18. On a Connection in an Areal Space.

By Akitsugu KAWAGUCHI and Yoshie KATSURADA.

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Introduction.

Let an m -dimensional subspace in an n -dimensional space X_n be given in the parameter forms:

$$(0.1) \quad x^i = x^i(u^a) \quad i=1, 2, \dots, n; \quad a=1, 2, \dots, m,$$

then it gives rise to the problem: How can we develop the theory of connections in the space where the area of the subspace is defined with an m -ple integral

$$(0.2) \quad O = \int \dots \int_{(m)} f(x^i, \partial x^i / \partial u^a) d(u^1 \dots u^m)$$

extended over a region of the subspace for which $f > 0$, f being an analytical function of x 's and $\frac{\partial x}{\partial u}$'s. This problem was treated first by one $([1] - [4])^1$ of the present authors.

For that the integral be invariant under coordinate and parameter transformations, the function f must be an absolute invariant for coordinate transformations, a scalar density of weight one for parameter transformations and positive homogeneous of degree one in $\frac{\partial x^i}{\partial u^a}$ for each value of a . If $p^{i_1 \dots i_m}$ denotes the simple m -vector obtained by omitting $n-k$ columns of the matrix $\left\| \frac{\partial x}{\partial u} \right\|$, it was proved by W. Gross ([5]) that the above integral can be expressed in the form

$$(0.3) \quad S = \int \dots \int_{(m)} F(x, p) d(u^1 \dots u^m),$$

where F is positively homogeneous of degree one in p 's. But, since $p^{i_1 \dots i_m}$ is simple, there are identical relations

$$(0.4) \quad p^{i_1 \dots i_m} p^{j_1 \dots j_m} = 0,$$

which show us the $\binom{n}{m}$ components of the m -vector are not all independent of each other. By this reason the functional form of F does not be definite, and its derivatives with respect to $p^{i_1 \dots i_m}$ have no meaning. We must therefore introduce a

1) Numbers in brackets refer to the references at the end of the paper.

proper method to study this problem.

In the particular case such as $n=5$, $m=2$ or 3 , this problem was solved by one ([6]—[8]) of the present authors and he treated the problem also in the general case from many points of view. Recently, the same problem is discussed by H. Iwamoto ([9]), V. Wagner ([10], [11]), E. T. Davies ([12], [13]) and R. Debever ([14]—[16]) from each different standpoints, but it seems to us that none of these treatments give the final form to the theory.

In the present paper we shall give only an attempt to establish the theory of connections in the space with an m -ple integral. Such a space will be called here an *areal space* A_n^m . We confine ourselves, for the sake of formal simplicity, to the case $m=2$. Introducing an intrinsic and symmetric tensor of four-index, A_{ijkl} , we shall determine the parameters of connection under some hypotheses in use of this tensor and derive curvature tensors of the space.

§ 1. The intrinsic and symmetric tensor with four-index.

As we confine ourselves to the case $m=2$, putting

$$(1.1) \quad p_a^i = \frac{\partial x^i}{\partial u^a}, \quad p^{ij} = 2 \frac{\partial x^i}{\partial u^{[1}} \frac{\partial x^j}{\partial u^{2]}} \quad a=1, 2,$$

and considering the metric bitensor $g_{ij,kl}$ introduced by A. Kawaguchi ([17]), the following equation is obtained

$$(1.2) \quad f^2 = \frac{1}{4} g_{ij,kl} p^{ij} p^{kl}.$$

$g_{ij,kl}$ are components of a covariant tensor of four-index for which the following relations hold good :

$$(1.3) \quad g_{(ij),kl} = g_{ij,(kl)} = 0, \quad g_{ij,kl} = g_{kl,ij}, \quad g_{i[j,kl]} = 0.$$

If the $\frac{1}{2}n(n-1)$ -rowed determinant g constructed from $g_{ij,kl}$ with respect to the pairs of indices (ij) , (kl) does not vanish in the considered domain, the quantities $g^{ij,lm}$ are determined uniquely by the equations

$$(1.4) \quad g_{ij,kh} g^{ij,lm} = 4 \delta_{[k}^l \delta_{h]}^m,$$

where δ_k^l denotes the Kronecker's delta. It is then evident that the quantities $g^{ij,lm}$ are components of a contravariant tensor of four-index and satisfy the relations

$$(1.5) \quad g^{(ij),kh} = g^{ij,(kh)} = 0, \quad g^{ij,kh} = g^{kh,ij}.$$

Now we put

$$(1.6) \quad A_{jh,pq} = \frac{1}{(n-1)(n-2)} \{ g_{i[j,(kl)h]} g^{il,km} g_{(p)l,(q)m} - g_{(j(p,h)q)} \} \quad n \geq 3,$$

which is a tensor whose components are symmetric with respect to j, h and p, q respectively and satisfy the relation

$$(1.7) \quad A_{jh,pq} = A_{pq,jh}.$$

Remark. If there is the relation

$$(1.8) \quad g^{[l,km]} = 0,$$

then we can prove as follows that the tensor:

$$(1.9) \quad A_{jh,pq}^* \equiv \frac{1}{(n-1)(n-2)} \{g_{ij,kh} g^{il,km} g_{pl,qm} - g_{jp,hq}\}$$

is symmetric with respect to j, h and similarly to p, q .

