8. On the Asymptotic Number of Latin Rectangles.

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Introduction.—Paul Erdős and Irving Kaplansky have proved in a celebrated paper (Amer. J. Math., **68** (1946), 230—236) the asymptotic relation for the number $f(n, k)$ of n by k Latin rectangles:

$$(1) \quad f(n, k) \sim (n!)^k \exp(-k(k-1)/2),$$

where

$$(2) \quad k < (\log n)^{2/2-\varepsilon}.$$

They expressed the conjecture that, making use of beginning terms of the asymptotic expansion, the same formula would be valid for k up to nearly $n^{1/3}$.

In the present paper the author wishes to confirm this conjecture, showing the validity of (1) for those values of k satisfying

$$(3) \quad k < n^{1/2-\delta},$$

where δ is a positive constant, or more generally, may be a positive-valued function of n tending to zero such that

$$(4) \quad n^{-\delta} \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

The method followed is a direct continuation and refinement of theirs. It seems interesting that our method has some application in related problems, e. g., in the *problème des ménages* beautifully solved by J. Touchard [2] and I. Kaplansky J. Riordan [1], and in a verification of A. Sade's enumeration [3] of 7-sided Latin squares.

1.—We may start from where Erdős and Kaplansky stopped.

Let L be an n by k Latin rectangle in the integers $1, 2, \dots, n$, and denote by N the number of ways of augmenting L to an n by $(k+1)$ Latin rectangle by adjoining the $(k+1)^{\text{th}}$ row to L . The Erdős-Kaplansky fundamental formula states that

$$(5) \quad N = \sum_{r=0}^n (-)^r \sum_{s=0}^r (-)^s \sum_t F(s, t) \binom{n-t}{r-t} k^{r-t} (n-r)!,$$

where their ultimate building block $F(s, t)$ denotes the number of ways of choosing s pairs of 1's, 2's, $\dots$, n 's which can be formed by using all of t elements in different columns of L . The summation without indication of variability domain

should be extended over the whole variability domain of the argument. We retain this convention throughout the present paper. The last formula may be transformed into

$$N = n! \sum_s (-)^s \sum_t (-)^t F(s, t) / (n)_t \sum_{u=0}^{n-t} (-k)^u / u!,$$

where $(n)_t = n!/(n-t)!$ is the Jordan factorial notation. We rewrite it as

$$(6) \quad N = n! \sum_{t=0}^n (-)^t G_t \sigma_{n-t} / (n)_t,$$

with

$$(7) \quad \sigma_m = \sum_{u=0}^m (-k)^u / u!$$

and

$$(8) \quad G_t = \sum_s (-)^s F(s, t).$$

Note that $\sigma_n \sim e^{-k}$ as $n \rightarrow \infty$ and that

$$(9) \quad G_0 = 1, \quad G_1 = 0.$$

In order to evaluate the other G_t , we make further investigation of the number $F(s, t)$. Let exactly u integers appear in a choice of s pairs, formed by all of t elements in different columns of L ; and let a_2 of these integers appear each in exactly two different columns, a_3 of them each in exactly three different columns, etc. Note that the maximal times of appearance of an integer is k . Each choice of s pairs in $F(s, t)$ thus belongs to a (restricted, non-unitary) bipartite partition π :

$$(10) \quad \begin{cases} t = 2a_2 + 3a_3 + \cdots + ka_k, \\ u = a_2 + a_3 + \cdots + a_k. \end{cases}$$

