17. *Specialization of the Theory of a Space of Higher Order I. On the Extended Non-holonomic System.*

By Yoshie KATSURADA.

(Received Jan. 10, 1952)

Introduction.

In order to find the geometrical meanings of results in the theory obtained in use of the extensor analysis, one should make clear the relation between the geometries in the space of higher order and in the original point space. By this reason, it seems to be desirable that the geometrical theory stated in words of the extensor analysis in the space of higher order is an extension of that of tensor analysis in the original point space and the part corresponding to tensor member of the former just coincides with the latter, so as an extensor introduced by H. V. Craig [1]¹⁾ is an extension of a tensor and the components of the extensor include the tensor member as a part. As the first step, in the previous paper [2], the present author obtained already the extended connection parameters in the above-mentioned sense, in the case that either the original point space is a space with a symmetric affine connection or a Riemannian space. It is the purpose of the present paper to establish the theory of non-holonomic systems and subspaces in the space of higher order based upon a point space with a non-holonomic system $\lambda_{i'}^i$, at the same standpoint.

In the present paper, the ideas, notations and results given in the previous papers ([2]—[5]) will be used without explanation. The present author wishes to express to Prof. A. Kawaguchi her appreciation for his helpful guidance.

§ 1. The extendent non-holonomic system.

Let us consider an n -dimensional point space K_n with a field of n mutually independent contravariant vectors $\lambda_{i'}^i$ ($i'=1, \dots, n$) called base vectors of the non-holonomic system, and with the group of non-holonomic transformations, of which any transformation is given by

$$(1.1) \quad \lambda_{i'}^i = N_{i'}^{a'} \lambda_{a'}^i \quad | N_{i'}^{a'} | \neq 0.$$

Then, differentiating the vectors $\lambda_{i'}^i$ at each point on a parameterized arc $x^i = x^i(t)$ of class P in the space $(\alpha - \gamma)$ times with respect to t and multiplying the binomial

1) Numbers in brackets refer to the references at the end of the paper.

coefficient $\binom{\alpha}{\gamma}$, we obtain the following quantities:

$$(1.2) \quad \begin{aligned} \lambda_{\tau i'}^{\alpha i} &= \binom{\alpha}{\tau} \lambda_{i'}^i (\alpha - \tau) & \text{for } \alpha \geq \gamma \\ &= 0 & \text{for } \alpha < \gamma \end{aligned}$$

at each point on the arc, where t is a fixed essential parameter, for which we can state the following theorem

Theorem 1.1. *The quantities $\lambda_{\tau i'}^{\alpha i}$ are changed by*

$$(1.3) \quad \lambda_{\tau i'}^{\alpha i} = \sum_{\theta=\tau}^{\alpha} X_{\theta \alpha}^{\alpha i} \lambda_{\tau i'}^{\theta \alpha}$$

under a transformation of the expoint ([1], p. 765).

Proof. Differentiating the transformation equation of the contravariant vector: $\lambda_{i'}^i = X_a^i \lambda_{i'}^a (\alpha - \gamma)$ times by Leibniz rule and multiplying the binomial coefficient $\binom{\alpha}{\tau}$, it follows that

$$\begin{aligned} \binom{\alpha}{\tau} \lambda_{i'}^i (\alpha - \tau) &= \binom{\alpha}{\tau} (X_a^i \lambda_{i'}^a)^{(\alpha - \tau)} = \binom{\alpha}{\tau} \sum_{\lambda=0}^{\alpha - \tau} \binom{\alpha - \tau}{\lambda} X_a^{i(\alpha - \tau - \lambda)} \lambda_{i'}^{a(\lambda)} \\ &= \sum_{\theta=\tau}^{\alpha} \binom{\theta}{\tau} \binom{\alpha}{\theta} X_a^{i(\alpha - \theta)} \lambda_{i'}^{a(\theta - \tau)} = \sum_{\theta=\tau}^{\alpha} X_{\theta \alpha}^{\alpha i} \binom{\theta}{\tau} \lambda_{i'}^{a(\theta - \tau)} \\ &\quad \text{(putting: } \theta = \gamma + \lambda) \quad \text{Q.E.D.} \end{aligned}$$

By virtue of the theorem, it is known that the quantities $\lambda_{\tau i'}^{\alpha i}$ for fixed γ and i' are components of an excontravariant extensor.

If we shall consider the space of line-elements of order M $K_n^{(M)} (M \leq P)$ based upon such the space, then the $n(M+1)$ mutually independent excontravariant extensors $\lambda_{\tau i'}^{\alpha i}$ of range M ($\gamma=0, \dots, M$; $i'=1, \dots, n$) and the reciprocal excovariant extensors $\lambda_{\alpha i}^{\beta j'}$ which satisfy the equation $\sum_{\alpha=0}^M \lambda_{\tau i'}^{\alpha i} \lambda_{\alpha i}^{\beta j'} = \delta_{\tau}^{\beta} \delta_{i'}^{j'}$ will be considered at each expoint in $K_n^{(M)}$, since the determinant $|\lambda_{\tau i'}^{\alpha i}| = |\lambda_{i'}^i|^{M+1} \neq 0$. Differentiating the equation of non-holonomic transformation (1.1) $(\alpha - \gamma)$ times by Leibniz rule along the arc and multiplying the binomial coefficient $\binom{\alpha}{\tau}$, the following result is obtained

$$(1.4) \quad \lambda_{\tau i'}^{\alpha i} = \sum_{\beta=\tau}^{\alpha} N_{\tau i'}^{\beta \alpha'} \lambda_{\beta \alpha'}^{\alpha i} \quad \alpha, \gamma=0, \dots, M$$

where $N_{\tau i'}^{\beta \alpha'} = \binom{\beta}{\tau} N_{i'}^{\alpha'(\beta - \tau)}$, whose determinant $|N_{\tau i'}^{\beta \alpha'}| = |N_{i'}^{\alpha'}|^{M+1} \neq 0$. Hence in $K_n^{(M)}$, we shall adopt a set of such the $n(M+1)$ excontravariant extensors $\lambda_{\tau i'}^{\alpha i}$ and of their $n(M+1)$ reciprocal extensors $\lambda_{\alpha i}^{\beta j'}$ as a non-holonomic system and transformation (1.4) as a non-holonomic transformation. These are called *extended non-holonomic system* (briefly denoted with ex.n.s.) and *extended non-holonomic transformation* (briefly ex.n.t.) respectively, by virtue of the reason that $\lambda_{\tau i'}^{\alpha i}$, $\lambda_{\alpha i}^{\beta j'}$ and

$N_{\tau i'}^{\beta \alpha'}$ are extensions of $\lambda_{i'}^i$, $\lambda_{i'}^{j'}$ and $N_{i'}^{\alpha'}$ respectively. Also, they are the special ones of what stated in the previous paper [3]. For the structure of the reciprocal excovariant extensor, the following theorem is given

Theorem 1.2. *If $\lambda_{i'}^i$ is the reciprocal covariant vector of $\lambda_{j'}^i$, then the relation:*

$$\lambda_{\alpha i}^{\tau i'} = \binom{\tau}{\alpha} \lambda_{i'}^{j'(\tau-\alpha)}$$

holds good.

