12. On a Theorem of Siegel.

By Tikao TATUZAWA.

(Received Nov. 15, 1951)

1. Introduction.

Let k be a positive integer. Let χ be a non-principal primitive real character to modulus k , and let $L(s, \chi)$ denote the corresponding L -series. We denote by $h = h(\pm k)$ the class number of the quadratic field with the fundamental discriminant $\pm k$. Then (see [2], p. 203) (numbers in square brackets refer to the references at the end of the paper), using Kronecker's symbol $\left(\frac{\pm k}{n}\right)$,

$$(1) \quad h(k) = \frac{\sqrt{k}}{2 \log \eta} \sum_{n=1}^{\infty} \left(\frac{k}{n}\right) \frac{1}{n}, \quad h(-k) = \frac{w \sqrt{k}}{2\pi} \sum_{n=1}^{\infty} \left(\frac{-k}{n}\right) \frac{1}{n},$$

where $\eta = \frac{a + b\sqrt{k}}{2} (> 1)$ is the fundamental unit and $w = 6, = 4$ and $= 2$ for $k = 3, = 4$ and > 4 respectively. Since $\eta - \eta^{-1} = a$ or $b\sqrt{k} \geq 1$ and $\eta + \eta^{-1} = \sqrt{(\eta - \eta^{-1})^2 + 4} \geq \sqrt{5}$, we obtain

$$\eta \geq \frac{1 + \sqrt{5}}{2}.$$

It is well known (see the proof of Lemma 4) that

$$(2) \quad \chi(n) = \left(\frac{\chi(-1)k}{n}\right).$$

Hence, by (1),

$$(3) \quad L(1, \chi) \geq 2 \log \frac{1 + \sqrt{5}}{2} \frac{1}{\sqrt{k}},$$

where the equality holds if and only if $\chi(n) = \left(\frac{5}{n}\right)$. The result is almost the same as the one obtained by Page [7].

Still deeper result of Siegel (see [10] and [1]) is stated as follows:

$$L(1, \chi) > \frac{c(\varepsilon)}{k^\varepsilon}$$

for any positive ε , where $c(\varepsilon)$ is a positive constant depending only on ε . But this constant cannot be given explicitly. In this paper, making considerable use of Estermann's method (see [1]), with a little additional argument, we shall replace his result by the following

Theorem 1. *For any positive ε ,*

$$L(1, \chi) > \frac{\varepsilon}{10 k^\varepsilon},$$

with one possible exception.

The constant $\frac{1}{10}$ of this theorem is not the best which can be obtained by our methods. In fact, our calculations are mainly devoted to prove the following one, which gives better result than the above theorem for the subsequent class number problem.

Theorem 2. *Let $\frac{1}{2} > \varepsilon > 0$ and $k \geq \text{Max}(e^{1/\varepsilon}, e^{11.2})$. Then*

$$L(1, \chi) > 0.655 \frac{\varepsilon}{k^\varepsilon},$$

with one possible exception.

For the imaginary quadratic field with the fundamental discriminant $-k$ where $k \geq \text{Max}(e^{1/\varepsilon}, e^{11.2})$, we obtain, from Theorem 2 and (1)

$$h > \frac{0.655 \varepsilon}{\pi} k^{1/2-\varepsilon},$$

except at most one field. On taking ε such that

$$(4) \quad \frac{1}{\varepsilon} - 2 \log \frac{1}{\varepsilon} \geq 2 \log \left(\frac{\pi e h_0}{0.655} \right), \quad \frac{1}{\varepsilon} \geq 11.2,$$

for a given positive $h_0 \geq 1$, we have

$$\begin{aligned} h &> \frac{0.655 \varepsilon}{\pi} k^{\varepsilon \left(\frac{1}{2\varepsilon} - 1 \right)} \geq \frac{0.655 \varepsilon}{\pi} k^{\varepsilon \log \frac{\pi h_0}{0.655 \varepsilon}} \\ &= \frac{0.655 \varepsilon}{\pi} \left(\frac{\pi h_0}{0.655 \varepsilon} \right)^{\varepsilon \log k} \geq h_0, \end{aligned}$$

with one possible exception, if $k \geq e^{1/\varepsilon}$. Our theorem is therefore effective for the problem investigated by Heilbronn and Linfoot [3] for $h_0 = 1$ and recently by Iseki [5] for the general case. Thus we conclude the following

Theorem 3. *Let h be the class number of imaginary quadratic field with the fundamental discriminant $-k$. If we take*

$$k \geq 2100h_0^2 \log^2(13h_0)$$

for a given positive $h_0 \geq 1$, then $h > h_0$ with one possible exception.

