8. On the Asymptotic Number of Latin Rectangles.

By Koichi YAMAMOTO.

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Introduction.—Paul Erdős and Irving Kaplansky have proved in a celebrated paper (Amer. J. Math., 68 (1946), 230—236) the asymptotic relation for the number $f(n, k)$ of $n$ by $k$ Latin rectangles:

\[(1) \quad f(n, k) \sim (n!)^k \exp(-k(k-1)/2),\]

where

\[(2) \quad k < (\log n)^{3/2-\varepsilon}.
\]

They expressed the conjecture that, making use of beginning terms of the asymptotic expansion, the same formula would be valid for $k$ up to nearly $n^{1/3}$.

In the present paper the author wishes to confirm this conjecture, showing the validity of (1) for those values of $k$ satisfying

\[(3) \quad k < n^{1/3-\delta},\]

where $\delta$ is a positive constant, or more generally, may be a positive-valued function of $n$ tending to zero such that

\[(4) \quad n^{-\delta} \to 0 \quad (as \ n \to \infty).
\]

The method followed is a direct continuation and refinement of theirs. It seems interesting that our method has some application in related problems, e.g., in the problème des ménages beautifully solved by J. Touchard [2] and I. Kaplansky J. Riordan [1], and in a verification of A. Sade's enumeration [3] of 7-sided Latin squares.

1.—We may start from where Erdős and Kaplansky stopped.

Let $L$ be an $n$ by $k$ Latin rectangle in the integers 1, 2, ..., $n$, and denote by $N$ the number of ways of augmenting $L$ to an $n$ by $(k+1)$ Latin rectangle by adjoining the $(k+1)^{st}$ row to $L$. The Erdős-Kaplansky fundamental formula states that

\[(5) \quad N = \sum_{r=0}^{n} (-1)^r \sum_{s=0}^{r} (-1)^s \sum_{t=0}^{s} F(s, t)(n-t)^{k-r}(n-r)!,\]

where their ultimate building block $F(s, t)$ denotes the number of ways of choosing $s$ pairs of 1's, 2's, ..., $n$'s which can be formed by using all of $t$ elements in different columns of $L$. The summation without indication of variability domain
should be extended over the whole variability domain of the argument. We retain this convention throughout the present paper. The last formula may be transformed into

$$N = n! \sum_{s=0}^{n} (-1)^s \sum_{t=0}^{n} (-1)^t F(s, t) / \binom{n}{t} \sum_{u=0}^{n-t} (-k)^u / u! ,$$

where $\binom{n}{t} = n! / (n-t)!$ is the Jordan factorial notation. We rewrite it as

$$N = n! \sum_{t=0}^{n} (-1)^t G_t \sigma_{n-t} / \binom{n}{t} ,$$

with

$$\sigma_m = \sum_{u=0}^{m} (-k)^u / u!$$

and

$$G_t = \sum_{s=0}^{n} (-1)^s F(s, t) .$$

Note that $\sigma_n \sim e^{-k}$ as $n \to \infty$ and that

$$G_0 = 1, \quad G_1 = 0 .$$

In order to evaluate the other $G_t$, we make further investigation of the number $F(s, t)$. Let exactly $u$ integers appear in a choice of $s$ pairs, formed by all of $t$ elements in different columns of $L$; and let $a_2$ of these integers appear each in exactly two different columns, $a_3$ of them each in exactly three different columns, etc. Note that the maximal times of appearance of an integer is $k$. Each choice of $s$ pairs in $F(s, t)$ thus belongs to a (restricted, non-unitary) bipartite partition $\pi$:

$$\begin{cases} t = 2a_2 + 3a_3 + \cdots + ka_k , \\ u = a_2 + a_3 + \cdots + a_k . \end{cases}$$