Proof. According to that $|\lambda_{\tau i'}^{\alpha i}| \neq 0$, the solution $\lambda_{\alpha i}^{\beta j'}$ of the equation:

$\sum_{\alpha=0}^M \lambda_{\tau i'}^{\alpha i} \lambda_{\alpha i}^{\beta j'} = \delta_{\tau}^{\beta} \delta_{i'}^{j'}$ is determined uniquely. In fact, calculating the expression

$\sum_{\alpha=0}^M \lambda_{\tau i'}^{\alpha i} \binom{\beta}{\alpha} \lambda_{i'}^{j'(\beta-\alpha)}$, it follows that

$$\begin{aligned} \sum_{\alpha=0}^M \lambda_{\tau i'}^{\alpha i} \binom{\beta}{\alpha} \lambda_{i'}^{j'(\beta-\alpha)} &= \sum_{\alpha=0}^M \binom{\alpha}{\tau} \lambda_{i'}^{i(\alpha-\tau)} \binom{\beta}{\alpha} \lambda_{i'}^{j'(\beta-\alpha)} \\ &= \sum_{\alpha=\tau}^{\beta} \binom{\alpha}{\tau} \binom{\beta}{\alpha} \lambda_{i'}^{i(\alpha-\tau)} \lambda_{i'}^{j'(\beta-\alpha)} = \sum_{\theta=0}^{\beta-\tau} \binom{\beta}{\tau} \binom{\beta-\tau}{\theta} \lambda_{i'}^{i(0)} \lambda_{i'}^{j'(\beta-\tau-\theta)} \\ &\quad \text{(putting: } \theta = \alpha - \tau) \\ &= \binom{\beta}{\tau} (\lambda_{i'}^i \lambda_{i'}^{j'})^{(\beta-\alpha)} = \binom{\beta}{\tau} \delta_{i'}^{j'(\beta-\tau)} = \delta_{i'}^{j'} \delta_{\tau}^{\beta}. \end{aligned}$$

Accordingly $\lambda_{\alpha i}^{\beta j'}$ must be $\binom{\beta}{\alpha} \lambda_{i'}^{j'(\beta-\alpha)}$.

§ 2. The non-holonomic quantities.

At first, we discuss the problem: how reduce the non-holonomic quantities introduced in the previous papers ([3], [4]) for ex.n.s.? The extended coefficients of a vector $C_{pj'}^{\tau i'}$ in the non-holonomic system ([3], p. 195) become as follows:

$$C_{pj'}^{\tau i'} = \sum_{\theta=p}^{\tau} \binom{\theta}{p} \lambda_{\theta i'}^{\tau i'} \lambda_{0j'}^{0i(\theta-p)} = \delta_p^{\tau} \delta_{j'}^{i'},$$

because of $\lambda_{\theta i'}^{\tau i'} = \binom{\tau}{\theta} \lambda_{i'}^{j'(\tau-\theta)}$ and $\lambda_{0j'}^{0i} = \lambda_{j'}^i$. Consequently, as the non-holonomic components of the extensor $v^{(\alpha)i}$ obtained by differentiating a vector v^i α times with respect to t , we get

$$v^{\tau i'} = C_{pj'}^{\tau i'} v^{j'(\rho)} = v^{i'(\tau)}.$$

This is the same as the structure of the holonomic system.

Likely, for the $\mathfrak{S}$, $\mathfrak{Z}^H$, $\mathfrak{Y}^H$ -operation coefficients of an excontravariant extensor: $C_{\delta j'}^{\tau i'}(\mathfrak{S})$ ([3], p. 197), $C_{\delta j'}^{\mu i'}(\mathfrak{Z}^H)$ ([3], p. 205), $C_{\delta j'}^{\tau i'}(\mathfrak{Y}^H)$ ([3], p. 211) and the $\mathfrak{S}^H$, $\mathfrak{Z}^H$, $\mathfrak{Y}^H$ -operation coefficients of an excovariant extensor: $C^{\rho \delta j'}_{\tau i'}(\mathfrak{S}^H)$ ([3], p. 201), $C^{\lambda \delta j'}_{\tau i'}(\mathfrak{Z}^H)$ ([3], p. 207), $C^{H-\rho \delta j'}_{\tau i'}(\mathfrak{Y}^H)$ ([3], p. 214), the following results exist

$$(2.1) \quad C_{\delta j'}^{\tau i'}(\mathfrak{S}) = \delta_{\delta-\rho}^{\tau} \delta_{j'}^{i'}, \quad C_{\delta j'}^{\mu i'}(\mathfrak{Z}^H) = \delta_{\delta}^{\mu-\nu} \delta_{j'}^{i'}, \quad C_{\delta j'}^{\tau i'}(\mathfrak{Y}^H) = \delta_{\delta}^{\tau-\rho} \delta_{j'}^{i'}$$

and

$$(2.2) \quad C^{\rho \delta j'}_{\tau i'}(\mathfrak{S}^H) = \delta_{\tau}^{\delta-\rho} \delta_{i'}^{j'}, \quad C^{\lambda \delta j'}_{\tau i'}(\mathfrak{Z}^H) = \delta_{\tau}^{\delta-\lambda-H} \delta_{i'}^{j'}, \quad C^{H-\rho \delta j'}_{\tau i'}(\mathfrak{Y}^H) = \delta_{\tau}^{H-\rho+\delta} \delta_{i'}^{j'}$$

respectively, by virtue of (1.2). Hence the following theorem is selfevident by this

fact

Theorem 2.1. *In ex.n.s., the structures $\mathfrak{S}^H$, $\mathfrak{S}^K$, $\mathfrak{Y}^L$ -operations of an extensor are as same as that in the holonomic system.*

If we consider the components of the displacement of an exponent in ex.n.s., then it follows that

$$\begin{aligned} ds^{a'} &= \sum_{\tau=0}^{\alpha} \lambda_{\tau}^{a'} dx^{(\tau)k} = \sum_{\tau=0}^{\alpha} \binom{\alpha}{\tau} \lambda_{\tau}^{i'(\alpha-\tau)} dx^{(\tau)k} \\ &= (\lambda_{\tau}^{i'} dx^k)^{(\alpha)} = ds^{i'(\alpha)}, \end{aligned}$$

accordingly they are equal to $(ds^{i'})^{(\alpha)}$. Moreover there exists the following theorem for the non-holonomic quantities $\omega_{\tau j'}^{\tau i'} \lambda_{\delta k'}^{\tau i'}$ ([4], p. 60).

