11. *On Fundamental Systems of Symmetric Functions.*

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I.

For n independent variables

$$x_1, x_2, \dots, x_n \quad (1)$$

we denote, as usual, the power sum

$$x_1^k + x_2^k + \dots + x_n^k \quad (2)$$

by S_k , k being a positive integer.

We can find n power sums

$$S_{k_1}, S_{k_2}, \dots, S_{k_n} \quad (3)$$

which has the nature that any rational symmetric function of (1) can be expressed as a rational function of (3). n power sums of such a nature is called a *fundamental system* (of power sums) in this paper.

Besides the well-known fundamental system

$$S_1, S_2, \dots, S_n, \quad (4)$$

Borchard⁽¹⁾ gave a system

$$S_1, S_3, \dots, S_{2n-1}. \quad (5)$$

Vahlen⁽²⁾ extended this, and gave a system

$$S_1, S_2, \dots, S_{\nu-1}, S_{\nu+1}, S_{\nu+2}, \dots, S_{2\nu-1}, S_{2\nu+1}, S_{2\nu+2}, \dots, S_{\mu} \quad (6)$$

where the suffixes of S are the first n numbers of the sequence of natural numbers, excluding the multiples of ν . Recently, Ludwig⁽³⁾ has given a system

$$S_1, S_2, \dots, S_k, S_{k+1}, S_{k+2}, \dots, S_{l_n}, \quad (7)$$

⁽¹⁾ Monatsberichte der Berliner Akademie (1857), 301.

⁽²⁾ Acta Math. 23 91.

⁽³⁾ Math. Ann. 83 67.

where k is such an integer that

$$k < n \leq 2k + 1$$

and $l_{k+1}, l_{k+2}, \dots, l_n$ are any $n - k$ integers among

$$k + 1, k + 2, \dots, 2k + 1.$$

Here I am going to give a kind of conditions for being a fundamental system and to prove, by the aid of it, the preceding theorems somewhat simpler than the original proof, and to give a new kind of fundamental system which may be seen as an extension of those of Vahlen and Ludwig.

II.

If n power sums form a fundamental system, then it is necessary that each of n fundamental symmetric functions

$$p_i = \sum x_1 x_2 \dots x_i, \quad i = 1, 2, \dots, n \quad (8)$$

can be expressed rationally by (3) in the form

$$p_i = \frac{A_i \{S_{k_1}, S_{k_2}, \dots, S_{k_n}\}}{B_i \{S_{k_1}, S_{k_2}, \dots, S_{k_n}\}}, \quad (9)$$

where A_i and B_i are polynomials of (3). Hence if the values of S_{k_i} 's of (3) is given, the values of p_i 's of (8) must, in general, be determined uniquely. The exception may occur only when some of B_i 's of (9) should vanish, in which case $x_1, x_2, \dots, x_n$ and consequently $p_1, p_2, \dots, p_n$ have some particular relations among themselves. And the unique determination of p_i 's is identical with the unique determination of the functional form

$$\begin{aligned} f(x) &= 1 + p_1 x + p_2 x^2 + \dots + p_n x^n \\ &= (1 + x_1 x)(1 + x_2 x) \dots (1 + x_n x). \end{aligned} \quad (10)$$

Conversely, if the form of (10) is determined uniquely, in general, whatever we give the fixed values to S_{k_i} 's of (3), then (3) must be a fundamental system. For each of p_i 's of (8) is evidently an algebraic function of S_{k_i} 's of (3), and it is uniform, in general, so that it must be a rational function of S_{k_i} 's.

Now it is easily seen that

$$\log f(x) = S_1x - \frac{1}{2}S_2x^2 + \frac{1}{3}S_3x^3 - \dots + (-1)^{n-1}\frac{1}{n}S_nx^n \dots \dots \dots \quad (11)$$

Consequently, if $f(x)$ is determined by S_{k_i} 's, then $\log g(x)$, where $g(x)$ is a polynomial of the form

$$g(x) = 1 + p_1'x + p_2'x^2 + \dots + p_n'x^n, \quad (12)$$

has the expansion of the form (11) in which the coefficients of

$$x^{k_1}, x^{k_2}, \dots, x^{k_n}$$

are identical with those of (11), when and only when $f(x)$ is identical with $g(x)$. That is to say, if in the expansion

$$\begin{aligned} \log f(x) - \log g(x) &= \log \frac{f(x)}{g(x)} \\ &= t_1x + t_2x^2 + \dots + t_nx^n + \dots, \end{aligned} \quad (13)$$

$t_{k_1}, t_{k_2}, \dots, t_{k_n}$ are all equal to zero, we must have $f(x) \equiv g(x)$.