Proof. For $h_0 < 2$, the result is true by the argument of [3]. So we assume $h_0 \geq 2$. Now we put

$$A = \log \left(\frac{\pi e h_0}{0.655} \right)^2, \quad A_0 = \log \left(\frac{\pi e 2}{0.655} \right)^2$$

and

$$\frac{1}{\varepsilon} = A + 2 \log A + 1.12.$$

It follows that

$$\begin{aligned} \frac{1}{\varepsilon} - 2 \log \frac{1}{\varepsilon} &= A + 2 \log A + 1.12 - 2 \log (A + 2 \log A + 1.12) \\ &= A + 1.12 - 2 \log \frac{A + 2 \log A + 1.12}{A} \\ &\geq A + 1.12 - 2 \log \frac{A_0 + 2 \log A_0 + 1.12}{A_0} > A \end{aligned}$$

and

$$\frac{1}{\varepsilon} = A + 2 \log A + 1.12 \geq A_0 + 2 \log A_0 + 1.12 > 11.2.$$

Hence, by (4), the result is true if

$$k \geq \exp(A + 2 \log A + 1.12).$$

But the right hand side is equal to the left side of the following inequalities.

$$e^{1.12} \left(\frac{\pi e}{0.655} \right)^2 h_0^2 4 \log^2 \left(\frac{\pi e h_0}{0.655} \right) < 2085 h_0^2 \log^2(13.05 h_0) < 2100 h_0^2 \log^2(13 h_0).$$

We have still to prove Theorem 1 and Theorem 2. This is done in the last paragraph. Throughout the paper, I have used the mathematical tables of exponential functions [9] for calculations. Finally, I should like to express my warmest thanks to Prof. Suetuna and Mr. Iseki for their valuable advices.

2. General lemmas.

Here and in the following we denote by σ the real part of the complex variable s .

Lemma 1. *Let $0 < r < s_0$. On the circle $|s - s_0| = r$,*

$$\frac{|s|}{\sigma} \leq \frac{s_0}{\sqrt{s_0^2 - \tau^2}}.$$

This can be easily verified by the geometrical consideration, whose device is due to Mr. Iseki.

In what follows, K_1 and K_2 denote the circles

$$|s-2| = \frac{15}{8}$$

and

$$|s-s_0| = s_0 - \frac{1}{25}$$

respectively, where $1 < s_0 < \frac{113}{112}$.

Using the above lemma, we have

$$(5) \quad \frac{|s|}{\sigma} < 2.88 \quad \text{on } K_1,$$

$$(6) \quad \frac{|s|}{\sigma} < 3.61 \quad \text{on } K_2.$$

Lemma 2. For $\sigma > 0$,

$$\zeta(s) = \frac{s+1}{2(s-1)} - s \int_1^\infty \left(u - [u] - \frac{1}{2}\right) u^{-s-1} du.$$

This is obtained by the well known formula ([4], p. 26, (3))

$$\zeta(s) - \frac{1}{s-1} = 1 - s \int_1^\infty (u - [u]) u^{-s-1} du = \frac{1}{2} - s \int_1^\infty \left(u - [u] - \frac{1}{2}\right) u^{-s-1} du.$$

Hence we have

$$|\zeta(s)| \leq \frac{1}{2} \left| \frac{s+1}{s-1} \right| + \frac{1}{2} \frac{|s|}{\sigma},$$

for $\sigma > 0$, $s \neq 1$. Accordingly, by (5), (6) and the geometrical consideration,

$$(7) \quad |\zeta(s)| < \frac{1}{2} \left| \frac{31/8+1}{31/8-1} \right| + \frac{1}{2} 2.88 < 2.29 \quad \text{on } K_1,$$

$$(8) \quad |\zeta(s)(s-1)| < \frac{1}{2} \left(2 + \frac{24}{25} + \frac{1}{56} \right) + \frac{1}{2} \left(\frac{24}{25} + \frac{1}{56} \right) 3.61 < 3.26 \quad \text{on } K_2.$$

For the non-principal character χ , we write

$$d_n = \sum_{m=1}^n \chi(m), \quad M(\chi) = \max |d_n|.$$

Lemma 3. Let χ be the non-principal primitive real character to modulus k . Then

$$M(\chi) \leq \frac{1}{2\pi} \sqrt{k} (\log k + 2 \log \log k + 2 \log 2 + 6 + \frac{\pi}{\sqrt{k}}).$$

This is obtained by Pólya-Landau's method. For any positive integer n , we have, by [6],

$$M(\chi) \leq \frac{1}{2} + \frac{\sqrt{k}}{\pi} (\log n + 2) + \frac{2}{\pi} \frac{k \log k}{n+1}.$$

On taking $n = [2\sqrt{k} \log k]$, the result follows at once.

Since

$$1.1x > 2 \log x + 2 \log 2 + 6 + \pi e^{-x/2}$$

for $x \geq 11.2$, we have

$$(9) \quad M(\chi) < \frac{2.1}{2\pi} \sqrt{k} \log k < k^{0.618}.$$

for $k \geq e^{11.2}$.