Theorem 2.2. *The relation between the non-holonomic quantities $\omega_{j'k'}^{i'}$ $\equiv \left(\frac{\partial \lambda_{j'}^{i'}}{\partial x^k} - \frac{\partial \lambda_{k'}^{i'}}{\partial x^j} \right) \lambda_{\tau j'}^{\tau i'} \lambda_{\delta k'}^{\tau i'}$ in K_n and the non-holonomic quantities*

$$\omega_{\tau j'}^{\tau i'} \lambda_{\delta k'}^{\tau i'} \equiv \sum_{\alpha, \beta=0}^{\tau} \left(\frac{\partial \lambda_{\alpha j'}^{\tau i'}}{\partial x^{(\beta)k}} - \frac{\partial \lambda_{\beta k'}^{\tau i'}}{\partial x^{(\alpha)j}} \right) \lambda_{\tau j'}^{\alpha j} \lambda_{\delta k'}^{\beta k} = \sum_{\beta=\delta}^{\tau-\gamma} \sum_{\alpha=\gamma}^{\tau-\beta} \left(\frac{\partial \lambda_{\alpha j'}^{\tau i'}}{\partial x^{(\beta)k}} - \frac{\partial \lambda_{\beta k'}^{\tau i'}}{\partial x^{(\alpha)j}} \right) \lambda_{\tau j'}^{\alpha j} \lambda_{\delta k'}^{\beta k}$$

in $K_n^{(M)}$ is given by

$$\omega_{\tau j'}^{\tau i'} \lambda_{\delta k'}^{\tau i'} = \binom{\tau}{\tau \delta} \omega_{j'k'}^{i'(\tau-\gamma-\delta)}$$

where $\binom{\tau}{\tau \delta}$ indicates the generalized binomial coefficient of Craig ([6], p. 336), i.e.,

$$\begin{aligned} \binom{\tau}{\tau \delta} &= \frac{\tau!}{\gamma! \delta! (\tau - \gamma - \delta)!} & \text{for } \tau \geq \gamma + \delta \\ &= 0 & \text{for } \tau < \gamma + \delta. \end{aligned}$$

Proof. Differentiating the expression: $\lambda_{\alpha j'}^{\tau i'} = \binom{\tau}{\alpha} \lambda_{j'}^{i'(\tau-\alpha)}$ with respect to $x^{(\beta)k}$, we have

$$\frac{\partial \lambda_{\alpha j'}^{\tau i'}}{\partial x^{(\beta)k}} = \binom{\tau}{\alpha} \binom{\tau-\alpha}{\beta} \left(\frac{\partial \lambda_{j'}^{i'(\tau-\alpha-\beta)}}{\partial x^k} \right) = \binom{\tau}{\alpha \beta} \left(\frac{\partial \lambda_{j'}^{i'(\tau-\alpha-\beta)}}{\partial x^k} \right),$$

similarly

$$\frac{\partial \lambda_{\beta k'}^{\tau i'}}{\partial x^{(\alpha)j}} = \binom{\tau}{\beta \alpha} \left(\frac{\partial \lambda_{k'}^{i'(\tau-\beta-\alpha)}}{\partial x^j} \right),$$

consequently

$$\frac{\partial \lambda_{\alpha j'}^{\tau i'}}{\partial x^{(\beta)k}} - \frac{\partial \lambda_{\beta k'}^{\tau i'}}{\partial x^{(\alpha)j}} = \binom{\tau}{\alpha \beta} \left(\frac{\partial \lambda_{j'}^{i'(\tau-\alpha-\beta)}}{\partial x^k} - \frac{\partial \lambda_{k'}^{i'(\tau-\alpha-\beta)}}{\partial x^j} \right),$$

moreover it follows that

$$\sum_{\alpha=\gamma}^{\tau-\beta} \left(\frac{\partial \lambda_{\alpha j'}^{\tau i'}}{\partial x^{(\beta)k}} - \frac{\partial \lambda_{\beta k'}^{\tau i'}}{\partial x^{(\alpha)j}} \right) \lambda_{\tau j'}^{\alpha j} = \sum_{\alpha=\gamma}^{\tau-\beta} \binom{\tau}{\alpha \beta} \binom{\tau-\alpha-\beta}{\tau} \left(\frac{\partial \lambda_{j'}^{i'(\tau-\alpha-\beta)}}{\partial x^k} - \frac{\partial \lambda_{k'}^{i'(\tau-\alpha-\beta)}}{\partial x^j} \right) \lambda_{j'}^{i'(\alpha-\gamma)}$$

$$\begin{aligned}
&= (\tau_{\tau\beta}) \sum_{\lambda=0}^{\tau-\tau-\beta} (\tau-\tau-\beta) \left(\frac{\partial \lambda_j^{i'}}{\partial x^k} - \frac{\partial \lambda_k^{i'}}{\partial x^j} \right) \lambda_{j'}^{j(\lambda)} \quad (\text{putting: } \lambda = \alpha - \gamma) \\
&= (\tau_{\tau\beta}) \left\{ \left(\frac{\partial \lambda_j^{i'}}{\partial x^k} - \frac{\partial \lambda_k^{i'}}{\partial x^j} \right) \lambda_{j'}^j \right\}^{(\tau-\tau-\beta)},
\end{aligned}$$

at last

$$\begin{aligned}
&\sum_{\beta=\delta}^{\tau-\tau} \sum_{\alpha=\tau}^{\tau-\beta} \left(\frac{\partial \lambda_{\alpha j}^{\tau i'}}{\partial x^{(\beta)k}} - \frac{\partial \lambda_{\beta k}^{\tau i'}}{\partial x^{(\alpha)j}} \right) \lambda_{\tau j'}^{\alpha j} \lambda_{\delta k'}^{\beta k} = \sum_{\beta=\delta}^{\tau-\tau} (\tau_{\tau\beta}) (\beta_{\delta}) \left\{ \left(\frac{\partial \lambda_j^{i'}}{\partial x^k} - \frac{\partial \lambda_k^{i'}}{\partial x^j} \right) \lambda_{j'}^j \right\}^{(\tau-\tau-\beta)} \lambda_{k'}^{k(\beta-\delta)} \\
&= (\tau_{\tau\delta}) \sum_{\mu=0}^{\tau-\tau-\delta} (\tau-\tau-\delta) \left\{ \left(\frac{\partial \lambda_j^{i'}}{\partial x^k} - \frac{\partial \lambda_k^{i'}}{\partial x^j} \right) \lambda_{j'}^j \right\}^{(\tau-\tau-\delta-\mu)} \lambda_{k'}^{k(\mu)} \\
&\quad (\text{putting: } \mu = \beta - \delta) \\
&= (\tau_{\tau\delta}) \left\{ \left(\frac{\partial \lambda_j^{i'}}{\partial x^k} - \frac{\partial \lambda_k^{i'}}{\partial x^j} \right) \lambda_{j'}^j \lambda_{k'}^k \right\}^{(\tau-\tau-\delta)} = (\tau_{\tau\delta}) \omega_{j'k'}^{i', \tau-\tau-\delta} \quad \text{Q.E.D.}
\end{aligned}$$

Corollary 1. If $\omega_{j'k'}^{i'} = 0$, then it follows that $\omega_{\tau j' \delta k'}^{\tau i'} = 0$.