The exception may occur only when some particular relations among the coefficients of $f(x)$ or of $g(x)$ exist. Thus we get:

Theorem. *The n power sums $S_{k_1}, S_{k_2}, \dots, S_{k_n}$ form a fundamental system when and only when the identity*

$$\log \frac{1 + p_1x + p_2x^2 + \dots + p_nx^n}{1 + p_1'x + p_2'x^2 + \dots + p_n'x^n} = t_1x + t_2x^2 + \dots + t_nx^n + \dots$$

where

$$t_{k_1}, t_{k_2}, \dots, t_{k_n} = 0,$$

gives the result that either

$$p_1 = p_1', p_2 = p_2', \dots, p_n = p_n'$$

or some particular relations among $p_1, p_2, \dots, p_n$ or among $p_1', p_2', \dots, p_n'$ exist.

It is to be remarked that, if from this identity we get an infinite number of values of $p_1', p_2', \dots, p_n'$ corresponding to the fixed values of $p_1, p_2, \dots, p_n$ or an infinite number of values of $p_1, p_2, \dots, p_n$ corresponding to the fixed values of $p_1', p_2', \dots, p_n'$, then some particular relations among $p_1, p_2, \dots, p_n$ or among $p_1', p_2', \dots, p_n'$ must exist. For in this case an algebraic function p_i of $S_{k_1}, S_{k_2}, \dots, S_{k_n}$ has an infinite number

of values, which shows that the arguments $(S_{k_1}, S_{k_2}, \dots, S_{k_n})$ correspond to a point of indeterminateness, namely have a relation among themselves.

In the application of the theorem, we have often to refer to the following lemma.

Lemma. *If we have the identity of the form*

$$\log \frac{1 + p_1 x + p_2 x^2 + \dots + p_m x^m + \dots}{1 + p_1' x + p_2' x^2 + \dots + p_m' x^m + \dots} = t_{k+1} x^{k+1} + t_{k+2} x^{k+2} + \dots,$$

then we must have

$$p_1 = p_1', \quad p_2 = p_2', \dots, \quad p_k = p_k'.$$

$$\begin{aligned} \text{For} \quad & \log \frac{1 + p_1 x + p_2 x^2 + \dots + p_m x^m + \dots}{1 + p_1' x + p_2' x^2 + \dots + p_m' x^m + \dots} \\ &= \log \left\{ 1 + \frac{(p_1 - p_1')x + (p_2 - p_2')x^2 + \dots + (p_m - p_m')x^m + \dots}{1 + p_1' x + p_2' x^2 + \dots + p_m' x^m + \dots} \right\} \end{aligned}$$

is a power series of

$$y = \frac{(p_1 - p_1')x + (p_2 - p_2')x^2 + \dots + (p_m - p_m')x^m + \dots}{1 + p_1' x + p_2' x^2 + \dots + p_m' x^m + \dots},$$

having the form

$$y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \dots,$$

order of whose zero at origin is equal to that of y . And if that order of y is not less than $k+1$, we must have all of $p_1 - p_1', p_2 - p_2', \dots, p_k - p_k'$ equal to zero.

III.

The fundamental system (4) and (5) are particular cases of (6). But since their proofs may be simpler than that of general case, we shall treat them separately.

To prove that (4) is a fundamental system, it is sufficient, by our theorem, to see that the identity of the form

$$\log \frac{1 + p_1 x + p_2 x^2 + \dots + p_n x^n}{1 + p_1' x + p_2' x^2 + \dots + p_n' x^n} = t_{n+1} x^{n+1} + t_{n+2} x^{n+2} + \dots \quad (14)$$

implies that

$$p_i = p_i', \quad i=1, 2, \dots, n.$$

But this is nothing other than the lemma itself given in the preceding paragraph. So we see that (4) is a fundamental system.