Lemma 4. *Let χ_1 and χ_2 be the non-principal primitive real characters to moduli k_1 and k_2 respectively, and $\chi_1 \neq \chi_2$. Then*

$$M(\chi_1 \chi_2) \leq \frac{1}{2\pi} \sqrt{k_1 k_2} \left(\log(k_1 k_2) + 2 \log \log(k_1 k_2) + 2 \log 2 + 6 + \frac{\pi}{\sqrt{k_1 k_2}} \right).$$

In [11], Suetuna has remarked that the modulus of the non-principal primitive real character χ is either of P , $4P$, or $8P$ where P is the product of odd distinct primes. Hence we can easily deduce that

$$(10) \quad \chi(n) = \begin{cases} \left(\frac{n}{P}\right) & \text{for } k = P, \\ \psi_1(n) \left(\frac{n}{P}\right) & \text{for } k = 4P, \\ \psi_2(n) \left(\frac{n}{P}\right) \text{ or } \psi_3(n) \left(\frac{n}{P}\right) & \text{for } k = 8P, \end{cases}$$

where

$$\psi_1(n) = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{for } 2 \nmid n, \\ 0 & \text{for } 2 \mid n, \end{cases} \quad \psi_2(n) = \begin{cases} (-1)^{\frac{n^2-1}{8}} & \text{for } 2 \nmid n, \\ 0 & \text{for } 2 \mid n, \end{cases}$$

and

$$\psi_3(n) = \psi_1(n) \psi_2(n),$$

whose conductors are 4, 8 and 8 respectively. Hence we can easily verify that the

quotient $\frac{k_1 k_2}{k_0}$ is the perfect square, where k_0 is the conductor of $\chi_1 \chi_2$. Consequently, by Landau's argument (see [6]),

$$\frac{M(\chi_1 \chi_2)}{\sqrt{k_1 k_2}} \leq \frac{M(\chi_0)}{\sqrt{k_0}},$$

where χ_0 is the primitive character corresponding to $\chi_1 \chi_2$. Then the result follows immediately by Lemma 3. Finally, we note that (2) follows from (10) by the law of reciprocity. (see also [11]).

Since

$$0.61x > 2 \log x + 2 \log 2 + 6 + \pi e^{-x/2}$$

for $x \geq 22.4$, we have

$$(11) \quad M(\chi_1 \chi_2) < \frac{1.61}{2\pi} \sqrt{k_1 k_2} \log(k_1 k_2) < (k_1 k_2)^{0.58}$$

for $k_1, k_2 \geq e^{11.2}$.

Lemma 5. *Let χ be the real non-principal character to modulus k . Then*

$$|L(s, \chi)| \leq \frac{|s|}{\sigma} \sum_{n=1}^{M(\chi)} n^{-\sigma}$$

for $\sigma > 0$.

Noting that

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \sum_{n=1}^{\infty} (d_n - d_{n-1}) n^{-s} = \sum_{n=1}^{\infty} d_n (n^{-s} - (n+1)^{-s})$$

and

$$|n^{-s} - (n+1)^{-s}| = \left| \int_n^{n+1} s u^{-s-1} du \right| \leq |s| \int_n^{n+1} u^{-\sigma-1} du = \frac{|s|}{\sigma} (n^{-\sigma} - (n+1)^{-\sigma})$$

for $\sigma > 0$, we obtain

$$\begin{aligned} |L(s, \chi)| &\leq \left(\sum_{n=1}^{m-1} + \sum_{n=m}^{\infty} \right) \frac{|s|}{\sigma} d_n (n^{-\sigma} - (n+1)^{-\sigma}) \\ &\leq \frac{|s|}{\sigma} \sum_{n=1}^{m-1} n (n^{-\sigma} - (n+1)^{-\sigma}) + \frac{|s|}{\sigma} \sum_{n=m}^{\infty} M(\chi) (n^{-\sigma} - (n+1)^{-\sigma}) \\ &= \frac{|s|}{\sigma} \left(\sum_{n=1}^m n^{-\sigma} - m^{1-\sigma} + M(\chi) m^{-\sigma} \right), \end{aligned}$$

for any positive integer m . The result follows by taking $m = M(\chi)$. It should be noticed that $\sum_{n=1}^{M(\chi)} n^{-\sigma}$ is steadily decreasing with σ .

As an immediate consequence of Lemma 5, we have

$$(12) \quad |L(s, \chi)| < \frac{|s|}{\sigma} \frac{M(\chi)^{1-\sigma}}{1-\sigma}$$

for $0 < \sigma < 1$. Further we have

$$(13) \quad L(1, \chi) \leq \sum_{n=1}^{M(\chi)} n^{-1} < \log M(\chi) + C + \frac{1}{2M(\chi)},$$

where C is Euler's constant. (see [8], p. 197).

It follows from (5), (6), (9), (12) and (13) that

$$(14) \quad |L(s, \chi)| < 3.3 k^{0.541} \quad \text{on } K_1,$$

$$(15) \quad |L(s, \chi)| < 3.8 k^{0.594} \quad \text{on } K_2,$$

$$(16) \quad L(1, \chi) < 0.67 \log k,$$

if χ is the character of Lemma 3 and $k \geq e^{11.2}$.