Corollary 2. In this case, the non-holonomic connection parameter $\omega_{\tau j' \delta k'}^{* \tau i'}$ ([4], p. 63) is equal to zero.

Corollary 3. In our system, the non-holonomic derivative ([4], p. 63) becomes partial derivative in the usual meaning.

Next if K_n is a space C_n with a symmetric affine connection Γ_{jk}^i , then in $C_n^{(M)}$, there exists the extended connection parameter $\Gamma_{\beta j \tau k}^{\alpha i} = (\alpha_{\beta \tau}) \Gamma_{jk}^{i(\alpha-\beta-\tau)}$ ([2], p. 18). A statement according to the non-holonomic connection parameter $\Gamma_{\beta j' \tau k'}^{\alpha i'}$ which is induced upon ex.n.s. from the extended connection parameter, i.e., so called exnon-holonomic connection parameter:

$$\Gamma_{\beta j' \tau k'}^{\alpha i'} = \sum_{\theta=\tau}^{\alpha-\beta-\theta} \sum_{\psi=\beta}^{\psi} \sum_{\varphi=\theta}^{\varphi} \Gamma_{\varphi j \theta k}^{\psi i} \lambda_{\psi i}^{\alpha i'} \lambda_{\beta j'}^{\varphi j} \lambda_{\tau k'}^{\theta k} + \sum_{\psi=\beta+\tau}^{\alpha} \frac{\partial \lambda_{\beta j'}^{\psi i}}{\partial x^{(\tau)k'}} \lambda_{\psi i}^{\alpha i'}$$

is given by the following theorem.

Theorem 2.3. When $\Gamma_{j'k'}^{i'}$ is the induced non-holonomic connection parameter of Γ_{jk}^i , it follows that

$$\Gamma_{\beta j' \tau k'}^{\alpha i'} = (\alpha_{\beta \tau}) \Gamma_{j'k'}^{i'(\alpha-\beta-\tau)}.$$

The verification of the theorem will be suggested by replacing the letter X in the proof of Theorem 1.1 of the previous paper [2] with λ .

At last, let us suppose K_n to be a Riemannian space with the metric tensor g_{ij} , then the structure of the non-holonomic components of metric extensor $g_{\alpha i \beta j}$ ([2], p. 21) in $R_n^{(M)}$ is offered by the following theorem.

Theorem 2.4. If $g_{i'j'}$ are the non-holonomic components of the metric tensor g_{ij} , then the non-holonomic components of the metric extensor $g_{\alpha i \beta j}$ are given by

$$g_{\alpha i \beta j} = (\alpha_{\beta}^M) g_{i'j'}^{(M-\alpha-\beta)}.$$

The theorem is proved as same as Theorem 2.2.

§ 3. The exsurface and the expseudonormal.

Let us consider an m -dimensional subspace K_m in K_n

$$x^i = x^i(u^{j'}) \quad j' = 1, \dots, m; {}^{2)} \quad i = 1, \dots, n$$

and its pseudonormal $K_n^{(n-m)}$ which means a set of $(n-m)$ mutually independent contravariant vectors $\lambda_{p''}^i(u^{j'})$ ($p'' = m+1, \dots, n$)³⁾ not belonging to K_m defined at each point in K_m , then along a parameterized arc of class P in K_m : $u^{j'} = u^{j'}(t)$, the excontravariant extensors defined by (1.2), that is,

$$(3.1) \quad \lambda_{\tau p''}^{\alpha i} = \binom{\alpha}{\tau} \lambda_{p''}^{i(\alpha-\tau)} \quad \text{for } \alpha \geq \tau \\ = 0 \quad \text{for } \alpha < \tau$$

are generated. Accordingly there exists the set of the $(n-m)(M+1)$ mutually independent excontravariant extensors $\lambda_{\tau p''}^{\alpha i}$ ($\gamma = 0, \dots, M$; $p'' = m+1, \dots, n$) of range M at each expoint in the exsurface of order M $K_m^{(M)}$, i.e.,

$$x^i = x^i(u^{j'}), \quad x^{i(\alpha+1)j} = \sum_{\beta=0}^{\alpha} B_{\beta j'}^{\alpha i} u^{(\beta+1)j'} \quad \alpha = 0, \dots, M-1$$

where $B_{\beta j'}^{\alpha i} \equiv \frac{\partial x^{i(\alpha+1)j}}{\partial u^{(\beta+1)j'}}$. Such the set of $\lambda_{\tau p''}^{\alpha i}$ may be adopted as the expseudonormal ([5], p. 31) and indicated by $K_n^{(M)(n-m)}$.

Since the $n(M+1)$ -rowed determinant $|\lambda_{\tau p''}^{\alpha i}, B_{\tau j'}^{\alpha i}|$ constructed from $B_{\tau j'}^{\alpha i}$ ($\gamma = 0, \dots, M$; $j' = 1, \dots, m$) and $\lambda_{\tau p''}^{\alpha i}$ ($\gamma = 0, \dots, M$; $p'' = m+1, \dots, n$) is equal to $|\lambda_{p''}^i, B_{j'}^i|^{M+1}$ that does not vanish at each expoint on the exsurface, the reciprocal excovariant extensors $B_{\alpha i}^{\tau j'}$ and $\lambda_{\alpha i}^{\tau p''}$ are determined uniquely as the solution of the following equations:

$$(3.3) \quad \sum_{\alpha=0}^M B_{\beta j'}^{\alpha i} B_{\alpha i}^{\tau k'} = \delta_{\beta}^{\tau} \delta_{j'}^{k'}, \quad \sum_{\alpha=0}^M B_{\beta j'}^{\alpha i} \lambda_{\alpha i}^{\tau p''} = 0, \\ \sum_{\alpha=0}^M \lambda_{\beta p''}^{\alpha i} B_{\alpha i}^{\tau k'} = 0, \quad \sum_{\alpha=0}^M \lambda_{\beta p''}^{\alpha i} \lambda_{\alpha i}^{\tau p''} = \delta_{\beta}^{\tau} \delta_{p''}^{q''}.$$

Then it is known from Theorem 1.2 that

$$(3.3) \quad B_{\alpha i}^{\tau k'} = \binom{\tau}{\alpha} B_i^{k'(\tau-\alpha)} \quad \text{and} \quad \lambda_{\alpha i}^{\tau p''} = \binom{\tau}{\alpha} \lambda_i^{p''(\tau-\alpha)},$$

$B_i^{k'}$ and $\lambda_i^{p''}$ being the reciprocal covariant vectors of the B_i^{τ} ($= \frac{\partial x^i}{\partial u^{i'}}$) and $\lambda_{p''}^i$, respectively. Then we get the following theorems referred to the operations of an extensor in the exsurface or the expseudonormal.