The number of choices of s pairs belonging to the partition π will be denoted by $H(s, \pi)$. And if we separate contributions

$$(11) \quad G(\pi) = \sum_s (-)^s H(s, \pi),$$

of individual partitions to the sum G_t , (8) may be written as

$$(12) \quad G_t = \sum_u \sum_{\pi} G(\pi).$$

Now the choice of t elements in u integers precedes the choice of s pairs, according to the partition π , but it is easy to see that the contributions to $H(s, \pi)$ and hence to $G(\pi)$, of each choice of elements is the same, say $h(s, \pi)$ and $g(\pi)$. In fact let in a choice of t elements, u integers $x_1, x_2, \cdots, x_u$ appear $c_1, c_2, \cdots, c_u$

times respectively, and a_2 of the c 's are equal to 2, a_3 of them to 3, $\dots$, a_k of them to k . Then we may divide the s pairs into the $s_1, \dots, s_u$ pairs in the integers $x_1, \dots, x_u$. Note that the s_i pairs use up the c_i elements in the integer x_i . Thus we have

$$h(s, \pi) = \sum_{s_1 + \dots + s_u = s} h(s_1, c_1) \cdots h(s_u, c_u),$$

and

$$(13) \quad g(\pi) = \sum_s (-)^s h(s, \pi) = \prod_{i=1}^u \sum_{s_i} (-)^{s_i} h(s_i, c_i) \\ = g(c_1) \cdots g(c_u) = (g(2))^{a_2} (g(3))^{a_3} \cdots (g(k))^{a_k},$$

where $g(m)$ is the special $g(\pi)$, where π is an elementary bipartite partition

$$(14) \quad \begin{cases} t = m = \sum_i \delta(m, i) i, \\ u = 1 = \sum_i \delta(m, i); \end{cases}$$

$\delta(m, i)$ denoting the Kronecker delta.

Now the numbers $g(m)$ are given very simply by the following combinatorial

LEMMA 1. $g(m) = (-)^{m-1} (m-1).$

PROOF. The number $g(m)$ is, by definition

$$g(m) = \sum_s (-)^s h(s, m),$$

where $h(s, m)$ is the number of ways of choosing s pairs using up all m elements given (whose integers are, in this case, the same). We apply the sieve process to obtain this number:

$$h(s, m) = \binom{\binom{m}{2}}{s} - \binom{m}{1} \binom{\binom{m-1}{2}}{s} + \cdots \\ = \sum_p (-)^p \binom{m}{p} \binom{\binom{m-p}{2}}{s}.$$

Changing order of summation we find

$$g(m) = \sum_p (-)^p \binom{m}{p} \sum_s (-)^s \binom{\binom{m-p}{2}}{s} \\ = \sum_p (-)^p \binom{m}{p} \delta\left(0, \binom{m-p}{2}\right).$$

The last factor is again the Kronecker delta, which vanishes except for $p = m-1$ and $p = m$, and we conclude the Lemma.

By Lemma 1 and (13), we obtain the exact values of $G(\pi)$ and G_i in (11),

(12), if only we know the number $J(\pi)$ of choices of t elements according to the bipartite pattern π .

$$(15) \quad \begin{aligned} G(\pi) &= (-)^u 1^{a_2} 2^{a_3} \cdots (k-1)^{a_k} J(\pi), \\ G_t &= \sum_u (-)^u \sum_{\pi} 1^{a_2} 2^{a_3} \cdots (k-1)^{a_k} J(\pi). \end{aligned}$$

This function $J(\pi)$ is the ultimate building block, from which we construct the exact value of N .

We take as an example the *problème des ménages*. In this problem $k=2$, and $J(\pi) = 0$ unless $\pi = (2^a)$, hence $G_t = 0$ unless $t = 2a$, $u = a$. In case $t = 2u = 2a$, $g(\pi) = (-1)^a$, $J(\pi) = \frac{n}{a} \binom{n-a-1}{a-1}$. Thus from (15) and (6) follows

$$(16) \quad \begin{aligned} G_t &= (-)^a \frac{n}{a} \binom{n-a-1}{a-1} && \text{for } t = 2a, \\ G_t &= 0 && \text{for odd } t\text{'s.} \\ \frac{N}{n!} &= \sum_{a=0}^{\lfloor n/2 \rfloor} (-)^a \frac{\sigma_{n-2a}}{a!(n-1)_a}. \end{aligned}$$