The number of choices of $s$ pairs belonging to the partition $\pi$ will be denoted by $H(s, \pi)$. And if we separate contributions

$$G(\pi) = \sum_{s=0}^{n} (-1)^s H(s, \pi) ,$$

of individual partitions to the sum $G_t$, (8) may be written as

$$G_t = \sum_{\pi} H(s, \pi) .$$

Now the choice of $t$ elements in $u$ integers precedes the choice of $s$ pairs, according to the partition $\pi$, but it is easy to see that the contributions to $H(s, \pi)$ and hence to $G(\pi)$, of each choice of elements is the same, say $h(s, \pi)$ and $g(\pi)$. In fact let in a choice of $t$ elements, $u$ integers $x_1, x_2, \ldots, x_u$ appear $c_1, c_2, \ldots, c_u$
times respectively, and \( a_2 \) of the \( c \)'s are equal to 2, \( a_3 \) of them to 3, \ldots, \( a_k \) of them to \( k \). Then we may divide the \( s \) pairs into the \( s_1, \ldots, s_u \) pairs in the integers \( x_1, \ldots, x_u \). Note that the \( s_i \) pairs use up the \( c_i \) elements in the integer \( x_i \). Thus we have

\[
h(s, \pi) = \sum_{s_1 + \cdots + s_u = s} h(s_1, c_1) \cdots h(s_u, c_u),
\]

and

\[
g(\pi) = \sum_s (-1)^s h(s, \pi) = \Pi_{i=1}^u (-1)^i h(s_i, c_i)
\]

\[
= g(c_1) \cdots g(c_u) = (g(2))^{s_2}(g(3))^{s_3} \cdots (g(k))^{s_k},
\]

where \( g(m) \) is the special \( g(\pi) \), where \( \pi \) is an elementary bipartite partition

\[
\begin{aligned}
\tau &= m = \sum_i \delta(m, i) i, \\
u &= 1 = \sum_i \delta(m, i).
\end{aligned}
\]

\( \delta(m, i) \) denoting the Kronecker delta.

Now the numbers \( g(m) \) are given very simply by the following combinatorial

**LEMMA 1.** \( g(m) = (-)^{m-1}(m-1) \).

**PROOF.** The number \( g(m) \) is, by definition

\[
g(m) = \sum_s (-1)^s h(s, m),
\]

where \( h(s, m) \) is the number of ways of choosing \( s \) pairs using up all \( m \) elements given (whose integers are, in this case, the same). We apply the sieve process to obtain this number:

\[
h(s, m) = \binom{m}{2} - \binom{m-1}{2} + \cdots
\]

\[
= \sum_p (-1)^p \binom{m}{p} \binom{m-p}{2}.
\]

Changing order of summation we find

\[
g(m) = \sum_p (-1)^p \binom{m}{p} \sum_s (-1)^s \binom{m-p}{2} \delta(0, \binom{m-p}{2}).
\]

The last factor is again the Kronecker delta, which vanishes except for \( p = m-1 \) and \( p = m \), and we conclude the Lemma.

By Lemma 1 and (13), we obtain the exact values of \( G(\pi) \) and \( G_t \) in (11),
(12), if only we know the number \( J(\pi) \) of choices of \( t \) elements according to the bipartite pattern \( \pi \).

\[
G(\pi) = (-)^n 1^{a_1} 2^{a_2} \cdots (k-1)^{a_k} J(\pi),
\]

\[
G_t = \sum_n (-)^n \sum_\pi 1^{a_1} 2^{a_2} \cdots (k-1)^{a_k} J(\pi).
\]

This function \( J(\pi) \) is the ultimate building block, from which we construct the exact value of \( N \).