From (1.9) it follows

$$(1.10) \quad A_{[jh],pq}^* = \frac{1}{(n-1)(n-2)} \{g_{i[j,k]h} g^{il,km} g_{pl,qm} - g_{[j]p,[h]q}\},$$

$$(1.11) \quad g_{i[j,k]h} g^{il,km} g_{pl,qm} = g_{[i[j,k]h]} g^{[il,k]m} g_{[p]q,[m]}.$$

Since the last equation of (1.3) and (1.8) give us

$$(1.12) \quad g_{[i[j,k]h]} = \frac{-1}{2} g_{ik,hj}, \quad g^{[il,k]m} = \frac{-1}{2} g^{ik,ml},$$

we obtain

$$(1.13) \quad \begin{aligned} g_{i[j,k]h} g^{il,km} g_{pl,qm} &= \frac{-1}{8} g_{ik,hj} g^{ik,ml} g_{pq,ml} \\ &= \frac{-1}{2} \delta_{[h}^m \delta_{j]}^l g_{pq,ml} = \frac{-1}{2} g_{pq,hj}. \end{aligned}$$

The above result gives rise to

$$(1.14) \quad \begin{aligned} A_{[jh],pq}^* &= \frac{1}{(n-1)(n-2)} \left(-\frac{1}{2} g_{pq,hj} - g_{[j]p,[h]q} \right) \\ &= \frac{-1}{2(n-1)(n-2)} (g_{jp,hq} + g_{ph,jq} + g_{pq,hj}) \\ &= \frac{-3}{2(n-1)(n-2)} g_{p[q,h]j} = 0, \end{aligned}$$

and likely

$$(1.15) \quad A_{jh,[pq]}^* = 0.$$

If the $\frac{1}{2}n(n+1)$ -rowed determinant A of the $A_{jh,pq}$ with respect to (jh) and (pq) does not vanish identically, the quantities $A^{ij,kh}$ determined from

$$(1.16) \quad A_{ij,kh} A^{il,km} = \delta_j^i \delta_h^m$$

are components of a contravariant tensor of four-index which are symmetric with respect to i, j and k, h respectively, i. e.,

$$(1.17) \quad A^{ij, kh} = A^{ij, [kh]} = 0.$$

In a Riemannian space, we can see $A_{ij, kh} = A_{ij, kh}^* = g_{ij}g_{kh}$ and $A^{ij, kl} = A^{*ij, kl} = g^{ij}g^{kl}$, for in this case

$$(1.18) \quad g_{ij, kl} = 2g_{[i, k}g_{j]l}, \quad g^{ij, kl} = 2g^{[i, k}g^{j]l}$$

and (1.8) holds good.

§ 2. The connection parameters.

Let all geometrical objects, for example vectors and tensors, considered in the present paper have analytical functions of x^i and $\frac{\partial x^i}{\partial u^a}$ as their components and let us introduce the absolute differential of a vector $v^i \left(x^i, \frac{\partial x^i}{\partial u^a} \right)$ in this space in the form:

$$(2.1) \quad \delta v^i = dv^i + \Gamma_{jk}^i v^j dx^k + C_{jk}^{i, a} v^j dp_a^k.$$

In use of the following quantities corresponding to Christoffel symbols in a Finsler space introduced by E. Cartan ([18]):

$$(2.2) \quad \gamma_{jk}^i \equiv \frac{1}{2n} [A^{ip, ql} \{ \partial_k A_{pq, jl} + \partial_j A_{pq, lk} - \partial_l A_{pq, jk} \\ - \frac{1}{2n^2} A^{rs, tu} (A_{pq, lk} \partial_j A_{rs, tu} + A_{pq, jl} \partial_k A_{rs, tu} - A_{pq, jk} \partial_l A_{rs, tu}) \}],$$

where the symbol ∂_k means the derivative with respect to x^k , Γ_{jk}^i can be written in the form

$$(2.3) \quad \Gamma_{jk}^i = \gamma_{jk}^i + S_{jk}^i.$$

Denoting $A_{pq, jk, l}^a \equiv \frac{\partial A_{pq, jk}}{\partial p_a^l}$ and putting

$$(2.4) \quad C_{jk; m}^{li, a} \equiv \frac{1}{2n} \{ A^{pq, li} A_{pq, jk, m}^a - \frac{1}{2n^2} A^{pq, li} A_{pq, jk} A^{rs, tu} A_{rs, tu, m}^a \}, \\ 3G_{ab}^i \equiv \gamma_{jk}^i p_a^j p_b^k$$

and

$$(2.5) \quad S_{jk}^i = C_{jk; l}^{ip, b} \frac{\partial G_{ab}^i}{\partial p_a^p} - C_{kp; l}^{ip, b} \frac{\partial G_{ab}^i}{\partial p_a^p},$$

let us suppose only the case, where the following relations hold good

$$(2.6) \quad A_{ij, kh} A^{il, pq} = \delta_j^l T_{kh}^{pq}, \quad C_{jk; m}^{ip, a} p_b^j = 0,$$

which can be seen easily to be satisfied in a Riemannian space, then

$$(2.7) \quad \Gamma_{jk}^{*i} = \gamma_{jk}^i + C_{jk; l}^{ip, b} \frac{\partial G_{ab}^i}{\partial p_a^p} - C_{jp; l}^{ip, b} \frac{\partial G_{ab}^i}{\partial p_a^p} - C_{kp; l}^{ip, b} \frac{\partial G_{ab}^i}{\partial p_a^p}$$

varies under coordinate transformations $x^i = x^i(x^\lambda)$ as follows

$$(2.8) \quad \Gamma_{\tau \nu}^{* \lambda} = B_i^\lambda A_\tau^j A_\nu^k \Gamma_{jk}^{*i} + A_{\tau \nu}^i B_i^\lambda,$$

where $A_{\lambda}^i = \frac{\partial x^i}{\partial x^{\lambda}}$, $B_i^{\lambda} = \frac{\partial x^{\lambda}}{\partial x^i}$, $A_{\tau\nu}^i = \frac{\partial^2 x^i}{\partial x^{\tau} \partial x^{\nu}}$, etc.