It follows from (5), (6), (11), (12) and (13) that

$$(17) \quad |L(s, \chi_1 \chi_2)| < 3.3 (k_1 k_2)^{0.598} \quad \text{on } K_1,$$

$$(18) \quad |L(s, \chi_1 \chi_2)| < 3.8 (k_1 k_2)^{0.557} \quad \text{on } K_2,$$

$$(19) \quad L(1, \chi_1 \chi_2) < 0.606 \log (k_1 k_2),$$

if $\chi_1 \chi_2$ is the character of Lemma 4 and $k_1, k_2 \geq e^{11.2}$.

Lemma 6. *If $H(s)$ is regular, $H(s_0) \neq 0$, $\left| \frac{H(s)}{H(s_0)} \right| < e^M$ in the circle $|s - s_0| \leq r$, and has no zeros in the right half of this circle, then*

$$-\Re \frac{H'}{H}(s_0) + \sum_{\rho} \Re \frac{1}{s_0 - \rho} < \frac{2M}{r} \left(1 + \left(\frac{r}{r_0} \log \frac{r}{r_0} \right)^{-1} \right),$$

where ρ runs over the zeros of $H(s)$ in $|s - s_0| \leq r_0 < r$.

Let ξ be any zero of $H(s)$ in the circle $|s - s_0| < r$. Since

$$(20) \quad \left| \frac{r^2 - (s - s_0)(\overline{\xi - s_0})}{s - \xi} \right| = r$$

on the circle $|s - s_0| = r$, we can deduce that

$$(21) \quad f(s) = H(s) \prod_{\xi} \frac{r^2 - (s - s_0)(\overline{\xi - s_0})}{s - \xi}$$

is regular and $f(s) \neq 0$ in the circle $|s - s_0| < r$, and

$$\left| \frac{f(s)}{f(s_0)} \right| = \left| \frac{H(s)}{H(s_0)} \right| \prod_{\xi} \frac{|s_0 - \xi|}{r} \leq \left| \frac{H(s)}{H(s_0)} \right| < e^M$$

on the circle. This inequality of course holds inside the circle. Hence, by Carathéodory's theorem ([4], p. 50, Theorem E)

$$\left| \frac{f'}{f}(s_0) \right| < \frac{2M}{r}.$$

This shows that

$$(22) \quad -\Re \frac{H'}{H}(s_0) + \sum_{\xi} \Re \left(\frac{1}{s_0 - \xi} - \frac{s_0 - \xi}{r^2} \right) < \frac{2M}{r},$$

by (21). Hence, noting that

$$\Re \left(\frac{1}{s_0 - \xi} - \frac{s_0 - \xi}{r^2} \right) = \Re(s_0 - \xi) \left(\frac{1}{|s_0 - \xi|^2} - \frac{1}{r^2} \right) > 0,$$

we obtain

$$(23) \quad -\Re \frac{H'}{H}(s_0) + \sum_p \Re \frac{1}{s_0 - \rho} < \frac{2M}{r} + \sum_p \Re \frac{s_0 - \rho}{r^2} \leq \frac{2M}{r} + \frac{r_0}{r^2} n,$$

by (22), where n is the number of ρ .

We write

$$g(s) = H(s) \prod_p \frac{r^2 - (s - s_0)(\overline{\rho - s_0})}{r(s - \rho)}.$$

On the circle $|s - s_0| = r$, $|g(s)| = |H(s)| < |H(s_0)| e^M$ by (20).

Since this inequality holds in the circle $|s - s_0| \leq r$ and

$$|g(s_0)| = |H(s_0)| \prod_p \frac{r}{|s_0 - \rho|}, \text{ we obtain}$$

$$\prod_p \frac{r}{|s_0 - \rho|} < e^M.$$

This, combined with the inequality $\left(\frac{r}{r_0}\right)^n \leq \prod_p \frac{r}{|s_0 - \rho|}$, gives

$$n < M \left(\log \left(\frac{r}{r_0} \right) \right)^{-1},$$

from which the result follows by (23).

3. Fundamental Lemmas.

We assume henceforth that χ, χ_1 and χ_2 are the primitive real characters with moduli k, k_1 and k_2 respectively and further k, k_1 and $k_2 \geq e^{11.2}$. We define

$$F(s) = \zeta(s)L(s, \chi), \quad G(s) = \zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_1\chi_2).$$

If we write

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad G(s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

for $\sigma > 1$, and

$$F(s) = \sum_{m=0}^{\infty} a_m^* (2-s)^m, \quad G(s) = \sum_{m=0}^{\infty} b_m^* (2-s)^m,$$

for $|s-2| < 1$, then we have the relation

$$a_m^* = \sum_{n=1}^{\infty} \frac{a_n \log^m n}{n^2 m!}, \quad b_m^* = \sum_{n=1}^{\infty} \frac{b_n \log^m n}{n^2 m!}.$$

It is well known (see [1]) that

$$a_n, b_n \geq \begin{cases} 0 & \text{always,} \\ 1 & \text{if } n \text{ is a perfect square.} \end{cases}$$

Therefore

$$(24) \quad \sum_{r=0}^{n-1} a_r^* (2-s)^r, \sum_{r=0}^{n-1} b_r^* (2-s)^r \geq \sum_{r=0}^{n-1} \sum_{m=1}^{\infty} \frac{(2 \log m)^r}{m^4 r!} (2-s)^r.$$

Now we denote by $E(n, s)$ the right hand side of (24).