IV.

To prove that (5) is a fundamental system, it is sufficient to see that the identity of the form

$$\log \frac{1 + p_1 x + p_2 x^2 + \dots + p_n x^n}{1 + p_1' x + p_2' x^2 + \dots + p_n' x^n} = t_2 x^2 + t_4 x^4 + \dots + t_{2n} x^{2n} + t_{2n+1} x^{2n+1} + t_{2n+2} x^{2n+2} + \dots, \quad (15)$$

where the coefficients of $x^1, x^3, \dots, x^{2n-1}$ are zero, implies

$$p_i = p_i', \quad i=1, 2, \dots, n$$

or the existence of some relations among p_i 's.

If we put $-x$ in place of x in (15) and subtract it from (15) side by side, we get

$$\begin{aligned} & \log \frac{(1 + p_1 x + p_2 x^2 + \dots + p_n x^n)(1 - p_1' x + p_2' x^2 - \dots + (-1)^n p_n' x^n)}{(1 + p_1' x + p_2' x^2 + \dots + p_n' x^n)(1 - p_1 x + p_2 x^2 - \dots + (-1)^n p_n x^n)} \\ & = 2t_{2n+1} x^{2n+1} + \dots, \end{aligned} \quad (16)$$

from which we see, by lemma, that corresponding coefficients of

$$P(x) = (1 + p_1 x + p_2 x^2 + \dots + p_n x^n)(1 - p_1' x + p_2' x^2 - \dots + (-1)^n p_n' x^n)$$

$$\text{and } P(-x) = (1 + p_1' x + p_2' x^2 + \dots + p_n' x^n)(1 - p_1 x + p_2 x^2 - \dots + (-1)^n p_n x^n)$$

up to x^{2n} must be equal, namely both the products must be identical. So we see that $P(x)$ is an even function. Hence we have either that

$$p_i = p_i', \quad i=1, 2, \dots, n$$

or that $1 + p_1 x + p_2 x^2 + \dots + p_n x^n$ possesses some pairs of zeros which have equal magnitudes and contrary signs. In the latter case, there is some relation among p_i 's.

Thus we see that (5) is a fundamental system.

V.

Now we proceed to prove the theorem of Vahlen, viz. the fact that the sequence (6) is a fundamental system.

In this case we are to start from the identity of the form

$$\log \frac{1+p_1x+p_2x^2+\dots+p_nx^n}{1+p_1'x+p_2'x^2+\dots+p_n'x^n} = t_vx^v + t_{2v}x^{2v} + \dots + t_{kv}x^{kv} + t_{\mu+1}x^{\mu+1} + t_{\mu+2}x^{\mu+2} + \dots \quad (17)$$

where the coefficients of

$$x^0, x^1, x^2, \dots, x^{v-1}, x^{v+1}, \dots, x^\mu$$

are all zero. Here evidently

$$\left[\frac{\mu}{v} \right] = k, \quad \mu = n + k. \quad (18)$$

Now, if we consider an aggregate of power series of the form

$$c_0 + c_1x^v + c_2x^{2v} + \dots + c_mx^{mv} + \dots, \quad (19)$$

then the sum, difference, product and quotient (where divisor does not vanish at origin) of any two functions of the aggregate must also belong to the aggregate. And the first n coefficients of the results are determined by the first n coefficients of the original two functions.

Hence the expansion

$$e^{-(t_vx^v+t_{2v}x^{2v}+\dots+t_{kv}x^{kv})} = 1 + r_vx^v + r_{2v}x^{2v} + \dots + r_{kv}x^{kv} + \dots$$

belongs to our aggregate. $r_v, r_{2v}, \dots, r_{kv}$ are determined by $t_v, t_{2v}, \dots, t_{kv}$ and conversely. Consequently we must have

$$\log(1 + r_vx^v + r_{2v}x^{2v} + \dots + r_{kv}x^{kv}) = -t_vx^v - t_{2v}x^{2v} - \dots - t_{kv}x^{kv} - \dots \quad (20)$$