In order to prove (2.8), we differentiate the both-hand sides of

$$(2.9) \quad A_{\alpha\beta, \tau\tau} = A_{\alpha}^i A_{\beta}^j A_{\tau}^k A_{\tau}^h A_{ij, kh}$$

with respect to x^{ν} , then we obtain

$$(2.10) \quad \begin{aligned} \partial_{\nu} A_{\alpha\beta, \tau\tau} = & \partial_{\tau} A_{ij, kh} A_{\alpha}^i A_{\beta}^j A_{\tau}^k A_{\tau}^h + A_{ij, kh, m} A_{\rho\nu}^m p_{\alpha}^{\rho} A_{\tau}^i A_{\beta}^j A_{\tau}^k A_{\tau}^h \\ & + A_{ij, kh} (A_{\alpha\nu}^i A_{\beta}^j A_{\tau}^k A_{\tau}^h + A_{\alpha}^i A_{\beta\nu}^j A_{\tau}^k A_{\tau}^h \\ & + A_{\alpha}^i A_{\beta}^j A_{\tau\nu}^k A_{\tau}^h + A_{\alpha}^i A_{\beta}^j A_{\tau}^k A_{\tau\nu}^h). \end{aligned}$$

On account of the first relation of (2.6), it follows

$$(2.11) \quad \begin{aligned} A^{\alpha\beta, \tau\lambda} (\partial_{\nu} A_{\alpha\beta, \tau\tau} + \partial_{\tau} A_{\alpha\beta, \tau\nu} - \partial_{\tau} A_{\alpha\beta, \tau\nu}) = & A^{lm, ni} (\partial_k A_{lm, jn} + \partial_j A_{lm, nk} \\ & - \partial_n A_{lm, jk}) B_i^{\lambda} A_{\tau}^j A_{\nu}^k + p_{\alpha}^{\rho} \{ A_{ij, kh, m} A_{\rho\nu}^m A_{\tau}^{ij, hp} A_{\rho\tau}^k B_p^{\lambda} \\ & + A_{ij, kh, m} A^{ij, hp} A_{\rho\tau}^m A_{\nu}^k B_p^{\lambda} - A_{ij, kh, m} A^{ij, pq} A_{\rho\tau}^m A_{\nu}^k A_{\tau}^p B_p^{\lambda} \} \\ & + 2(A_{\alpha\nu}^i B_i^{\lambda} \delta_{\tau}^{\lambda} + A_{\alpha\tau}^i B_i^{\lambda} \delta_{\nu}^{\lambda} + n A_{\tau\nu}^k B_k^{\lambda}) - \frac{2}{n} A_{\alpha\beta, \tau\nu} A^{\alpha\beta, \tau\lambda} A_{\delta\tau}^q B_q^{\delta}, \end{aligned}$$

and

$$(2.12) \quad \begin{aligned} & \frac{1}{2n^2} A^{\alpha\beta, \tau\lambda} A^{\xi\eta, \zeta\varphi} (A_{\alpha\beta, \tau\tau} \partial_{\nu} A_{\xi\eta, \zeta\varphi} + A_{\alpha\beta, \tau\nu} \partial_{\tau} A_{\xi\eta, \zeta\varphi} - A_{\alpha\beta, \tau\nu} \partial_{\tau} A_{\xi\eta, \zeta\varphi}) \\ & = \frac{1}{2n^2} [A^{lm, ni} A^{pq, rs} (A_{lm, jn} \partial_k A_{pq, rs} + A_{lm, nk} \partial_j A_{pq, rs} - A_{lm, jk} \partial_n A_{pq, rs}) A_{\tau}^j A_{\nu}^k B_i^{\lambda} \\ & \quad + 4n^2 (\delta_{\tau}^{\lambda} A_{\xi\nu}^p B_p^{\xi} + \delta_{\nu}^{\lambda} A_{\xi\tau}^p B_p^{\xi}) - 4n A^{\alpha\beta, \tau\lambda} A_{\alpha\beta, \tau\nu} A_{\xi\tau}^q B_q^{\xi} \\ & \quad + A^{pq, rs} A_{pq, ln} A_{ij, kh, m} A^{ij, kh} \{ \delta_{\tau}^n (B_s^{\lambda} A_{\tau}^i A_{\rho\nu}^m + B_s^{\lambda} A_{\nu}^i A_{\rho\tau}^m \\ & \quad - B_{\tau}^s B_s^{\lambda} A_{\tau}^i A_{\nu}^m A_{\rho\tau}^m) p_{\alpha}^{\rho} \}. \end{aligned}$$

(2.11) and (2.12) give us

$$(2.13) \quad \begin{aligned} \gamma_{\tau\nu}^{\lambda\tau} = & \gamma_{jk}^i A_{\tau}^j A_{\nu}^k B_i^{\lambda} + A_{\tau\nu}^i B_i^{\lambda} + \frac{1}{2n} [A_{ij, kh, m} A^{ij, hp} (A_{\rho\nu}^m A_{\tau}^k B_p^{\lambda} + A_{\rho\tau}^m A_{\nu}^k B_p^{\lambda}) \\ & - A_{ij, kh, m} A^{ij, pq} A_{\rho\tau}^m A_{\nu}^k A_{\tau}^p B_q^{\lambda} \\ & - \frac{1}{2n^2} \{ A^{pq, rs} A_{pq, lr} A_{ij, kh, m} A^{ij, kh} (B_s^{\lambda} A_{\tau}^i A_{\rho\nu}^m + B_s^{\lambda} A_{\nu}^i A_{\rho\tau}^m) \\ & - A^{pq, rs} A_{pq, ln} A_{ij, kh, m} A^{ij, kh} B_{\tau}^s B_s^{\lambda} A_{\tau}^i A_{\nu}^m A_{\rho\tau}^m \}] p_{\alpha}^{\rho} \\ & = \gamma_{jk}^i A_{\tau}^j A_{\nu}^k B_i^{\lambda} + A_{\tau\nu}^i B_i^{\lambda} + \frac{1}{2n} \{ A^{ij, hp} A_{ij, kh, m} \\ & - \frac{1}{2n^2} A^{ij, hp} A_{ij, kh} A^{rs, tu} A_{rs, tu, m} \} (A_{\rho\nu}^m A_{\tau}^k B_p^{\lambda} + A_{\rho\tau}^m A_{\nu}^k B_p^{\lambda}) p_{\alpha}^{\rho} \\ & - \frac{1}{2n} \{ A^{ij, pq} A_{ij, kh, m} - \frac{1}{2n^2} A^{ij, pq} A_{ij, kh} A^{rs, tu} A_{rs, tu, m} \} A_{\rho\tau}^m A_{\nu}^k A_{\tau}^p B_q^{\lambda} p_{\alpha}^{\rho} \end{aligned}$$

$$= \gamma_{jk}^i A_\tau^j A_\nu^k B_i^\lambda + A_{\tau\nu}^i B_i^\lambda + C_{kt;m}^{\tau i} (A_{p\nu}^m A_\tau^k B_p^\lambda + A_{p\tau}^m B_p^\lambda A_\nu^k) p_a^\rho \\ - C_{kh;m}^{ij} A_{p\tau}^m A_\nu^k A_\tau^h B_i^\lambda B_j^\rho p_a^\rho.$$