Lemma 7. *If $\frac{1}{2} < s < 1$, then*

$$E(n, s) > \zeta(2s) - \frac{1}{(2s-1)l^{2s-1}} - \frac{\zeta(2s)-1}{\sqrt{2n\pi}} \left(\frac{2e(2-s) \log l}{n} \right)^n,$$

for any positive integer l .

Noting that

$$\sum_{m=0}^{n-1} \frac{x^m}{m!} > e^x \left(1 - \frac{x^n}{n!} \right) > e^x \left(1 - \frac{1}{\sqrt{2n\pi}} \left(\frac{ex}{n} \right)^n \right)$$

for positive x , and inverting the order of the summations, we obtain

$$\begin{aligned} E(n, s) &> \sum_{r=0}^{n-1} \sum_{m=1}^l = \sum_{m=1}^l \sum_{r=0}^{n-1} \frac{(2 \log m)^r}{m^4 r!} (2-s)^r \\ &> 1 + \sum_{m=2}^l \frac{\exp(2(2-s) \log m)}{m^4} \left(1 - \frac{1}{\sqrt{2n\pi}} \left(\frac{2e(2-s) \log m}{n} \right)^n \right) \\ &> \sum_{m=1}^l m^{-2s} - \left(\sum_{m=2}^l m^{-2s} \right) \frac{1}{\sqrt{2n\pi}} \left(\frac{2e(2-s) \log l}{n} \right)^n. \end{aligned}$$

The result follows at once.

Lemma 8. *Let $0 < \varepsilon < \frac{1}{2}$ and let $L(1, \chi) \leq 0.376 \frac{\varepsilon}{k^\varepsilon}$. Then $L(s, \chi)$ has at least one real zero in the interval*

$$1 - \frac{\varepsilon}{4} < s < 1.$$

Proof. On the circle K_1

$$|F(s)| < 7.56 k^{0.541}$$

by (7) and (14),

$$\left| \frac{L(1, \chi)}{s-1} \right| < 0.77 \log k < 0.04 k^{0.541}$$

by (16). Hence, in the circle $|s-2| \leq \frac{15}{8}$,

$$(25) \quad \left| F(s) - \frac{L(1, \chi)}{s-1} \right| < 7.6 k^{0.541}.$$

By Taylor's theorem

$$(26) \quad F(s) - \frac{L(1, \chi)}{s-1} = \sum_{m=0}^{\infty} (a_m^* - L(1, \chi))(2-s)^m$$

for $|s-2| < 1$. But since the left hand side is regular, (26) must hold for $|s-2| < 2$, $s \neq 1$. Hence, by Cauchy's inequality

$$(27) \quad |a_m^* - L(1, \chi)| < 7.6 k^{0.541} \left(\frac{8}{15} \right)^m.$$

It follows from (24), (26), (27) and Lemma 7 that

$$(28) \quad \begin{aligned} F(s) &> \frac{L(1, \chi)}{s-1} + \sum_{m=0}^{n-1} a_m^* (2-s)^m - L(1, \chi) \frac{(2-s)^n - 1}{1-s} \\ &\quad - 7.6 k^{0.541} \left(\frac{8}{15} \right)^n (2-s)^n \left\{ 1 - \frac{8}{15} (2-s) \right\}^{-1} \\ &> E(n, s) - L(1, \chi) \frac{(2-s)^n}{1-s} - 19 k^{0.541} \left(\frac{8}{15} \right)^n (2-s)^n, \end{aligned}$$

for any positive integer n , if $\frac{7}{8} < s < 1$. On taking

$$n = [2 \log k],$$

and noting that

$$2 \log k \geq n > 2 \log k - 1, \quad n \geq 22,$$

we obtain

$$\begin{aligned} \left(\frac{8}{15} \right)^n &< \left(\frac{8}{15} \right)^{2 \log k - 1} = \frac{15}{8} k^{-2 \log(15/8)} < \frac{15}{8} k^{-1.256}, \\ (2-s)^n &\leq (2-s)^{(2 \log k)} = k^{2 \log(3-s)} < k^{2(1-s)}. \end{aligned}$$

Further, on taking $l = 35$ in Lemma 7,

$$\begin{aligned} E(n, s) &> E(n, 1) \geq E(22, 1) > \frac{\pi^2}{6} - \frac{1}{35} - \frac{0.645}{\sqrt{44\pi}} \left(\frac{2e \log 35}{22} \right)^{22} \\ &> \frac{\pi^2}{6} - \frac{1}{35} - 0.0034 > 1.612. \end{aligned}$$

Inserting these results in (28), we get

$$F(s) > 1.612 - 19 k^{2(1-s)-0.715} - \frac{L(1, \chi)}{1-s} k^{2(1-s)}$$

for $\frac{7}{8} < s < 1$, whence

$$(29) \quad F\left(1 - \frac{\varepsilon}{4}\right) > 1.612 - 19 k^{\varepsilon/2 - 0.715} - L(1, \chi) 4 \varepsilon^{-1} k^{2(1-s)},$$

provided that $0 < \varepsilon < \frac{1}{2}$.