Adding (17) and (20) side by side, we get an identity of the form

$$\begin{aligned} & \log \frac{(1+p_1x+p_2x^2+\dots+p_nx^n)(1+r_vx^v+r_{2v}x^{2v}+\dots+r_{kv}x^{kv})}{1+p_1'x+p_2'x^2+\dots+p_n'x^n} \\ & = \tau_{\mu+1}x^{\mu+1} + \dots \end{aligned} \quad (21)$$

Hence, by lemma, we must have the k coefficients of $x^{n+1}, x^{n+2}, \dots, x^\mu$ in

$$(1+p_1x+p_2x^2+\dots+p_nx^n)(1+r_vx^v+r_{2v}x^{2v}+\dots+r_{kv}x^{kv})$$

equal to the corresponding coefficients of

$$1 + p_1'x + p_2'x^2 + \dots + p_n'x^n,$$

namely they are all zero. Expressing this fact, we get k linear and homogeneous equations with respect to k numbers $r_v, r_{2v}, \dots, r_{kv}$, whose coefficients are functions of p_i 's. So we have in general

$$r_{iv} = 0, \quad i = 1, 2, \dots, k. \quad (22)$$

In that case (21) becomes

$$\log \frac{1 + p_1x + p_2x^2 + \dots + p_nx^n}{1 + p_1'x + p_2'x^2 + \dots + p_n'x^n} = \tau_{\mu+1}x^{\mu+1} + \dots \quad (23)$$

$$\mu \geq n,$$

and, by lemma, we have $p_i = p_i', \quad i = 1, 2, \dots, n.$

If there are any solution other than (22) of the said linear equations, then we must have an infinite number of solutions, which shows that there are an infinite number of systems $(p_1', p_2', \dots, p_n')$ satisfying the identity of the form (17). In this case we must have a relation among p_i 's.

Thus we see that (6) is a fundamental system.

VI.

We are now able to extend Vahlen's theorem.

Let a sequence which consists of zero and some of positive integers, having the nature that the sum of any two integers (they may be equal) of the sequence also belongs to the sequence, be called *additive sequence*.

It is very evident that an aggregate of the power series of the form

$$c_0 + c_1x^{\mu_1} + c_2x^{\mu_2} + \dots \quad (24)$$

where $(0, \mu_1, \mu_2, \dots)$ is an additive sequence, has the nature which is exactly same as the nature stated about the aggregate (19). Hence we get the following theorem by the same reasoning as before.

Theorem. *If $\nu_1, \nu_2, \dots, \nu_n$ are the first n positive integers which do not belong to an additive sequence, then*

$$S_{\nu_1}, S_{\nu_2}, \dots, S_{\nu_n}$$

form a fundamental system.

All multiples of an integer ν , in the theorem of Vahlen, evidently form an additive sequence, so that our theorem include that of Vahlen as a special case.

It is very evident that, in our case, the sequence consisting of all non-negative integers which are different from $\nu_1, \nu_2, \dots, \nu_n$ form an additive sequence. Hence our theorem may be enunciated with equal generality as follows.

Theorem. *If a sequence consisting of all non-negative integers which are different from the given n positive integers $\nu_1, \nu_2, \dots, \nu_n$ is additive, then $S_{\nu_1}, S_{\nu_2}, \dots, S_{\nu_n}$ form a fundamental system.*

VII.

By the way, we shall explain some precise nature of the additive sequence.

Let $0, \mu_1, \mu_2, \dots$ be an additive sequence. Take a number μ_1 . Then all multiples of μ_1 , namely all the numbers having the form

$$m_1 \mu_1, \quad (25)$$

where m_1 is a non-negative integer, must belong to the sequence. If there is any other number of the sequence which does not have the form (25), we take one of them, say μ_2 , then all the numbers of the form

$$m_1 \mu_1 + m_2 \mu_2 \quad (26)$$

where m_1 and m_2 are non-negative integers, must belong to the sequence. If d_1 is the greatest common divisor of μ_1 and μ_2 , then $\mu_1 > d_1$, since μ_2 is not a multiple of μ_1 . And the aggregate of the numbers of the form (26) (we may say simply the aggregate (26)) consists of numbers which are multiples of d_1 . It contains all the multiples of d_1 , greater than some fixed number.