Contracting $p_a^\tau p_b^\nu$ to the both sides of (2.13) and applying the second relation of (2.6), it is obtained

$$(2.14) \quad \gamma_{\tau\nu}^\lambda p_a^\tau p_b^\nu = \gamma_{jk}^i p_a^j p_b^k B_i^\lambda + A_{\tau\nu}^i B_i^\lambda B_p^\tau B_q^\nu p_a^p p_b^q.$$

Accordingly the second equation of (2.4) leads us to

$$(2.15) \quad G_{ab}^\lambda = G_{ab}^i B_i^\lambda + \frac{1}{3} A_{\tau\nu}^i B_i^\lambda B_p^\tau B_q^\nu p_a^p p_b^q.$$

Differentiating the both sides of (2.15) with respect to p_a^ω and summing with respect to a , we get the following result

$$(2.16) \quad \frac{\partial G_{ab}^\theta}{\partial p_a^\omega} = \frac{\partial G_{ab}^i}{\partial p_a^\omega} A_\omega^i B_i^\theta + A_{\omega\nu}^i B_i^\theta p_a^\nu.$$

If we multiply the left- and right-hand sides of equation (2.16) by $C_{\tau\nu;\theta}^{\lambda\omega b}$ and $B_p^\lambda B_q^\omega A_\tau^s A_\nu^t A_\theta^m C_{rs;m}^{pq b}$ respectively, and sum up with respect to θ , ω and b , we have

$$(2.17) \quad \frac{\partial G_{ab}^\theta}{\partial p_a^\omega} C_{\tau\nu;\theta}^{\lambda\omega b} = B_i^\lambda A_\tau^j A_\nu^k C_{jk;m}^{it b} \frac{\partial G_{ab}^m}{\partial p_a^i} + C_{kh;m}^{pq a} A_{p\tau}^m A_\nu^k A_\tau^h B_q^\lambda B_p^\rho p_a^\rho,$$

which gives us with aid of (2.13)

$$(2.18) \quad \gamma_{\tau\nu}^\lambda + C_{\tau\nu;\theta}^{\lambda\omega b} \frac{\partial G_{ab}^\theta}{\partial p_a^\omega} - C_{\tau\omega;\theta}^{\lambda\omega b} \frac{\partial G_{ab}^\theta}{\partial p_a^\nu} - C_{\nu\omega;\theta}^{\lambda\omega b} \frac{\partial G_{ab}^\theta}{\partial p_a^\tau} \\ = B_i^\lambda A_\tau^j A_\nu^k \left\{ \gamma_{jk}^i + C_{jk;m}^{it b} \frac{\partial G_{ab}^m}{\partial p_a^i} - C_{jt;m}^{it b} \frac{\partial G_{ab}^m}{\partial p_a^i} - C_{kt;m}^{it b} \frac{\partial G_{ab}^m}{\partial p_a^i} \right\} + A_{\tau\nu}^i B_i^\lambda,$$

hence we can see the following result

$$(2.19) \quad \Gamma_{\tau\nu}^{*\lambda} = B_i^\lambda A_\tau^j A_\nu^k \Gamma_{jk}^{*i} + A_{\tau\nu}^i B_i^\lambda.$$

Equation (2.16) shows that the quantities $\frac{\partial G_{ab}^i}{\partial p_a^i}$ can be adopted as the

base connection parameters in the n -dimensional space X_n . Put

$$(2.20) \quad \delta^* p_a^k = dp_a^k + \frac{\partial G_{ab}^k}{\partial p_b^j} dx^j,$$

then $\delta^* p_a^k$ have the nature of a contravariant vector with respect to the index k , but not to the index a . By this reason we must modify the quantities $\delta^* p_a^k$ so that they have the nature of a vector with respect to a also. For this purpose, starting from $A_{ij,kl} p_a^i p_b^j p_c^k p_d^l = A_{ab,cd}$, we derive $A^{ab,cd}$ as like as $A^{ij,kl}$ and can introduce γ_{bc}^a by the same method as equation (2.2) has got. Then the quantities

$$(2.21) \quad \delta p_a^k = dp_a^k + \frac{\partial G_{ab}^k}{\partial p_b^j} dx^j - \gamma_{ab}^c p_c^k du^b$$

have the nature of a contravariant vector in the n -dimensional space and of a

covariant vector in the m -dimensional space with respect to the index k and the index a respectively.