If $L(1, \chi) \leq 0.376 \frac{\varepsilon}{k^\varepsilon}$, then

$$F\left(1 - \frac{\varepsilon}{4}\right) > 1.612 - 19 k^{0.25 - 0.715} - 1.504 > 0.108 - 19 e^{-0.465 \times 11.2} > 0$$

by (29). The result is obvious, since $\lim_{s \rightarrow 1-0} F(s) = -\infty$.

Lemma 9. Let $0 < \varepsilon < \frac{1}{2}$, $k \geq e^{1/\varepsilon}$, and let $L(1, \chi) \leq 0.655 \frac{\varepsilon}{k^\varepsilon}$.

Then $L(s, \chi)$ has at least one real zero in the interval

$$1 - \frac{\varepsilon}{4} < s < 1.$$

Proof. We use also (29) in the previous lemma. If

$$L(1, \chi) \leq 0.655 \frac{\varepsilon}{k^\varepsilon},$$

then

$$(30) \quad F\left(1 - \frac{\varepsilon}{4}\right) > 1.612 - 19 k^{\varepsilon/2 - 0.715} - 2.62 k^{-\varepsilon/2}.$$

Noting that as a function of x

$$19 k^{-0.715} x + 2.62 x^{-1}$$

is convex for $x > 0$, we obtain

$$\begin{aligned} 19 k^{\varepsilon/2 - 0.715} + 2.62 k^{-\varepsilon/2} &\leq \text{Max}(19 k^{0.25 - 0.715} + 2.62 k^{-0.25}, 19 e^{0.5} k^{-0.715} + 2.62 e^{-0.5}) \\ &= 19 e^{0.5} k^{-0.715} + 2.62 e^{-0.5}, \end{aligned}$$

since $k \geq e^{1/\varepsilon}$ and $0 < \varepsilon < \frac{1}{2}$.

It follows therefore from (30) that

$$\begin{aligned} F\left(1 - \frac{\varepsilon}{4}\right) &> 1.612 - 19 e^{0.5} k^{-0.715} - 2.62 e^{-0.5} \\ &> 1.612 - 19 e^{-7.5} - 2.62 e^{-0.5} > 0. \end{aligned}$$

Hence we get the required result by the same argument as in the previous lemma.

Lemma 10. For $\frac{7}{8} < s < 1$,

$$G(s) > 1.637 - L(1, \chi_1)L(1, \chi_2)L(1, \chi_1\chi_2)\frac{(k_1k_2)^{3(1-s)}}{1-s}.$$

Proof. On the circle K_1

$$\begin{aligned} |G(s)| &< 2.29(3.3k_1^{0.541})(3.3k_2^{0.541})(3.3(k_1k_2)^{0.508}) \\ &< 82.8(k_1k_2)^{1.049} \end{aligned}$$

by (7), (14) and (17). If we write

$$L = L(1, \chi_1)L(1, \chi_2)L(1, \chi_1\chi_2),$$

then

$$\begin{aligned} \left| \frac{L}{s-1} \right| &< \frac{8}{7}(0.67 \log k_1)(0.67 \log k_2)(0.606 \log(k_1k_2)) \\ &< 0.2(k_1k_2)^{1.049}. \end{aligned}$$

By the same argument as in Lemma 8, we have

$$(31) \quad G(s) > E(n, s) - L\frac{(2-s)^n}{1-s} - 208(k_1k_2)^{1.049}\left(\frac{3}{5}\right)^n,$$

for any positive integer n and $\frac{7}{8} < s < 1$. On taking

$$n = [3 \log(k_1k_2)],$$

and noting that

$$3 \log(k_1k_2) \geq n > 3 \log(k_1k_2) - 1, \quad n \geq 67,$$

we obtain

$$\left(\frac{3}{5}\right)^n < \left(\frac{3}{5}\right)^{3 \log(k_1k_2)-1} = \frac{5}{3}(k_1k_2)^{-3 \log(5/3)} < \frac{5}{3}(k_1k_2)^{-1.53},$$

$$(2-s)^n \leq (2-s)^{3 \log(k_1k_2)} = (k_1k_2)^{3 \log(2-s)} < (k_1k_2)^{3(1-s)}.$$

Further, on taking $l = [e^{10}]$ in Lemma 7,

$$\begin{aligned} E(n, s) &> E(n, 1) \geq E(67, 1) \\ &> \frac{\pi^2}{6} - \frac{1}{2.10^4} - \frac{0.645}{\sqrt{132}\pi} \left(\frac{20e}{67}\right)^{67} > 1.6448. \end{aligned}$$

Inserting these results in (31), we get

$$\begin{aligned} G(s) &> 1.6448 - 347(k_1k_2)^{-0.481} - \frac{L}{1-s}(k_1k_2)^{3(1-s)} \\ &> 1.637 - \frac{L}{1-s}(k_1k_2)^{3(1-s)}. \end{aligned}$$

This is the desired result.