The formula is very simple and the most appropriate for the asymptotic expansion.

$$N/n! \sim e^{-2} \sum_{a=0}^{\lfloor n/2 \rfloor} (-)^a \frac{1}{a!(n-1)_a}$$

as given by Kaplansky-Riordan [1]. Moreover the generating function

$$\varphi_n(x) = \sum_{t=0}^n G_t x^{n-t}$$

is easily found

$$\varphi_n(x) = 2 \cos n\theta, \quad \theta = \cos^{-1}(x/2),$$

or

$$(17) \quad \varphi_n(x) = 2T_n(x/2)$$

with the Tchebycheff polynomial $T_n(x)$ of degree n . The last equation seems not fully recognized by Touchard [2] and Kaplansky-Riordan [1].

2.—We now proceed to study the asymptotic behavior of N . The ultimate building block $J(\pi)$ does, unlike $g(\pi)$, depend on the internal structure of the Latin rectangle L . For our purpose, however, it is sufficient to make use of the crude inequality

$$(18) \quad J(\pi) \leq \frac{(n)_u}{a_2! a_3! \cdots a_k!} \binom{k}{2}^{a_2} \binom{k}{3}^{a_3} \cdots \binom{k}{k}^{a_k}.$$

In fact the first factor of the right hand member represents the number of ways of choosing u integers, while the ensuing product represents the number of ways of

determining t elements (not necessarily in different columns), both after the prescribed pattern π . The exact value of $J(\pi)$ is constructed by the sieve process using this number as the first term.

Using this fundamental inequality and (15), (10) we arrive at the following inequality for G_t :

$$(19) \quad G_t \leq \sum_u \sum_{\pi} k^t (n)_u B(\pi),$$

where

$$(20) \quad B(\pi) = \frac{1}{a_2! a_3! \dots a_k!} \left(\frac{1}{2!}\right)^{a_2} \left(\frac{2}{3!}\right)^{a_3} \dots \left(\frac{k-1}{k!}\right)^{a_k}.$$

These numbers $B(\pi)$ are rather small. Indeed,

$$\text{LEMMA 2.} \quad \sum_t \sum_u \sum_{\pi} B(\pi) = e.$$

The summation should be extended over all non-restricted, non-unitary bipartite partitions π .

PROOF. The presented sum factors into

$$\sum_{a_i=0}^{\infty} \frac{1}{a_i!} \left(\frac{i-1}{i!}\right)^{a_i} = \exp\left(\frac{i-1}{i!}\right), \quad i = 2, 3, \dots;$$

because of the unrestrictedness of the bipartite partitions. The Lemma follows immediately from

$$\sum_{i=2}^{\infty} \frac{i-1}{i!} = 1.$$

Now it follows from (6) and (9) that

$$(21) \quad \left| N/n! - \sigma_n \right| \leq \sum_{t=2}^n k^t \sigma_{n-t} \sum_u \frac{(n-t)!}{(n-u)!} \sum_{\pi} B(\pi).$$

But it can be further transformed by making approximate substitution

$$(22) \quad \sigma_m = e^{-k} + \theta_m, \quad \left| \theta_m \right| < \frac{k^{m+1}}{(m+1)!}.$$

Thus

$$(23) \quad \left| N/n! - e^{-k} \right| \leq \frac{k^{n+1}}{(n+1)!} + \sum_{t=2}^n k^t \left(e^{-k} + \frac{k^{n-t+1}}{(n-t+1)!} \right) \sum_u \frac{(n-t)!}{(n-u)!} \sum_{\pi} B(\pi),$$

$$\left| Ne^k/n! - 1 \right| \leq \sum_{t=2}^n \sum_u \sum_{\pi} k^t \frac{(n-t)!}{(n-u)!} B(\pi) + e^k k^{n+1} \sum_t \sum_u \sum_{\pi} \frac{B(\pi)}{(n-u)!}.$$

Let us denote the two sums by S_1 and S_2 respectively, and show that for

$$(24) \quad k < n^{1/2-\varepsilon}, \quad \varepsilon: \text{positive constant},$$

both S_1 and S_2 tend to be small for sufficiently large n .