From the above results, the absolute differential can be defined in the n -dimensional space as follows, by adopting the quantities Γ_{jk}^{*i} , $C_{jt;i}^{it\ a}$ given by (2.2), (2.4) and (2.7) or the quantities Γ_{jk}^i , $C_{jt;i}^{it\ a}$ given by (2.2), (2.3), (2.4) and (2.5),

$$\begin{aligned} \delta v^i &= dv^i + \Gamma_{jk}^{*i} v^j dx^k + C_{jt;i}^{it\ a} \delta p_a^i v^j \\ (2.22) \quad &= dv^i + \Gamma_{jk}^{*i} v^j dx^k + C_{jt;i}^{it\ a} v^j \delta^* p_a^i \quad (\text{because } C_{jt;i}^{it\ a} p_a^i = 0) \\ &= dv^i + \Gamma_{jk}^i v^j dx^k + C_{jt;i}^{it\ a} v^j dp_a^i \end{aligned}$$

for a vector $v^i \left(x, \frac{\partial x}{\partial u} \right)$ and

$$\begin{aligned} \delta T_{j_1 \dots j_q}^{i_1 \dots i_p} &= dT_{j_1 \dots j_q}^{i_1 \dots i_p} + \left(\sum_{s=1}^p \Gamma_{jk}^{*i_s} T_{j_1 \dots j_q}^{i_1 \dots i_{s-1} j_{s+1} \dots i_p} \right. \\ &\quad \left. - \sum_{r=1}^q \Gamma_{jr}^{*i} T_{j_1 \dots j_{r-1}^i j_{r+1} \dots j_q}^{i_1 \dots i_p} \right) dx^k + \left(\sum_{s=1}^p C_{jt;i}^{i_s\ ta} T_{j_1 \dots j_q}^{i_1 \dots i_{s-1} j_{s+1} \dots i_p} \right. \\ &\quad \left. - \sum_{r=1}^q C_{jr;t;i}^{i\ ta} T_{j_1 \dots j_{r-1}^i j_{r+1} \dots j_q}^{i_1 \dots i_p} \right) \delta p_a^i \\ (2.23) \quad &= dT_{j_1 \dots j_q}^{i_1 \dots i_p} + \left(\sum_{s=1}^p \Gamma_{jk}^{*i_s} T_{j_1 \dots j_q}^{i_1 \dots i_{s-1} j_{s+1} \dots i_p} \right. \\ &\quad \left. - \sum_{r=1}^q \Gamma_{jr}^{*i} T_{j_1 \dots j_{r-1}^i j_{r+1} \dots j_q}^{i_1 \dots i_p} \right) dx^k + \left(\sum_{s=1}^p C_{jt;i}^{i_s\ ta} T_{j_1 \dots j_q}^{i_1 \dots i_{s-1} j_{s+1} \dots i_p} \right. \\ &\quad \left. - \sum_{r=1}^q C_{jr;t;i}^{i\ ta} T_{j_1 \dots j_{r-1}^i j_{r+1} \dots j_q}^{i_1 \dots i_p} \right) \delta^* p_a^i \\ &= dT_{j_1 \dots j_q}^{i_1 \dots i_p} + \left(\sum_{s=1}^p \Gamma_{jk}^{i_s} T_{j_1 \dots j_q}^{i_1 \dots i_{s-1} j_{s+1} \dots i_p} \right. \\ &\quad \left. - \sum_{r=1}^q \Gamma_{jr}^i T_{j_1 \dots j_{r-1}^i j_{r+1} \dots j_q}^{i_1 \dots i_p} \right) dx^k + \left(\sum_{s=1}^p C_{jt;i}^{i_s\ ta} T_{j_1 \dots j_q}^{i_1 \dots i_{s-1} j_{s+1} \dots i_p} \right. \\ &\quad \left. - \sum_{r=1}^q C_{jr;t;i}^{i\ ta} T_{j_1 \dots j_{r-1}^i j_{r+1} \dots j_q}^{i_1 \dots i_p} \right) dp_a^i, \end{aligned}$$

for a tensor $T_{j_1 \dots j_q}^{i_1 \dots i_p} \left(x, \frac{\partial x}{\partial u} \right)$.

On the other hand the following relations between $A_{ij,kl}$ and $A^{ij,kl}$ can be found from the first condition of (2.6).

$$(2.24) \quad \begin{aligned} A_{lm,nq} A^{lm,rs} A^{nq,tu} &= \eta^2 A^{rs,tu}, & A_{pq,ij} A_{us,kl} A^{pq,us} &= \eta^2 A_{ij,kl}, \\ A^{pq,rs} A^{lm,tu} &= A^{rs,tu} A^{lm,pq}, & A_{ij,kl} A_{lm,pq} &= A_{ij,pq} A_{lm,kl}, \end{aligned}$$

of which the last relation produces

$$(2.25) \quad A_{lm,tu} A_{lm,tu} = A_{lm,lm} A_{tu,tu} \quad (\text{not summing}),$$

i. e.

$$(2.26) \quad A_{lm,tu} = \sqrt{A_{lm,lm}} \sqrt{A_{tu,tu}} \quad (\text{not summing}).$$

The above relation shows that the quantities $g_{lm} = \sqrt{A_{lm,lm}}$ are components of a symmetric covariant tensor of two-index and that the quantities $A_{lm,tu}$ can be decomposed as $A_{lm,tu} = g_{lm}g_{tu}$. Accordingly we can conclude that the first relation of (2.6) is the necessary and sufficient condition for that the quantities $A_{lm,tu}$ decompose to a product of two symmetric covariant tensors of two-index. Also the connection parameters Γ_{jk}^{*i} and $C_{jk;l}^{ih\ a}$ have the properties

$$(2.27) \quad \Gamma_{[jk]}^{*i} = 0, \quad C_{[jk;l]}^{ih\ a} = C_{jk;l}^{ih\ a} = 0,$$

as we can see easily.

