8. On the Mean Modulus of Regular Functions.

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In the conclusion of his interesting paper "Ueber Potenzreihen mit vorgeschriebenen Anfangsgliedern," Acta Math., 42, Prof. *F. Riesz* has deduced *Carathéodory-Fejér's* theorem⁽¹⁾ concerning the power series with given first $(n+1)$ coefficients; it seems to me, however, as *Carathéodory-Fejér's* theorem is now classical while *Riesz's* problem has not yet been solved completely, rather desirable to solve *Riesz's* problem by the aid of *Carathéodory-Fejér's* theorem. I will try it in the present paper.

Theorem. Let us consider a set of functions $\{f(z)\}$ defined by the power series :

$$f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots, \quad |z| \leq 1$$

whose first $(n+1)$ coefficients satisfy the condition

$$(A) \quad C_n a_0 + C_{n-1} a_1 + \dots + C_0 a_n = 1$$

in which C_ν ($\nu=0, 1, \dots, n$) are given constants, not all zero.

Among such functions, the one which gives the minimum of the integral

$$I(f) = \frac{1}{2\pi} \int_{|z|=1} |f(z)| |dz|$$

is given by

$$f^*(z) = \frac{2\pi}{m \int_{|z|=1} |P(z)|^2 |dz|} \cdot (P(z))^2,$$

where $P(z)$ is a polynomial of the form

$$P(z) = \prod_{\nu=1}^n (1 - \bar{\alpha}_\nu z), \quad (|\alpha_\nu| \leq 1, \nu=1, 2, \dots, n),$$

and

⁽¹⁾ *Carathéodory and Fejér*, Ueber den Zusammenhang der Extremen von harmonischen Funktionen usw., Rend. Palermo, 33 (1911), 218-239.

$$F(z) = m \frac{z^n \bar{P}\left(\frac{1}{z}\right)}{P(z)} = C_0 + C_1 z + \dots + C_n z^n + \dots, \quad (m > 0)$$

is Carathéodory-Fejér's function whose initial $(n+1)$ coefficients are $C_\nu (\nu=0, 1, \dots, n)$.

Proof. By the assumption, we have

$$\begin{aligned} 1 &= C_n \alpha_0 + C_{n-1} \alpha_1 + \dots + C_0 \alpha_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{F(z) f'(z)}{z^{n+1}} dz \\ &= \frac{1}{2\pi} \int_{|z|=1} \frac{F(z) f'(z)}{z^n} |dz| \\ &\leq \frac{1}{2\pi} \max_{|z|=1} |F(z)| \int_{|z|=1} |f'(z)| |dz| = m I(f), \end{aligned}$$

that is

$$1 \leq m I(f) \quad \text{or} \quad \frac{1}{m} \leq I(f),$$

so that the minimum of $I(f)$ is not smaller than $\frac{1}{m}$.

But if we put

$$f(z) = f^*(z) = \frac{2\pi}{m \int_{|z|=1} |P(z)|^2 |dz|} \cdot (P(z))^2,$$

we have obviously

$$I(f^*) = \frac{1}{m}.$$

Therefore $f^*(z)$ is a function which satisfies the condition (A) and gives the minimum of $I(f)$.

If therefore we can show that there is no such function other than $f^*(z)$, the theorem is completely proved.

For this purpose, let us suppose that there is another such function, say $g(z)$. Then if we take two real positive numbers λ and μ such that $\lambda + \mu = 1$, the function

$$h(z) = \lambda f^*(z) + \mu g(z)$$

also satisfies the condition (A).

Since $I(f^*)$ and $I(g)$ give the minimum of $I(f)$ and are equal to $\frac{1}{m}$, $I(h)$ must also be equal to $\frac{1}{m}$, that is, we must have

$$I(h) = \lambda I(f^*) + \mu I(g) = \frac{1}{m}.$$

Hence it must be

$$\lambda f^*(z) + \mu g(z) = \lambda f^*(z) + \mu g(z)$$

for all points on $|z|=1$, so that $f^*(z)$ and $g(z)$ must have equal amplitude for all points on $|z|=1$.

Consequently $g(z)/f^*(z)$ is real and not negative for all points on $|z|=1$.

Since, by a theorem given by *Riesz* in the paper mentioned above, the function whose first $(n+1)$ coefficients are given and gives the minimum of the mean modulus $I(f)$ is a polynomial of at most $2n$ degree, $g(z)/f^*(z)$ is a rational function of at most $2n$ degree.

On the other hand, the following theorem is due to Prof. *Kakeya*:

"In order that a rational function $R(z)$ is real and not negative along the unit circle, it is necessary and sufficient that it has the form

$$R(z) = k \prod \frac{(z-\beta)(1-\bar{\beta}z)}{(z-\alpha)(1-\bar{\alpha}z)}, \quad (\alpha \leq 1, |\beta| \leq 1),$$

where k is a real and non-negative constant" ⁽²⁾.

In our case therefore, we have

$$\frac{g(z)}{f^*(z)} = k \prod \frac{(z-\beta_v)(z-\beta'_v)(1-\bar{\beta}_v z)(1-\bar{\beta}'_v z)}{(z-\alpha_v)^2(1-\bar{\alpha}_v z)^2}, \quad (\alpha \leq 1, |\beta| \leq 1),$$

or replacing $f^*(z)$ by its actual expression, we have

$$g(z) = \frac{km}{2\pi} \int_{|z|=1} p(z)^{-2} dz \prod_{v=1}^n \frac{(z-\beta_v)(z-\beta'_v)(1-\bar{\beta}_v z)(1-\bar{\beta}'_v z)}{(z-\alpha_v)^2}.$$

Since $\alpha \leq 1$ and $g(z)$ is regular for all values of z ($|z| \leq 1$), we must have

⁽²⁾ *Kakeya*, On the mean modulus of polynomials, Proc. Phys.-Math. Society of Japan, Ser. 3, 5 (1923), 82.

$$\alpha_v = \beta_v = \beta'_v, \quad (v=1, 2, \dots, n),$$

so that

$$k=1, \quad g(z) \equiv f^*(z).$$

Thus the theorem is completely proved.

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