For the solution of the indeterminate equation

$$m_1 \mu_1 + m_2 \mu_2 = m d_1$$

has the form

$$m_1 = m M_1 - p \mu_2, \quad (27)$$

$$m_2 = -m M_2 + p \mu_1,$$

where p is any integer and M_1, M_2 are such positive integers that

$$M_1\mu_1 - M_2\mu_2 = d_1. \quad (28)$$

In order that we can have m_1, m_2 which are not negative, it is necessary and sufficient that an integer p exists for which

$$m \frac{M_1}{\mu_2} \geq p \geq m \frac{M_2}{\mu_1}. \quad (29)$$

This is always possible for sufficiently large value of m , for

$$\frac{M_1}{\mu_2} > \frac{M_2}{\mu_1}$$

in virtue of (28).

If there is an infinite number of numbers of the sequence which does not belong to the aggregate (26), we can take one, say μ_i , which is not a multiple of d_1 , so that the greatest common divisor of d_1 and μ_i , say d_2 , is less than d_1 .

Consider the aggregate of the numbers of the form

$$m_1\mu_1 + m_2\mu_2 + m_3\mu_3. \quad (30)$$

It consists of the multiples of d_2 . Since the aggregate (26) contains all the numbers of the form

$$N_1d_1 + n_1d_1,$$

where N is a sufficiently large positive integer and n_1 is a non-negative integer, and the numbers of the form

$$n_1d_1 + m_3\mu_3$$

includes all the multiples of d_2 , as before, which are greater than some value, we see that the aggregate (30) includes all the multiples of d_2 , greater than some value.

If there is an infinite number of the numbers of the sequence, which does not belong to the aggregate (30), we can take one, say, μ_4 , which is not a multiple of d_2 , so that the greatest common divisor of μ_4 and d_2 , say d_3 , is less than d_2 .

Similary, as before, we can see that the aggregate of the numbers of the form

$$m_1\mu_1 + m_2\mu_2 + m_3\mu_3 + m_4\mu_4$$

consists of the multiples of d_3 , and it includes all the multiples of d_3 , greater than some value.

Proceeding in this way, we finally arrive at such an aggregate of the form

$$m_1 \mu_1 + m_2 \mu_2 + \dots + m_l \mu_l \quad (31)$$

that there is at most finite number of numbers of the sequence which does not belong to the aggregate. For the decreasing succession of the positive integers $\mu_1, d_1, d_2, d_3, \dots$ can not contain an infinite number of integers.

Now if we take all these finite number of integers of the sequence which do not belong to (31), say $\mu_{l+1}, \mu_{l+2}, \dots, \mu_k$, and construct the aggregate of the form

$$m_1 \mu_1 + m_2 \mu_2 + \dots + m_k \mu_k, \quad (32)$$

then evidently it is contained in the sequence and contains the sequence, so that it is identical with the sequence.

Conversely, it is evident that any aggregate of the form (32) forms an additive sequence.

We shall call, in (32), $\mu_1, \mu_2, \dots, \mu_k$ the bases of the sequence.

If one of the bases, say μ_k , is expressed in the form

$$\mu_k = n_1 \mu_1 + n_2 \mu_2 + \dots + n_{k-1} \mu_{k-1}, \quad (33)$$

then (32) is identical with the aggregate

$$m_1 \mu_1 + m_2 \mu_2 + \dots + m_{k-1} \mu_{k-1}, \quad (34)$$

having $k-1$ bases. In such a case we say that k bases $\mu_1, \mu_2, \dots, \mu_k$ are dependent.

We can evidently express the sequence by a system of independent bases. Such expression is unique. For if $\mu_1, \mu_2, \dots, \mu_k$ are independent bases of a sequence, which is such that

$$\mu_1 < \mu_2 < \dots < \mu_k,$$

then evidently μ_1 is the smallest number of the sequence. μ_2 is the smallest number of the sequence which does not belong to the aggregate $m_1 \mu_1$. μ_3 is the smallest number of the sequence, which does not belong to the aggregate $m_1 \mu_1 + m_2 \mu_2$. And so on. Thus each of $\mu_1, \mu_2, \dots, \mu_k$ is determined uniquely from the nature of the sequence. Thus we get the theorem :

Theorem. *Additive sequence is an aggregate with certain number of bases. The independent system of bases of an additive sequence is uniquely determined.*

An additive sequence consists of multiples of the greatest common divisor of the bases and contains all those multiples which are greater than some value.