It is stated that the absolute differential of $A_{pq,rs}$ vanishes identically

$$(2.28) \quad \delta A_{pq,rs} = 0.$$

We shall proceed to prove (2.28). On account of (2.23) we obtain

$$(2.29) \quad \begin{aligned} \delta A_{pq,rs} &= dA_{pq,rs} - (\Gamma_{pu}^t A_{tq,rs} + \Gamma_{qu}^t V_{pt,rs} + \Gamma_{ru}^t A_{pq,ts} + \Gamma_{su}^t A_{pq,rt}) dx^u \\ &\quad - (C_{pt;l}^{ti\ a} A_{tq,rs} + C_{qt;l}^{ti\ a} A_{pt,rs} + C_{rt;l}^{ti\ a} A_{pq,ts} + C_{st;l}^{ti\ a} A_{pq,rt}) dp_a^l \\ &= \{ \partial_u A_{pq,rs} - (\Gamma_{pu}^t A_{tq,rs} + \Gamma_{qu}^t A_{pt,rs} + \Gamma_{ru}^t A_{pq,ts} + \Gamma_{su}^t A_{pq,rt}) \} dx^u \\ &\quad + \{ A_{pq,rs,l} - (C_{pt;l}^{ti\ a} A_{tq,rs} + C_{qt;l}^{ti\ a} A_{pt,rs} + C_{rt;l}^{ti\ a} A_{pq,ts} + C_{st;l}^{ti\ a} A_{pq,rt}) \} dp_a^l \\ &= [\partial_u A_{pq,rs} - \{ (\gamma_{pu}^t + S_{pu}^t) A_{tq,rs} + (\gamma_{qu}^t + S_{qu}^t) A_{pt,rs} \\ &\quad + (\gamma_{ru}^t + S_{ru}^t) A_{pq,ts} + (\gamma_{su}^t + S_{su}^t) A_{pq,rt} \}] dx^u + \{ A_{pq,rs,l} \\ &\quad - (C_{pt;l}^{ti\ a} A_{tq,rs} + C_{qt;l}^{ti\ a} A_{pt,rs} + C_{rt;l}^{ti\ a} A_{pq,ts} + C_{st;l}^{ti\ a} A_{pq,rt}) \} dp_a^l. \end{aligned}$$

On the other hand the first equation of (2.4) gives us with help of the first relation of (2.24)

$$(2.30) \quad C_{jk;l}^{ih\ b} = \frac{1}{n} C_{j\bar{l};\ i}^{m\bar{t}\ b} A_{pq,mk} A^{pq,th}.$$

Multiplying the both sides of the above equation by $A_{it,us}$ and summing with respect to the index i , we get in use of the second relation of (2.24)

$$(2.31) \quad C_{jk;l}^{ih\ b} A_{it,us} = \delta_l^b C_{j\bar{n};\ i}^{m\bar{n}\ b} A_{mk,us}.$$

Contracting $\frac{\partial G_{ab}^l}{\partial p_a^k}$ to the both sides of the last equation, it follows

$$(2.32) \quad C_{jk;l}^{ih\ b} A_{it,us} \frac{\partial G_{ab}^l}{\partial p_a^k} = C_{j\bar{n};\ i}^{m\bar{n}\ b} A_{mk,us} \frac{\partial G_{ab}^l}{\partial p_a^k}.$$

Consequently

$$(2.33) \quad S_{jk}^i A_{it,us} = C_{j\bar{n};\ i}^{m\bar{n}\ b} A_{mk,us} \frac{\partial G_{ab}^l}{\partial p_a^k} - C_{kn;l}^{in\ b} A_{it,us} \frac{\partial G_{ab}^l}{\partial p_a^j}.$$

As it follows that $C_{kn;l}^{in\ b} A_{it,us} = C_{ln;l}^{in\ b} A_{ik,us}$ from the definition of the quantities $C_{kn;l}^{in\ b}$ we can see

$$(2.34) \quad S_{jk}^i A_{it,us} = C_{j\bar{n};\ i}^{m\bar{n}\ b} A_{mk,us} \frac{\partial G_{ab}^l}{\partial p_a^k} - C_{ln;l}^{in\ b} A_{ik,us} \frac{\partial G_{ab}^l}{\partial p_a^j}.$$

Consequently, the quantities of the right-hand side of equation (2.34) are skew-symmetric with respect to the indices j and t . This fact leads us to the relation

$$(2.35) \quad S_{jk}^i A_{it,us} + S_{ik}^i A_{ij,us} = 0.$$

On account of the last relation, (2.29) can be written in the form

$$(2.36) \quad \delta A_{pq,rs} = \{ \partial_u A_{pq,rs} - (\gamma_{ru}^t A_{tq,rs} + \gamma_{qu}^t A_{pt,rs} + \gamma_{ru}^t A_{pq,ts} + \gamma_{su}^t A_{pq,rt}) \} dx^u \\ + \{ A_{pq,rs,i} - (C_{pi;l}^{ti\ a} A_{tq,rs} + C_{qi;l}^{ti\ a} A_{pt,rs} + C_{ri;l}^{ti\ a} A_{pq,ts} + C_{si;l}^{ti\ a} A_{pq,rt}) \} dp_a^i.$$

Since

$$(2.37) \quad \partial_u A_{pq,rs} - (\gamma_{ru}^t A_{tq,rs} + \gamma_{qu}^t A_{pt,rs} + \gamma_{ru}^t A_{pq,ts} + \gamma_{su}^t A_{pq,rt}) = 0, \\ A_{pq,rs,i} - (C_{pi;l}^{ti\ a} A_{tq,rs} + C_{qi;l}^{ti\ a} A_{pt,rs} + C_{ri;l}^{ti\ a} A_{pq,ts} + C_{si;l}^{ti\ a} A_{pq,rt}) = 0,$$

the definitions of the quantities γ_{pu}^t and $C_{pi;l}^{ti\ a}$ give rise to $\delta A_{pq,rs} = 0$. The relations (2.27), (2.28) and (2.35) may be considered to be analogous to the postulates ([18]) for euclidean connection in a Finsler space that have been adopted by E. Cartan.

§ 3. The curvature tensors and torsion tensor.

We proceed now to calculate the curvature tensors and the torsion tensor of the space. For this purpose we make use of E. Cartan's symbols.