Lemma 11. $G(s)$ has at most two real zeros in the interval

$$1 - \frac{1}{8 \log(k_1 k_2)} < s < 1.$$

Proof. Let

$$s_0 = 1 + \frac{1}{5 \log(k_1 k_2)}$$

and let $H(s) = (s-1)G(s)$. On the circle K_2 ,

$$\begin{aligned} |H(s)| &< 3.26(3.8 k_1^{0.594})(3.8 k_2^{0.594})(3.8(k_1 k_2)^{0.557}) \\ &< 180(k_1 k_2)^{1.151} < (k_1 k_2)^{1.383}, \end{aligned}$$

by (8), (15) and (18). Noting that

$$\frac{1}{s_0 - 1} < \zeta(s_0) < \frac{s_0}{s_0 - 1},$$

we have

$$\begin{aligned} \left| \frac{1}{H(s_0)} \right| &< \zeta(s_0)^3 < \frac{s_0^3}{(s_0 - 1)^3} < \left(1 + \frac{1}{112}\right)^3 5^3 \log^3(k_1 k_2) \\ &< (k_1 k_2)^{0.634}. \end{aligned}$$

Hence, in the circle $|s - s_0| \leq s_0 - \frac{1}{25}$,

$$\left| \frac{H(s)}{H(s_0)} \right| < (k_1 k_2)^{2.017}.$$

It follows therefore from Lemma 6 that

$$\begin{aligned} & -\frac{1}{s_0 - 1} - \Re \frac{\zeta'}{\zeta}(s_0) - \Re \frac{L'}{L}(s_0, \chi_1) - \Re \frac{L'}{L}(s_0, \chi_2) - \Re \frac{L'}{L}(s_0, \chi_1 \chi_2) \\ (32) \quad & + \sum_{\rho} \Re \frac{1}{s_0 - \rho} < 4.034 \frac{25}{24} \left(1 + \left(2 \frac{24}{25 r_0} \log \frac{24}{25 r_0}\right)^{-1}\right) \log(k_1 k_2), \end{aligned}$$

where ρ runs over the zero in the circle $|s - s_0| \leq r$. On taking

$$(33) \quad r_0 = \left(\frac{1}{5} + \frac{1}{8}\right) \frac{1}{\log(k_1 k_2)},$$

and noting that

$$0 \leq -\Re \frac{\zeta'}{\zeta}(s_0) - \Re \frac{L'}{L}(s_0, \chi_1) - \Re \frac{L'}{L}(s_0, \chi_2) - \Re \frac{L'}{L}(s_0, \chi_1 \chi_2),$$

we get from (32)

$$\begin{aligned} \sum_{\rho} \Re \frac{1}{s_0 - \rho} &< \left(5 + 4.21 + \frac{4.21}{128 \log 64}\right) \log(k_1 k_2) \\ (34) \quad &< 9.22 \log(k_1 k_2), \end{aligned}$$

since

$$r_0 < \left(\frac{1}{5} + \frac{1}{8} \right) \frac{1}{22.4} < \frac{3}{200}.$$

If $H(s)$ has at least three real zeros in the interval

$$1 - \frac{1}{8 \log(k_1 k_2)} < s < 1$$

then we have

$$\frac{3}{r_0} < 9.22 \log(k_1 k_2)$$

by (34). But this is a contradiction by (33). Hence we get the desired result.

4. Proof of the main theorem.

In the first place we shall prove Theorem 1. Noting that $\frac{k^\varepsilon}{\varepsilon}$ is a convex function of positive ε and attains its minimum value $e \log k$ at $\varepsilon = \frac{1}{\log k}$, and using (3), we obtain

$$L(1, \chi) > \frac{0.96}{\sqrt{k}} \geq \frac{0.96}{\sqrt{k}} e \log k \frac{\varepsilon}{k^\varepsilon} > 0.96 e \frac{11.2}{e^{5.6}} \frac{\varepsilon}{k^\varepsilon} > \frac{\varepsilon}{10k^\varepsilon},$$

for $0 < \varepsilon$ and $k < e^{11.2}$. Further, following the above remark, we see that the proof for $\frac{1}{2} \leq \varepsilon$ and $e^{11.2} \leq k$ is not necessary. Finally, we must prove the theorem for $0 < \varepsilon < \frac{1}{2}$ and $e^{11.2} \leq k$. In this case, the proof is similar to the one for Theorem 2, for which we shall work out all details in the sequel. We only replace Lemma 9 by Lemma 8. So we shall not dwell longer upon it.