2) Afterward Latin index with one prime runs from 1 to m .

3) Latin index with two primes takes the values $m+1, \dots, n$.

Theorem 3.1. *When an extensor belongs to $K_n^{(M)}$, the projections of its $\mathfrak{S}^H$, $\mathfrak{Z}^K$, $\mathfrak{Y}^L$ -operations regarded as an extensor of $K_n^{(M)}$ along $K_m^{(M)}$, upon $K_m^{(M)}$ and $K_n^{(M)(n-m)}$ are equal to $\mathfrak{S}^H$, $\mathfrak{Z}^K$, $\mathfrak{Y}^L$ -operations constructed of the components referred to the u -coordinate system and the λ -coordinate system respectively.*

The theorem is proved by replacing $\lambda_{\tau p}^{xi}$, $B_{xi}^{\tau k'}$ and $\lambda_{xi}^{\tau p''}$ in the proof of Theorems 2.2 ([5], p. 33) and 2.4 ([5], p. 35) of the previous paper with the right-hand members of (3.1) and (3.3) and by using (2.1) and (2.2).

Let us consider the space C_n with the symmetric affine connection, and let its induced connection parameters in the m -dimensional subspace C_m and in the pseudo-normal $C_n^{(n-m)}$ be

$$(3.4) \quad {}'G_{j'k'}^{i'} = B_i^{i'} \left\{ \Gamma_{jk}^i B_{j'}^j B_{k'}^k + \frac{\partial B_{j'}^i}{\partial u_{k'}} \right\}$$

and

$$(3.5) \quad {}''G_{q'k'}^{p''} = \lambda_i^{p''} \left\{ \Gamma_{jk}^i \lambda_{q'}^j B_{k'}^k + \frac{\partial \lambda_{q'}^k}{\partial u_{k'}} \right\}$$

respectively. Then similarly, the connection parameters in $C_m^{(M)}$ and $C_n^{(M)(n-m)}$ induced from the extended connection parameter

$$(3.6) \quad \Gamma_{\beta j \tau k}^{\alpha i} = \binom{\alpha}{\beta \tau} \Gamma_{jk}^i (\alpha - \beta - \tau)$$

are given by

$${}'G_{\beta j' \tau k'}^{\alpha i'} = \sum_{\psi=\beta+\tau}^{\alpha} B_{\psi i}^{\alpha i'} \left\{ \frac{\partial B_{\beta j'}^{\psi i}}{\partial u_{(\tau)k'}} + \sum_{\theta=\tau, \varphi=\beta}^{\psi-\beta, \psi-\theta} \Gamma_{\varphi j \theta k}^{\psi i} B_{\beta j'}^{\varphi j} B_{\tau k'}^{\theta k} \right\}$$

and

$${}''G_{\beta q' \tau k'}^{\alpha p''} = \sum_{\psi=\beta+\tau}^{\alpha} \lambda_{\psi i}^{\alpha p''} \left\{ \frac{\partial \lambda_{\beta q'}^{\psi i}}{\partial u_{(\tau)k'}} + \sum_{\theta=\tau, \varphi=\beta}^{\psi-\beta, \psi-\theta} \Gamma_{\varphi j \theta k}^{\psi i} \lambda_{\beta q'}^{\varphi j} B_{\tau k'}^{\theta k} \right\}$$

Theorem 3.2. *In $C_m^{(M)}$ and $C_n^{(M)(n-m)}$, the induced connection parameter coincides with the extended connection parameter, that is,*

$$(3.7) \quad {}'G_{\beta j' \tau k'}^{\alpha i'} = \binom{\alpha}{\beta \tau} {}'G_{j'k'}^{i'(\alpha-\beta-\tau)}, \quad {}''G_{\beta q' \tau k'}^{\alpha p''} = \binom{\alpha}{\beta \tau} {}''G_{q'k'}^{p''(\alpha-\beta-\tau)}.$$

Proof. Differentiating (3.4) $(\alpha - \beta - \gamma)$ times with respect to t and multiplying the generalized binomial coefficient $\binom{\alpha}{\beta \tau}$, we get

$$(3.8) \quad \binom{\alpha}{\beta \tau} {}'G_{j'k'}^{i'(\alpha-\beta-\tau)} = \binom{\alpha}{\beta \tau} \left\{ \Gamma_{jk}^i B_i^{i'} B_{j'}^j B_{k'}^k + B_i^{i'} \frac{\partial B_{j'}^i}{\partial u_{k'}} \right\}^{(\alpha-\beta-\tau)}$$

Calculating the first term of the right-hand member of the above equation, it follows that

$$\binom{\alpha}{\beta \tau} (\Gamma_{jk}^i B_i^{i'} B_{j'}^j B_{k'}^k)^{(\alpha-\beta-\tau)} = \binom{\alpha}{\beta \tau} \sum_{\lambda=0}^{\alpha-\beta-\tau} \binom{\alpha-\beta-\tau}{\lambda} (\Gamma_{jk}^i B_i^{i'} B_{j'}^j)^{(\alpha-\beta-\tau-\lambda)} B_{k'}^k^{(\lambda)}$$