As $G_{ij,kl,m} p_b^m = 0$ offers us $C_{ik;m}^{jk\ a} p_b^m = 0$,

$$(3.1) \quad \omega_i^j = \Gamma_{ik}^{*j} dx^k + C_{it;k}^{jt\ a} \delta p_a^k$$

can be written as follows

$$(3.2) \quad \omega_i^j = \Gamma_{ik}^{*j} dx^k + C_{it;k}^{jt\ a} \delta^* p_a^k.$$

Calculating $[\omega_i^j dx^i]$ ([19]), we have hence

$$(3.3) \quad [\omega_i^j dx^i] = C_{it;k}^{jt\ a} [dx^i \delta^* p_a^k] = C_{it;k}^{jt\ a} [dx^i \delta p_a^k]$$

which shows that quantities $C_{it;k}^{jt\ a}$ are the components of the torsion tensor of the space.

We are now on a step to introduce the curvature tensors of the space. Calculate Ω_i^j symbol of E. Cartan, then we have

$$(3.4) \quad \Omega_i^j = (\omega_i^j)' - [\omega_i^k \omega_k^j] \\ = [d_1 \{ \Gamma_{ik}^{*j} dx^k + C_{ip;k}^{jp\ a} \delta_2^* p_a^k \}] \\ - [\{ \Gamma_{ik}^{*k} dx^l + C_{ip;m}^{kp\ a} \delta_1^* p_a^m \} \{ \Gamma_{km}^{*j} dx^m + C_{kp;n}^{jp\ a} \delta_2^* p_a^n \}] \\ = S_{ikl}^{jab} [\delta_1^* p_b^l \delta_2^* p_a^k] + P_{ikl}^{ja} [\delta_1^* p_a^l dx_2^k] + R_{ilk}^j [d_1 x^l dx_2^k],$$

putting

$$(3.5) \quad S_{ikl}^{jab} = C_{ip;k,l}^{jp\ a\ b} - C_{ip;l}^{jp\ b} C_{qr;k}^{jr\ a}, \\ P_{ikl}^{ja} = \Gamma_{ik,l}^{*j\ b} - \partial_k C_{ip;l}^{jp\ b} + C_{ip;l,m}^{jp\ b\ c} G_{ca,k}^m + C_{ad,k,l}^{jp\ d} C_{jq;p}^a + \Gamma_{ik}^{*p} C_{pt;l}^{jt\ b} - \Gamma_{pk}^{*j} C_{it;l}^{pt\ b}, \\ R_{ilk}^j = \partial_l \Gamma_{ik}^{*j} - \Gamma_{ik,m}^{*j\ a} G_{ab,l}^m + C_{it;p}^{jt\ a} \partial_l G_{ab,k}^b - C_{it;p}^{jt\ a} G_{ab,k,l}^b C_{cd,i}^a - \Gamma_{ik}^{*p} \Gamma_{pk}^{*j}.$$

Equation (3.4) can be written in the form

$$(3.6) \quad \Omega_i^j = S_i^j [ab] [\delta_1^* p_b^l \delta_2^* p_a^k] + P_{ikl}^{ja} [\delta_1^* p_b^l dx_2^k] + R_{[ik]l}^j [d_1 x^l dx_2^k],$$

where

$$(3.7) \quad S_i^j[ab] = \frac{1}{2}(S_{ikl}^{jab} - S_{ilk}^{jba}).$$

The first equation of (2.4) gives

$$(3.8) \quad C_{it;k}^{jt\ a} = \frac{1}{2n} \{ A^{pq,rj} A_{pq,ri,k}^a - \frac{1}{2n} \delta_i^j A^{pq,rs} A_{pq,rs,k}^a \},$$

from which partial differentiation with respect to p_b^i gives us

$$(3.9) \quad \begin{aligned} C_{it;k,l}^{jt\ a\ b} = & \frac{1}{2n} \{ A^{pq,rj} A_{pq,ri,k,l}^{a\ b} + A^{pq,rj} A_{pq,ri,k}^b A_{pq,rs,l}^a \\ & - \frac{1}{2n} \delta_i^j (A^{pq,rs} A_{pq,rs,k}^b + A^{pq,rs} A_{pq,rs,l}^a) \}, \end{aligned}$$

consequently

$$(3.10) \quad \begin{aligned} C_{it;k,l}^{jt\ a\ b} - C_{it;l,k}^{jt\ b\ a} = & \frac{1}{2n} [A^{pq,rj} A_{pq,ri,k}^b A_{pq,rs,l}^a - A^{pq,rj} A_{pq,ri,l}^a A_{pq,rs,k}^b \\ & - \frac{1}{2n} \delta_i^j (A^{pq,rs} A_{pq,rs,k}^b - A^{pq,rs} A_{pq,rs,l}^a)]. \end{aligned}$$

By reason of $A_{pq,rs,k}^a p_b^k = 0$ and $A^{pq,rs} p_b^k = 0$ followed from $G_{ij,kh,m}^a p_b^m = 0$, we obtain the relation

$$(3.11) \quad S_i^j[ab] [\delta_i^* p_b^l \delta_j^* p_a^k] = S_i^j[ab] [\delta_i p_b^l \delta_j p_a^k].$$

Furthermore, since the quantities Γ_{ik}^{*j} are the homogeneous functions of degree zero in p 's, it follows that

$$(3.12) \quad \Gamma_{ik}^{*j} p_a^i = 0.$$

The definition of the quantities $C_{it;l}^{jt\ b}$ shows us

$$(3.13) \quad \partial_k C_{it;l}^{jt\ b} p_c^i = 0,$$

because the quantities $\partial_k A_{pq,rs}$ are homogeneous of degree zero in p . Consider the identity

$$(3.14) \quad \begin{aligned} C_{it;l,m}^{jt\ b\ c} G_{ca,k}^m + G_{ad,k,l}^p C_{it;p}^{jt\ a} = & \frac{\partial}{\partial p_b^i} (C_{it;m}^{jt\ c} G_{ca,k}^m) \\ & + (C_{it;l,m}^{jt\ b\ c} G_{ca,k}^m - C_{it;m,l}^{jt\ c\ b} G_{ca,k}^m). \end{aligned}$$

