tilde{\rho}$, therefore the uniform structure $\tilde{\mathbb{U}}$ of $\tilde{\mathbb{F}}$ can be described by the metric $\tilde{\rho}(x, y)$; because the uniform structure on a compact space is (essentially) unique(20). Evidently, $\tilde{\mathbb{F}} = \tilde{\mathbb{F}}(\tilde{\mathbb{U}})$ is the (unique) completion of $\mathbb{F}$ with respect to $\tilde{\mathbb{U}}$ or $\tilde{\rho}$. (cp. $\mathbb{F}$ is complete with respect to $\mathbb{U}$ or $\rho$).

$\tilde{\mathbb{F}}$ may be regarded as a sort of "Gauss-Riemann sphere" in the following sense: First, $\tilde{\mathbb{F}}$ is compact. Moreover, for $a, b, c, d \in \tilde{\mathbb{F}}$ with $a^{-1}b$

(19) The stronger inequality $\tilde{\rho}(x, z) \leq \max \{\tilde{\rho}(x, y), \tilde{\rho}(y, z)\}$ does not hold if and only if $|x| > |y| > |z|$ or $|x| < |y| < |z|$.

(20) Cf. Weil, loc. cit., § 3, Th. IV.
$-c^{-1}d = 0 (\text{or } ba^{-1} - dc^{-1} = 0)$

$x \rightarrow (ax + b)(cx + d)^{-1}$ (or $x \rightarrow (xc + dy(xa + b)$)

is a homeomorphic transformation of $F$ onto itself. By

$$x \rightarrow T_{a_1, a_2, a_3}(x) = (a_3 - a_2)(a_3 - a_1)^{-1} \cdot (x - a_1)(x - a_2)^{-1}$$

any three different points $a_1, a_2, a_3 \in F$ are transformed into $0, \infty, 1$ respectively, hence they are transformed into another set of three different points $b_1, b_2, b_3$ by $T_{b_1, b_2, b_3} \cdot T_{a_1, a_2, a_3}$

$$
(T_{a_1, a_2, a_3}^{-1}(x) = (x - (a_3 - a_2)(a_3 - a_1)^{-1})^{-1} \cdot (xa_2 - (a_3 - a_2)(a_3 - a_1)^{-1}a_1)
$$

Therefore, we can see that $F$ is 3-ply homogeneous by the "topological (= homeomorphic) linear transformations". (1)

Note. (Added at proof-reading)

The theorems (especially the necessity of the countability axioms) resumed in italics in § 1 are of essential importance in the general theory of "representations of topological-algebraic structures" in the widest sense of the words (21); for example, in the theories of

a) vector spaces, measures and probabilities on Boole algebras, analytically representable unctions, continuous geometry, vector lattices, Banach spaces, combinatorial geometry, functional analysis, tensorial geometry, potentials; b) analytic functions on Riemann surfaces, almost periodic functions on groups, Lie groups, unitary geometry, normed rings; c) Lorentz group, relativistic quantum mechanics; d) relative abelian fields; e) non-Pascal geometry; etc. . In the usual representation theories of topological algebras, $F(i)$ is an effective substrative structure, because it is the unique connected and algebraically closed l. c. field. Cf. Th. of (9). Though $P$ is not algebraically closed, $P$ is more fundamental, because it is an image by projection and a subfield of $F(i)$ and it satisfies a new simple categorical system of order-theoretical axioms. That is, $P$ (resp. $P$) as an archimedean ordered field uniquely admits the unique maximal (resp. minimal) one of the "residual-isomorphic" and dual-automorphic linear orderings. The ordering (=order type) $\omega$ of all natural numbers is the unique minimal residual-isomorphic linear ordering. (Definition: An ordered set $S$ is called "residual-isomorphic", if $\{x | x > a, x \in S\}$ is order-isomorphic with $S$ for any $a \in S$.), Cf. p. 198, Corollary and (15), (23).

Historical note. The author proved the theorems of (3) by proving without transfinite numbers that the additive group $K$ of real numbers mod 1 does not form a topological ring, on July 19, 1943 before knowing those of Taussky and Jacobson; and delivered the Note on Sept. 20, 1943; both at the meetings of Phys.-Math. Soc. Japan. My paper cited in (23) was presented in December, 1940 to Prof. Iyanaga. (The terminology there differs than here.)

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(21) Cf. Dantzig, loc. cit., T. 26 for the definition of "multiplicity of homogeneity".

(22) This concept corresponds to the "reprsentations de composes" of G. W. Leibniz, Die philosophische Schriften (ed. Gerhardt), IV, pp. 427-53; VI, 598-9, 697-16; VII, 263-4; 392-8. Of course, we must interpret him from the standpoint of modern times.

(23) Cf. also H. Weyl, Philosophie der Mathematik und Naturwissenschaft, München (1936), Gruppentheorie un Quantenmechanik, Leipzig (1931), The classical groups, Princeton (1939); T. Takagi, Sūgaku Zatsudan, Tokyo (1935); P. A. M. Dirac, The principles of quantum mechanics, Oxford (1935); G. Birkhoff, Lattice theory, New York (1940); Y. Otobe, On the regularities of topological spaces Japanese, Shizyo-Sūgaku (1941). The author has in view to develop and realize the ideas and plans suggested in the Note in the future.