$$\begin{aligned}
&= \sum_{\theta=\gamma}^{\alpha-\beta} \binom{\alpha}{\beta\theta} \binom{\theta}{\gamma} (\Gamma_{jk}^i B_i^{j'} B_{j'}^j)^{(\alpha-\beta-\theta)} B_{k'}^k \binom{\theta-\gamma}{\gamma} \quad (\text{putting: } \theta=\gamma+\lambda) \\
&= \sum_{\theta=\gamma}^{\alpha-\beta} \binom{\alpha}{\beta\theta} (\Gamma_{jk}^i B_i^{j'} B_{j'}^j)^{(\alpha-\beta-\theta)} B_{\gamma k'}^{0k}, \\
&= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\mu=0}^{\alpha-\beta-\theta} \binom{\alpha}{\beta\theta} \binom{\alpha-\beta-\theta}{\mu} (\Gamma_{jk}^i B_i^{j'})^{(\alpha-\beta-\theta-\mu)} B_{j'}^{j(\mu)} B_{\gamma k'}^{0k}, \\
&= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \binom{\alpha}{\theta\varphi} (\Gamma_{jk}^i B_i^{j'})^{(\alpha-\theta-\varphi)} B_{\beta j'}^{\varphi j} B_{\gamma k'}^{0k}, \quad (\text{putting: } \varphi=\beta+\mu) \\
&= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\nu=0}^{\alpha-\theta-\varphi} \binom{\alpha}{\theta\varphi} \binom{\alpha-\theta-\varphi}{\nu} \Gamma_{jk}^{i(\alpha-\theta-\varphi-\nu)} B_i^{j(\nu)} B_{\beta j'}^{\varphi j} B_{\gamma k'}^{0k}, \\
&= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\theta+\varphi}^{\alpha} \binom{\psi}{\varphi\theta} \Gamma_{jk}^{i(\psi-\varphi-\theta)} B_{\psi i}^{\varphi j} B_{\beta j'}^{\varphi j} B_{\gamma k'}^{0k} \quad (\text{putting: } \psi=\alpha-\nu) \\
&= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\theta+\varphi}^{\alpha} \Gamma_{\varphi j \theta k}^{\psi i} B_{\psi i}^{\varphi j} B_{\beta j'}^{\varphi j} B_{\gamma k'}^{0k},
\end{aligned}$$

the second term of the right-hand member of (3.8) becomes as follows

$$\begin{aligned}
\binom{\alpha}{\beta\gamma} \left(B_i^{j'} \frac{\partial B_{j'}^i}{\partial u^{k'}} \right)^{(\alpha-\beta-\gamma)} &= \sum_{\lambda=0}^{\alpha-\beta-\gamma} \binom{\alpha}{\beta\gamma} \binom{\alpha-\beta-\gamma}{\lambda} \left(\frac{\partial B_{j'}^i}{\partial u^{k'}} \right)^{(\alpha-\beta-\gamma-\lambda)} B_i^{j'(\lambda)} \\
&= \sum_{\psi=\beta+\gamma}^{\alpha} \binom{\psi-\beta}{\gamma} \binom{\psi}{\beta} \left(\frac{\partial B_{j'}^i}{\partial u^{k'}} \right)^{(\psi-\beta-\gamma)} B_{\psi i}^{j'} \quad (\text{putting: } \psi=\alpha-\lambda) \\
&= \sum_{\psi=\beta+\gamma}^{\alpha} \frac{\partial B_{\beta j'}^{\psi i}}{\partial u^{(\gamma)k'}} B_{\psi i}^{\alpha j'},
\end{aligned}$$

further by virtue of that

$$\sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\theta+\varphi}^{\alpha} = \sum_{\psi=\beta+\gamma}^{\alpha} \sum_{\theta=\gamma}^{\psi-\beta} \sum_{\varphi=\beta}^{\psi-\theta},$$

we have

$$\binom{\alpha}{\beta\gamma} \Gamma_{j'k'}^{i(\alpha-\beta-\gamma)} = \sum_{\psi=\beta+\gamma}^{\alpha} \sum_{\theta=\gamma}^{\psi-\beta} \sum_{\varphi=\beta}^{\psi-\theta} \Gamma_{\varphi j \theta k}^{\psi i} B_{\psi i}^{\varphi j} B_{\beta j'}^{\varphi j} B_{\gamma k'}^{0k} + \sum_{\psi=\beta+\gamma}^{\alpha} \frac{\partial B_{\beta j'}^{\psi i}}{\partial u^{(\gamma)k'}} B_{\psi i}^{\alpha j'}.$$

Similarly, the second expression of the theorem is proved.

As shown in the previous paper ([5], p. 38), we shall consider D -symbols referred to the excovariant differentiation by means of these extended connection parameters. Then we have the following theorems.

Theorem 3.3. When P is a scalar, and v^j , $v^{j'}$ and $v^{j''}$ express a contravariant vector in C_n , C_m and $C_n^{(n-m)}$ respectively, the following relations exist

$$\begin{aligned}
D_{\alpha i} P^{(\beta)} &= \binom{\beta}{\alpha} (D_i P)^{(\beta-\alpha)} & D_{\alpha i'} P^{(\beta)} &= \binom{\beta}{\alpha} (D_{i'} P)^{(\beta-\alpha)} & D_{\alpha p''} P^{(\beta)} &= \binom{\beta}{\alpha} (D_{p''} P)^{(\beta-\alpha)} \\
D_{\alpha i} v^{(\beta)j} &= \binom{\beta}{\alpha} (D_i v^j)^{(\beta-\alpha)} & D_{\alpha i'} v^{(\beta)j} &= \binom{\beta}{\alpha} (D_{i'} v^j)^{(\beta-\alpha)} & D_{\alpha p''} v^{(\beta)j} &= \binom{\beta}{\alpha} (D_{p''} v^j)^{(\beta-\alpha)} \\
D_{\alpha i} v^{(\beta)j'} &= \binom{\beta}{\alpha} (D_i v^{j'})^{(\beta-\alpha)} & D_{\alpha i'} v^{(\beta)j'} &= \binom{\beta}{\alpha} (D_{i'} v^{j'})^{(\beta-\alpha)} & D_{\alpha p''} v^{(\beta)j'} &= \binom{\beta}{\alpha} (D_{p''} v^{j'})^{(\beta-\alpha)} \\
D_{\alpha i} v^{(\beta)p''} &= \binom{\beta}{\alpha} (D_i v^{p''})^{(\beta-\alpha)} & D_{\alpha i'} v^{(\beta)p''} &= \binom{\beta}{\alpha} (D_{i'} v^{p''})^{(\beta-\alpha)} & D_{\alpha p''} v^{(\beta)p''} &= \binom{\beta}{\alpha} (D_{p''} v^{p''})^{(\beta-\alpha)}
\end{aligned}$$

$$D_{\alpha i} \omega^{\beta j} = \binom{\beta}{\alpha} \delta_i^{j(\beta-\alpha)} \quad D_{\alpha i'} \omega^{\beta j} = \binom{\beta}{\alpha} (D_{i'} x^j)^{(\beta-\alpha)} \quad D_{\alpha p'} \omega^{\beta j} = \binom{\beta}{\alpha} (D_{p'} x^j)^{(\beta-\alpha)}.$$

The theorem is proved from (3.1), (3.3), (3.6) and (3.7) following the method of the verification of the preceding theorem.

Theorem 3.4. *If $H_{i'j'}$ and $H_{\alpha i' \beta j'}$ ([5], p. 38) are the Euler-Schouten curvature tensor in C_m and extensor in $C_m^{(M)}$ respectively, then*

$$H_{\beta j' \alpha i'}^{\tau j} = \binom{\tau}{\beta \alpha} H_{j' i'}^{\tau j}{}^{(\tau-\beta-\alpha)}$$

holds good.