Now we proceed to prove Theorem 2. Suppose that

$$(35) \quad L(1, \chi_1) \leq 0.655 \frac{\varepsilon}{k_1^\varepsilon}, \quad L(1, \chi_2) \leq 0.655 \frac{\varepsilon}{k_2^\varepsilon},$$

for $\chi_1 \neq \chi_2$, $0 < \varepsilon < \frac{1}{2}$ and $k_1, k_2 \geq \text{Max}(e^{1/\varepsilon}, e^{11.2})$. Then, by Lemma 9, $L(s, \chi_1)$ and $L(s, \chi_2)$ have real zeros in the interval

$$1 - \frac{\varepsilon}{4} < s < 1,$$

whence $G(s)$ must have at least two real zeros in this interval. For brevity we write

$$s_1 = 1 - \frac{1}{8 \log(k_1 k_2)}, \quad s_2 = 1 - \frac{\varepsilon}{4}.$$

Noting that $\frac{(k_1 k_2)^\varepsilon}{\varepsilon^2} \geq \frac{e^2}{4} \log^2(k_1 k_2)$, and using (19) and (35), we obtain

$$\begin{aligned} & L(1, \chi_1) L(1, \chi_2) L(1, \chi_1 \chi_2) \frac{(k_1 k_2)^{3(1-s_1)}}{1-s_1} \\ & < (0.655)^2 \frac{\varepsilon^2}{(k_1 k_2)^\varepsilon} 0.606 \log(k_1 k_2) \{8 \log(k_1 k_2) e^{3/8}\} \\ (36) \quad & < 8 (0.655)^2 0.606 e^{3/8} \frac{4}{e^2} < 1.637. \end{aligned}$$

Noting that $(k_1 k_2)^{\varepsilon/4} \left/ \frac{\varepsilon}{4} \right. \geq e \log(k_1 k_2)$, and using (19) and (35), we obtain

$$\begin{aligned} & L(1, \chi_1) L(1, \chi_2) L(1, \chi_1 \chi_2) \frac{(k_1 k_2)^{3(1-s_2)}}{1-s_2} \\ & < (0.655)^2 \frac{\varepsilon^2}{(k_1 k_2)^\varepsilon} 0.606 \log(k_1 k_2) \left\{ \frac{4}{\varepsilon} (k_1 k_2)^{3\varepsilon/4} \right\} \\ (37) \quad & < 16 (0.655)^2 0.606 e^{-1} < 1.637. \end{aligned}$$

Since $\frac{(k_1 k_2)^{3(1-s)}}{1-s}$ is convex for $0 < s < 1$, we can see from (36), (37) and Lemma 10 that $G(s)$ is positive in the interval $s_2 \leq s \leq s_1$. Combining this with the previous argument, we are able to deduce that $G(s)$ has at least two real zeros in the interval

$$1 - \frac{1}{8 \log(k_1 k_2)} < s < 1.$$

But, since $\lim_{s \rightarrow 1-0} G(s) = -\infty$ and $G(s_1) > 0$, $G(s)$ must have at least three real zeros in the interval cited above. Hence, by Lemma 11, we can infer that our assumption (35) leads to a contradiction. The statement of the theorem is therefore certainly true.

Gakushuin University, Tokyo.

References.

- [1] T. Estermann: On Dirichlet's L -functions; Journal of London Math. Soc. **23** (1948), 275-279.
- [2] E. Hecke: Theorie der algebraischen Zahlen; Leipzig, 1923.
- [3] H. Heilbronn and E. H. Linfoot: On the imaginary quadratic corpora of class-number one; The Quarterly Journal of Math., Oxford Series, **5** (1934), 293-301.
- [4] A. E. Ingham: The distribution of prime numbers; Cambridge, 1932.
- [5] K. Iseki: On the imaginary quadratic fields of class-number one or two; Jap. Journal of Math., **21** (1951), 145-162.
- [6] E. Landau: Abschätzungen von Charactersummen, Einheiten und Klassenzahlen; Göttinger Nachrichten, 1918.
- [7] A. Page: On the number of primes in an arithmetic progression; Proc. London Math. Soc. **39** (1935), 116-141.
- [8] G. Polya und G. Szegő: Aufgaben und Lehrsätze aus der Analysis. Bd. I, Springer 1925.
- [9] W. Sibagaki: Eight-figure Math. Tables vol. 1. Exponential Functions, Maruzen Tokyo 1949.
- [10] C. L. Siegel: Über die Classenzahl quadratischer Zahlkörper: Acta. Arith., **1** (1935).
- [11] Z. Suetuna: Bemerkung über das Produkt von L -Funktionen; Tôhoku Math. Journal, **27** (1926), 248-257.