VIII

Returning to original aim, there remains to consider the theorem of Ludwig. His theorem is a particular case of our theorem obtained in the paragraph 6.

If we take all numbers among

$$k+1, k+2, \dots, 2k+1,$$

which are different from

$$l_{k+1}, l_{k+2}, \dots, l_n \quad (35)$$

and denote them by

$$\mu_1, \mu_2, \dots, \mu_{2k+1-n}, \quad (36)$$

then the additive sequence with the bases (36) has the nature that the first n positive integers which do not belong to the sequence are

$$1, 2, \dots, k, l_{k+1}, l_{k+2}, \dots, l_n.$$

For, since all of (36) are greater than k , sum of any two or more numbers of (36) (they may be equal), namely the number of sequence different from any of (36), is greater than $2k+1$, so also is greater than any of (35).

But we shall give here a direct proof of the theorem.

To prove (7) is a fundamental system, we are to consider the identity

$$\log \frac{1+p_1x+p_2x^2+\dots+p_nx^n}{1+p_1'x+p_2'x^2+\dots+p_n'x^n} = t_1x+t_2x^2+\dots \quad (37)$$

where $t_i=0$ for $i=1, 2, \dots, k, l_{k+1}, l_{k+2}, \dots, l_n$.

From $t_i=0$, $i=1, 2, \dots, k$,

we get $p_i=p_i'$, $i=1, 2, \dots, k$. (38)

Hence

$$\log \frac{1+p_1x+p_2x^2+\dots+p_nx^n}{1+p_1'x+p_2'x^2+\dots+p_n'x^n} = \log \left\{ 1 + \frac{(p_{k+1}-p_{k+1}')x^{k+1}+\dots+(p_n-p_n')x^n}{1+p_1'x+p_2'x^2+\dots+p_n'x^n} \right\}$$

is a power series of

$$y = \frac{(p_{k+1} - p'_{k+1})x^{k+1} + \dots + (p_n - p'_n)x^n}{1 + p'_1x + p'_2x^2 + \dots + p'_nx^n}$$

of the form

$$y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \dots \quad (39)$$

Since y has a zero at origin, whose order is not less than $k+1$, each of $y^2, y^3, \dots$ has a zero at origin, whose order is not less than $2k+2$. Hence the coefficients of $x^{k+1}, x^{k+2}, \dots, x^{2k+1}$ in the expansion of (39) according to the power of x are equal to those of y .

So we see that from

$$t_i = 0, \quad i = l_{k+1}, l_{k+2}, \dots, l_n,$$

that the coefficients of $x^{l_{k+1}}, x^{l_{k+2}}, \dots, x^{l_n}$ in

$$y = \frac{(p_{k+1} - p'_{k+1})x^{k+1} + \dots + (p_n - p'_n)x^n}{1 + p'_1x + p'_2x^2 + \dots + p'_nx^n}$$

are all zero. Expressing this fact, we get $n-k$ linear homogeneous equations with respect to $n-k$ quantities

$$p_{k+1} - p'_{k+1}, p_{k+2} - p'_{k+2}, \dots, p_n - p'_n.$$

Thus, in general, we must have

$$p_{k+1} - p'_{k+1} = 0, \quad p_{k+2} - p'_{k+2} = 0, \dots, p_n - p'_n = 0. \quad (40)$$

In the exceptional case, where the discriminant $\Delta(p'_1, p'_2, \dots, p'_n)$ formed by the coefficients of the said equation vanishes, we have an infinite number of values of $p_{k+1} - p'_{k+1}, p_{k+2} - p'_{k+2}, \dots, p_n - p'_n$ so also of $p_{k+1}, p_{k+2}, \dots, p_n$. This shows that there exists some relation among $p'_1, p'_2, \dots, p'_n$.

Thus we see that (7) is a fundamental system.