Proof. Differentiating the expression:

$$H_{j' i'}^{\tau j} = \frac{\partial B_{i'}^j}{\partial u^{j'}} + \Gamma_{ki}^j B_{i'}^k B_{j'}^i - {}' \Gamma_{i' j'}^{k'} B_{k'}^j$$

$(\gamma-\beta-\alpha)$ times with respect to t and multiplying $\binom{\tau}{\beta \alpha}$, we have

$$\binom{\tau}{\beta \alpha} H_{j' i'}^{\tau j}{}^{(\tau-\beta-\alpha)} = \binom{\tau}{\beta \alpha} \left(\frac{\partial B_{i'}^j}{\partial u^{j'}} + \Gamma_{ki}^j B_{i'}^k B_{j'}^i - {}' \Gamma_{i' j'}^{k'} B_{k'}^j \right)^{(\tau-\beta-\alpha)},$$

the first, second and third term of the right-hand member of the above equation become

$$\begin{aligned} \binom{\tau}{\beta \alpha} \left(\frac{\partial B_{i'}^j}{\partial u^{j'}} \right)^{(\tau-\beta-\alpha)} &= \binom{\tau}{\alpha} \binom{\tau-\alpha}{\beta} \left(\frac{\partial B_{i'}^j}{\partial u^{j'}} \right)^{(\tau-\alpha-\beta)} = \binom{\tau}{\alpha} \frac{\partial B_{i'}^j{}^{(\tau-\alpha)}}{\partial u^{(\beta)j'}} = \frac{\partial B_{\alpha i'}^{\tau j}}{\partial u^{(\beta)j'}}, \\ \binom{\tau}{\beta \alpha} (\Gamma_{ki}^j B_{i'}^k B_{j'}^i)^{(\tau-\beta-\alpha)} &= \binom{\tau-\alpha-\beta}{\beta \alpha} \sum_{\lambda=0}^{\tau-\alpha-\beta} \binom{\tau-\alpha-\beta}{\lambda} (\Gamma_{ki}^j B_{i'}^k)^{(\tau-\alpha-\beta-\lambda)} B_{j'}^i{}^{(\lambda)} \\ &= \sum_{\theta=\beta}^{\tau-\alpha} \binom{\tau}{\alpha \theta} (\Gamma_{ki}^j B_{i'}^k)^{(\tau-\alpha-\theta)} B_{\beta j'}^{\theta i} \quad (\text{putting: } \theta = \lambda + \beta) \\ &= \sum_{\theta=\beta}^{\tau-\alpha} \binom{\tau}{\alpha \theta} \sum_{\mu=0}^{\tau-\alpha-\theta} \binom{\tau-\alpha-\theta}{\mu} \Gamma_{ki}^{j(\tau-\alpha-\theta-\mu)} B_{i'}^{k(\mu)} B_{\beta j'}^{\theta i} \\ &= \sum_{\theta=\beta}^{\tau-\alpha} \binom{\tau}{\theta \varphi} \sum_{\varphi=\alpha}^{\tau-\theta} \Gamma_{ki}^{j(\tau-\theta-\varphi)} B_{\alpha i'}^{\varphi k} B_{\beta j'}^{\theta i} \quad (\text{putting: } \varphi = \mu + \alpha) \\ &= \sum_{\theta=\beta}^{\tau-\alpha} \sum_{\varphi=\alpha}^{\tau-\theta} \Gamma_{\varphi k \theta i}^j B_{\alpha i'}^{\varphi k} B_{\beta j'}^{\theta i} \end{aligned}$$

and

$$\begin{aligned} \binom{\tau}{\beta \alpha} ({}' \Gamma_{i' j'}^{k'} B_{k'}^j)^{(\tau-\beta-\alpha)} &= \binom{\tau-\alpha-\beta}{\beta \alpha} \sum_{\lambda=0}^{\tau-\alpha-\beta} \binom{\tau-\alpha-\beta}{\lambda} {}' \Gamma_{i' j'}^{k'(\tau-\alpha-\beta-\lambda)} B_{k'}^j{}^{(\lambda)} \\ &= \sum_{\theta=\alpha+\beta}^{\tau} \binom{\theta}{\alpha \beta} {}' \Gamma_{i' j'}^{k'(\theta-\alpha-\beta)} B_{\theta k'}^{\tau j} = \sum_{\theta=\alpha+\beta}^{\tau} {}' \Gamma_{\alpha i' \beta j'}^{\theta k'} B_{\theta k'}^{\tau j} \quad (\text{putting: } \theta = \gamma - \lambda) \end{aligned}$$

respectively. Consequently we have

$$\binom{\tau}{\beta \alpha} (H_{j' i'}^{\tau j})^{(\tau-\beta-\alpha)} = \frac{\partial B_{\alpha i'}^{\tau j}}{\partial u^{(\beta)j'}} + \sum_{\theta=\beta}^{\tau-\alpha} \sum_{\varphi=\alpha}^{\tau-\theta} \Gamma_{\varphi k \theta i}^j B_{\alpha i'}^{\varphi k} B_{\beta j'}^{\theta i} - \sum_{\theta=\alpha+\beta}^{\tau} {}' \Gamma_{\alpha i' \beta j'}^{\theta k'} B_{\theta k'}^{\tau j} \equiv H_{\beta j' \alpha i'}^{\tau j}.$$

Corollary 1. *Let $H_{i'j'}^{p''}$ and $H_{\alpha i' \beta j'}^{0p''}$ ([5], p. 39) be the components of $H_{i'j'}$*

referred to the λ -system in $C_n^{(n-m)}$ and of $H_{\alpha i' \beta j'}^{0j}$, referred to $\lambda_{\tau p'}^{2i}$ system in $C_n^{(M)(n-m)}$ respectively, then it follows that

$$H_{\alpha i' \beta j'}^{0p''} = \binom{0}{\alpha \beta} H_{i' j'}^{p''(0-\alpha-\beta)}.$$

Corollary 2. If $H_{i' j'}^j = 0$, then it follows that $H_{\alpha i' \beta j'}^{\tau j} = 0$.

Theorem 3.5. There exists

$$L_{\alpha i' \beta p'}^{\tau j} = \binom{\tau}{\alpha \beta} L_{i' p'}^{j(\tau-\alpha-\beta)},$$

where $L_{\alpha i' \beta p'}^{\tau j} \equiv D_{\alpha i'} \lambda_{\beta p'}^{\tau j}$, and $L_{i' p'}^j = D_{i'} \lambda_{p'}^j$.