We take as an example the problème des ménages. In this problem \( k=2 \), and \( J(\pi) = 0 \) unless \( \pi = (2^a) \), hence \( G_t = 0 \) unless \( t = 2a \), \( u = a \). In case \( t = 2a \), \( g(\pi) = (-1)^a \), \( J(\pi) = \frac{n}{a} \binom{n-a-1}{a-1} \). Thus from (15) and (6) follows

\[
G_t = (-)^a \frac{n}{a} \binom{n-a-1}{a-1} \quad \text{for } t = 2a,
\]

\[
G_t = 0 \quad \text{for odd } t's.
\]

\[
\frac{N}{n!} = \sum_{a=0}^{[n/2]} (-)^a \frac{\sigma_{n-2a}}{a!(n-1)_a}.
\]

The formula is very simple and the most appropriate for the asymptotic expansion

\[
N/n! \sim e^{-\sum_{a=0}^{[n/2]} (-)^a \frac{1}{a!(n-1)_a}}
\]

as given by Kaplansky-Riordan [1]. Moreover the generating function

\[
\varphi_n(x) = \sum_{t=0}^n G_t x^{n-t}
\]

is easily found

\[
\varphi_n(x) = 2 \cos n\theta, \quad \theta = \cos^{-1}(x/2),
\]

or

\[
\varphi_n(x) = 2T_n(x/2)
\]

with the Tchebycheff polynomial \( T_n(x) \) of degree \( n \). The last equation seems not fully recognized by Touchard [2] and Kaplansky-Riordan [1].

2. — We now proceed to study the asymptotic behavior of \( N \). The ultimate building block \( J(\pi) \) does, unlike \( g(\pi) \), depend on the internal structure of the Latin rectangle \( L \). For our purpose, however, it is sufficient to make use of the crude inequality

\[
J(\pi) \leq \frac{(n)_u}{a_1! a_2! \cdots a_k!} \left( \frac{k}{2} \right)^{a_1} \cdots \left( \frac{k}{3} \right)^{a_2} \cdots \left( \frac{k}{k} \right)^{a_k}.
\]

In fact the first factor of the right hand member represents the number of ways of choosing \( u \) integers, while the ensuing product represents the number of ways of
determining \( t \) elements (not necessarily in different columns), both after the pre
scribed pattern \( \pi \). The exact value of \( J(\pi) \) is constructed by the sieve process using
this number as the first term.

Using this fundamental inequality and (15), (10) we arrive at the following
inequality for \( G_t \):

\[
G_t \leq \sum_{u} \sum_{\pi} k^t (n) u B(\pi),
\]

where

\[
B(\pi) = \frac{1}{a_2! a_3! \cdots a_k!} \left( \frac{1}{2!} \right)^{a_2} \left( \frac{2}{3!} \right)^{a_3} \cdots \left( \frac{k-1}{k!} \right)^{a_k}.
\]

These numbers \( B(\pi) \) are rather small. Indeed,

**Lemma 2.** \[
\sum_{u} \sum_{\pi} B(\pi) = e.
\]

The summation should be extended over all non-restricted, non-unitary bipartite
partitions \( \pi \).

**Proof.** The presented sum factors into

\[
\sum_{i=0}^{\infty} \frac{1}{a_i!} \left( \frac{i-1}{i!} \right)^{a_i} = \exp \left( \frac{i-1}{i!} \right), \quad i = 2, 3, \ldots;
\]

because of the unrestrictedness of the bipartite partitions. The Lemma follows im-
mediately from

\[
\sum_{i=1}^{\infty} \frac{i-1}{i!} = 1.
\]

Now it follows from (6) and (9) that

\[
|N/n! - \sigma_n| \leq \sum_{t=2}^{n} k^t \sigma_{n-t} \sum_{u} \frac{(n-t)!}{(n-u)!} \sum_{\pi} B(\pi).
\]

But it can be further transformed by making approximate substitution

\[
\sigma_m = e^{-k} + \theta_m, \quad |\theta_m| < \frac{k^m}{(m+1)!}.
\]

Thus

\[
|N/n! - e^{-k}| \leq \frac{k^n}{(n+1)!} + \sum_{t=2}^{n} k^t \left( e^{-k} + \frac{k^{n-t+1}}{(n-t+1)!} \right) \sum_{u} \frac{(n-t)!}{(n-u)!} \sum_{\pi} B(\pi),
\]

\[
|Ne^k/n! - 1| \leq \sum_{t=2}^{n} \sum_{u} \sum_{\pi} k^t \left( \frac{(n-t)!}{(n-u)!} \right) B(\pi) + e^k \sum_{t=2}^{n} \sum_{u} \sum_{\pi} B(\pi).
\]