Indeed, recalling from (10) that $u \leq t/2$, we can estimate individual terms of S_1 as follows, making use of the Stirling's formula:

$$(25) \quad \begin{aligned} k^t \frac{(n-t)!}{(n-u)!} &\leq k^t \frac{(n-t)!}{(n-t/2)!} \leq k^t e^{t/2} (n-t/2)^{-t/2} \\ &\leq (2e)^{1/2} n^{-2\varepsilon} \leq 2e n^{-2\varepsilon}, \end{aligned}$$

for $2 \leq t \leq n$ and for sufficiently large n . (Factorial for the fractional argument suggests gamma function.) By Lemma 2 and the inequality above we obtain

$$S_1 \leq 2e^2 n^{-2\varepsilon},$$

if n is sufficiently large.

In the same manner we can estimate the individual terms of S_2 :

$$(26) \quad \begin{aligned} e^k k^{n+1}/(n-u)! &\leq e^k k^{n+1}/(n/2)! \\ &\leq ((2e)^{1/2} n^{-\varepsilon})^n \exp(n^{1/2}) < 2e n^{-2\varepsilon}, \end{aligned}$$

and hence

$$S_2 < 2e^2 n^{-2\varepsilon}$$

for sufficiently large n .

We have eventually proved

THEOREM 1. For $k < n^{1/2-\varepsilon}$, ε being positive constant, the inequality

$$|Ne^k/n! - 1| < cn^{-2\varepsilon}, \quad c: \text{absolute constant},$$

is valid for sufficiently large n .

REMARK. The ε need not remain constant as $n \rightarrow \infty$, but may be a positive-valued function of n such that

$$n^{-\varepsilon} \text{ tends to } 0 \text{ as } n \rightarrow \infty.$$

Our proof remains valid without any modification. For instance we may take

$$\varepsilon = \log^{(m)} n / \log n,$$

where m is a fixed positive integer and $\log^{(m)} n$ stands for the m times iterated logarithm of n .

3.—By the preceding Theorem we obtain immediately

THEOREM 2. For $k < n^{1/3-\delta}$, δ being positive constant, we have the asymptotic relation

$$f(n, k) \sim (n!)^k \exp(-k(k-1)/2)$$

for the number $f(n, k)$ of n by k Latin rectangles.

PROOF. We may take $\varepsilon = 1/6 + \delta$ in the previous Theorem and apply it for the first $k-1$ rows to obtain

$$f(n, 0) = 1,$$

$$1 - cn^{-2\epsilon} < e^i (n!)^{-1} f(n, i+1) / f(n, i) < 1 + cn^{-2\epsilon}$$

for sufficiently large n , and for $i = 0, 1, \dots, k-1$. Taking their product we have

$$(1 - cn^{-2\epsilon})^k < \exp(k(k-1)/2) (n!)^{-k} f(n, k) < (1 + cn^{-2\epsilon})^k.$$

The last member tends to 1, as is seen by taking its logarithm:

$$(27) \quad k \log(1 + cn^{-2\epsilon}) < cn^{-2.1/6+\delta} n^{1/3-\delta} = cn^{-3\delta},$$

for sufficiently large n . The same is valid for the first member, proving the Theorem.

REMARK. The same generalization as for the previous Theorem is immediate.

The limitations $k < n^{1/2-\epsilon}$ and $k < n^{1/3-\delta}$ for the two Theorems are the boundary of our method, as is readily seen from (25) and (27); but they seem very likely the "natural boundary" of our problem, as were observed by Erdős and Kaplansky.

References

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