Corollary. If $L_{i' p'}^{j'}$ and $L_{\alpha i' \beta p'}^{\tau j'}$ are the components of $L_{i' p'}^j$ referred to the u -coordinate system in C_m and of $L_{\alpha i' \beta p'}^{\tau j}$ referred to the u -coordinate system in $C_m^{(M)}$ respectively, then the following relation consists

$$L_{\alpha i' \beta p'}^{\tau j'} = \binom{\tau}{\alpha \beta} L_{i' p'}^{j'(\tau-\alpha-\beta)}.$$

The method of these proof is essentially similar to that of Theorem 3.4.

Let us define an *exgeodesic* in $C_n^{(M)}$ as a set of expoints $(x^i(t), x^{(1)i}(t), \dots, x^{(M)i}(t))$ which satisfy the equation

$$(3.9) \quad \frac{\delta^* x^{(\alpha+1)i}}{\delta t} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \rho^{(\alpha-\beta)} x^{(\beta+1)i} \quad \alpha=0, \dots, M,$$

where the symbol δ^* denotes excovariant differential ([2], p. 19) and ρ is an arbitrary function of t , being of class P , then putting $M=0$, the above equation becomes

$$\frac{\delta^* x^{(1)i}}{\delta t} = \rho x^{(1)i}$$

whose solution $x^i = x^i(t)$ is the geodesic in C_n . We can state the following theorems

from the fact that $\frac{\delta x^{(\alpha+1)i}}{\delta t} = \left(\frac{\delta x^{(1)i}}{\delta t} \right)^{(\alpha)}$ ([2], p. 20).

Theorem 3.6. When a curve of class P : $x^i = x^i(t)$ is geodesic in C_n , the set of the expoints $(x^i(t), x^{(1)i}(t), \dots, x^{(M)i}(t))$ along the geodesic is an *exgeodesic* in $C_n^{(M)}$.

Theorem 3.7. A necessary and sufficient condition for that an *exgeodesic* in $C_m^{(M)}$ be an *exgeodesic* in $C_n^{(M)}$, is that $H_{j' k'}^i = 0$.

We shall call such the exsurface $C_m^{(M)}$ as *exgeodesic exsurface*.

Proof. The extensor member of an expoint in $C_m^{(M)}$ is indicated by

$$x^{(\alpha+1)i} = \sum_{\beta=0}^{\alpha} B_{\beta j'}^{\alpha i} u^{(\beta+1)j'}, \quad \alpha=0, \dots, M-1,$$

differentiating excovariantly the above equation, we have

$$\frac{\delta^* x^{(\alpha+1)i}}{\delta t} = \sum_{\beta, \delta=0}^{\alpha} B_{\beta j', \delta k'}^{\alpha i} u^{(\beta+1)j'} u^{(\delta+1)k'} + B_{\beta j'}^{\alpha i} \frac{\delta^* u^{(\beta+1)j'}}{\delta t}$$

$$= H_{\delta k' \beta j'}^{\alpha i} u^{(\beta+1)j'} u^{(\delta+1)k'} + B_{\beta j'}^{\alpha i} \frac{\delta^* u^{(\beta+1)j'}}{\delta t}.$$

If a set of expoints: $x^{(\alpha)i}$ ($\alpha=0, \dots, M$) is an exgeodesic in $C_m^{(M)}$, then the following relations consist:

$$\frac{\delta^* u^{(\beta+1)j'}}{\delta t} = \sum_{\tau=0}^{\beta} \binom{\beta}{\tau} \rho^{(\beta-\tau)} u^{(\tau+1)j'}$$

and

$$\begin{aligned} \sum_{\beta=0}^{\alpha} B_{\beta j'}^{\alpha i} \frac{\delta^* u^{(\beta+1)j'}}{\delta t} &= \sum_{\beta=0}^{\alpha} B_{\beta j'}^{\alpha i} \sum_{\tau=0}^{\beta} \binom{\beta}{\tau} \rho^{(\beta-\tau)} u^{(\tau+1)j'} \\ &= \sum_{\tau=0}^{\alpha} \sum_{\lambda=0}^{\alpha-\tau} \binom{\alpha}{\tau} \binom{\alpha-\tau}{\lambda} B_{j'}^{i(\alpha-\tau-\lambda)} \rho^{(\lambda)} u^{(\tau+1)j'} \quad (\text{putting: } \lambda = \beta - \tau) \\ &= \sum_{\tau=0}^{\alpha} \binom{\alpha}{\tau} (\rho B_{j'}^i)^{(\alpha-\tau)} u^{(\tau+1)j'} = (\rho B_{j'}^i u^{(1)j'})^{(\alpha)} = (\rho x'^i)^{(\alpha)} \\ &= \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \rho^{(\alpha-\beta)} x^{(1+\beta)i}. \end{aligned}$$

Accordingly we can see that necessary and sufficient condition for that

$$\frac{\delta^* x^{(\alpha+1)i}}{\delta t} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \rho^{(\alpha-\beta)} x^{(1+\beta)i},$$

is that $H_{\delta k' \beta j'}^{\alpha i} = 0$, $\alpha, \beta, \delta = 0, \dots, M$. Our theorem is proved by virtue of this fact and Corollary 2 of Theorem 3.4.

Corollary. A necessary and sufficient condition for that $C_m^{(M)}$ becomes an exgeodesic exsurface is that C_m is a geodesic surface.

We call $K_n^{(M)}$ extended space of K_n (or briefly *exspace* of K_n), if all geometrical objects of $K_n^{(M)}$ are those extended from all geometrical objects in K_n . In fact, when K_n is an affinely connected space C_n , the space of line-elements of order M $C_n^{(M)}$ which has the properties given by the above statement may be called the *affinely connected exspace*. Similarly we can consider on a *Riemannian exspace*. Further it will be an interesting problem to discuss the exspaces of many spaces of various kinds.

(December, 1951)

Mathematical Institute,

Hokkaido University, Sapporo.

References.

- [1] H. V. Craig: On tensor relative to the extended point transformation, Amer. J. of Math., 59 (1937), 764-774.
- [2] Y. Katsurada: On the extended connection parameters in a space with affine connection and in Riemannian space, Jour. Fac. of Sci., Hokkaido Univ., 12 (1951), 17-28.

- [3] Y. Katsurada: Non-holonomic system in a space of higher order I. On the operations of extensors, Jour. Fac. of Sci., Hokkaido Univ., 11 (1950), 190-217.
- [4] Y. Katsurada: On the non-holonomic connection of extensors. Tensor, New Series 1 (1951), 60-66.
- [5] Y. Katsurada: Non-holonomic system in a space of higher order II. On the theory of extensors on the subspace, Jour. Fac. of Sci., Hokkaido Univ., 12 (1951), 29-41.
- [6] H. V. Craig: On the structure of intrinsic derivatives, Bulletin Amer. Math. Soc. 52 (1947 , 332-342.