Let us denote the two sums by \( S_1 \) and \( S_2 \) respectively, and show that for

\[
k < n^{1/2-\varepsilon}, \quad \varepsilon: \text{positive constant},
\]

both \( S_1 \) and \( S_2 \) tend to be small for sufficiently large \( n \).
Indeed, recalling from (10) that \( u \leq t/2 \), we can estimate individual terms of \( S_1 \) as follows, making use of the Stirling’s formula:

\[
\frac{k^t (n-t)!}{(n-u)!} \leq k^t \frac{(n-t)!}{(n-t/2)!} \leq k^t e^{t/2} (n-t/2)^{t/2} \leq (2e)^{t/2} n^{-t} \leq 2e n^{-2},
\]

for \( 2 \leq t \leq n \) and for sufficiently large \( n \). (Factorial for the fractional argument suggests gamma function.) By Lemma 2 and the inequality above we obtain

\[
S_1 \leq 2e^2 n^{-2},
\]

if \( n \) is sufficiently large.

In the same manner we can estimate the individual terms of \( S_2 \):

\[
e^k \frac{k^{n+1}}{(n-u)!} \leq e^k \frac{k^{n+1}}{(n/2)!} \leq (2e)^{1/2} n^{-1} \exp(n^{1/2}) < 2e n^{-2},
\]

and hence

\[
S_2 < 2e^2 n^{-2}
\]

for sufficiently large \( n \).

We have eventually proved

**THEOREM 1.** For \( k < n^{1/2-\varepsilon} \), \( \varepsilon \) being positive constant, the inequality

\[
| N \varepsilon^k n! - 1 | < c n^{-2}, \quad c: \text{absolute constant},
\]

is valid for sufficiently large \( n \).

**REMARK.** The \( \varepsilon \) need not remain constant as \( n \to \infty \), but may be a positive-valued function of \( n \) such that

\[
n^{-\varepsilon} \text{ tends to 0 as } n \to \infty.
\]

Our proof remains valid without any modification. For instance we may take

\[
\varepsilon = \log(m) n / \log n,
\]

where \( m \) is a fixed positive integer and \( \log(m) n \) stands for the \( m \) times iterated logarithm of \( n \).

3.---By the preceding Theorem we obtain immediately

**THEOREM 2.** For \( k < n^{1/3-\delta} \), \( \delta \) being positive constant, we have the asymptotic relation

\[
f(n, k) \sim (n!)^k \exp(-k(k-1)/2)
\]

for the number \( f(n, k) \) of \( n \) by \( k \) Latin rectangles.

**PROOF.** We may take \( \varepsilon = 1/6 + \delta \) in the previous Theorem and apply it for the first \( k-1 \) rows to obtain

\[
f(n, 0) = 1,
\]
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for sufficiently large $n$, and for $i = 0, 1, \ldots, k-1$. Taking their product we have

\[(1 - cn^{-2i})^k < \exp \left( k \frac{(k-1)/2}{n} \right) \frac{n!}{k} f(n, k) < (1 + cn^{-2i})^k.\]

The last member tends to 1, as is seen by taking its logarithm:

\[k \log (1 + cn^{-2i}) < cn^{-2(1/6+\varepsilon)} n^{1/3-\delta} = cn^{-3\delta},\]

for sufficiently large $n$. The same is valid for the first member, proving the Theorem.

REMARK. The same generalization as for the previous Theorem is immediate.

The limitations $k < n^{1/2-\varepsilon}$ and $k < n^{1/3-\delta}$ for the two Theorems are the boundary of our method, as is readily seen from (25) and (27); but they seem very likely the "natural boundary" of our problem, as were observed by Erdös and Kaplansky.

References


Department of Mathematics,
Kanazawa University.