It follows at first $\frac{\partial}{\partial p_b^i} (C_{it;m}^{jt\ c} G_{ca,k}^m) p_a^i = 0$ in use of the homogeneity of degree zero in p 's of the quantities $C_{it;m}^{jt\ c} G_{ca,k}^m$. The second term of the right-hand side of (3.14) goes into

$$(3.15) \quad C_{it;l,m}^{jt\ b\ c} G_{ca,k}^m - C_{it;m,l}^{jt\ c\ b} G_{ca,k}^m = (C_{it;l,m}^{jt\ b\ c} - C_{it;m,l}^{jt\ c\ b}) G_{ca,k}^m$$

and making use of the right-hand side of (3.10), we have

$$(3.16) \quad (C_{it;l,m}^{jt\ b\ c} - C_{it;m,l}^{jt\ c\ b}) p_a^i = 0,$$

consequently

$$(3.17) \quad (C_{it;l,m}^{jt \ b \ c} G_{ca,k}^m - C_{it;m,l}^{jt \ c \ b} G_{ca,k}^m) p_a^l = 0.$$

Accordingly we can conclude

$$(3.18) \quad (C_{it;l,m}^{jt \ b \ c} G_{ca,k}^m + G_{ac,k,l}^p C_{it;p}^{jt \ a}) p_a^l = 0.$$

Making use of the above relation, it can be proved $P_{ikl}^{j \ b} p_c^l = 0$ from which we obtain

$$(3.19) \quad P_{ikl}^{j \ b} [\delta_1^* p_b^l d_2 x^k] = P_{ikl}^{j \ b} [\delta_1 p_b^l d_2 x^k],$$

with help of the last result we can get the following expression

$$(3.20) \quad \mathcal{D}_i^j = S_i^{j \ [ab]} [\delta_1 p_b^l \delta_2 p_a^k] + P_{ikl}^{j \ b} [\delta_1 p_b^l d_2 x^k] + R_{i[uk]}^j [d_1 x^l d_2 x^k].$$

After an easy calculation we have from (3.8) and (3.15)

$$(3.21) \quad S_i^{j \ [ab]} = \frac{-1}{8n^2} \{ (2n A^{rp, sq} A^{tu, vj} - A^{rt, sv} A^{pu, qj}) (A_{pu, qv, i} A_{rt, si, k}^b - A_{pu, qv, k} A_{rt, si, i}^b) \},$$

$$P_{ikl}^{j \ b} = \Gamma_{i \ k, l}^{*j \ b} - \partial_k C_{it;l}^{jt \ b} + (C_{it;m}^{jt \ c} G_{ca,k}^m)_{,l}^b + (C_{it;l,m}^{jt \ b \ c} - C_{it;m,l}^{jt \ c \ b}) G_{ca,k}^m$$

$$+ \Gamma_{i \ k}^{*p} C_{pt;l}^{jt \ b} - \Gamma_{p \ k}^{*j} C_{it;l}^{pt \ b}.$$

Putting

$$(3.22) \quad \frac{1}{2} \delta p^{mn} = 2 \delta p_{[i} p_{s]}^{mn} = \omega^{mn}(d),$$

we get

$$(3.23) \quad \frac{\partial \varphi}{\partial p_a^l} \delta p_a^l = \frac{\partial \varphi}{\partial p^{mn}} \omega^{mn}(d),$$

where φ denotes any function of p^{mn} . Consequently it follows

$$(3.24) \quad A_{pq,rs,i} \delta p_b^i = \frac{\partial A_{pq,rs}}{\partial p^{mn}} \omega^{mn}(d).$$

Let us denote, for brevity, the partial derivative with respect to p^{mn} with the symbol

,mn as $\frac{\partial \varphi}{\partial p^{mn}} = \varphi_{,mn}$, then we have

$$(3.25) \quad S_i^{j \ [ab]} [\delta_1 p_b^l \delta_2 p_a^k] = S_{imntu}^j [\omega^{mn}(d_1) \omega^{tu}(d_2)],$$

$$(3.26) \quad P_{ikl}^{j \ b} [\delta_1 p_b^l d_2 x^k] = P_{ikmn}^j [\omega^{mn}(d_1) d_2 x^k],$$

where

$$(3.27) \quad S_{imnzw}^j = \frac{-1}{8n^2} (2n A^{rp, sq} A^{tu, vj} - A^{rt, sv} A^{pu, qj})$$

$$\times (A_{pu, qv, mn} A_{rt, si, zw} - A_{pu, qv, zw} A_{rt, si, mn}),$$

$$P_{ikmn}^j = \Gamma_{i \ k, mn}^{*j} - \partial_k C_{it;mn}^{jt} + (C_{it;l}^{jt \ a} G_{ab,k}^l)_{,mn}$$

$$- \frac{1}{2n} A^{rp, sq} A^{tu, vj} (A_{pu, qv, zw} A_{rt, si, mn} - A_{pu, qv, mn} A_{rt, si, zw}) G_k^{zw}$$

$$+ \Gamma_{i \ k}^{*p} C_{pt;mn}^{jt} - \Gamma_{p \ k}^{*j} C_{it;mn}^{pt},$$

$$G_k^{zv} = 4 \frac{\partial G_{a1}^z}{\partial p_c^k} p_c^{zv}, \quad C_{pt;mn}^{jt} = \frac{1}{2n} (A^{rs,tj} A_{rs,tp,mn} - \frac{1}{2n} \delta_r^j A^{rs,tu} A_{rs,tu,mn}).$$

Consequently, equation (3.20) is written in the form

$$(3.28) \quad \Omega_i^j = S_{imntu}^j [\omega^{mn}(d_1) \omega^{tu}(d_2)] + P_{ikmi}^j [\omega^{mn}(d_1) d_2 x^k] + R_{ikl}^j [d_1 x^k d_2 x^l]$$

and likewise equation (3.3) is given by

$$(3.29) \quad [\omega_i^j dx^i] = C_{it,mn}^{jt} [dx^i \omega^{mn}].$$

We obtain thus the intrinsic curvature tensors of three kinds S_{imntu}^j , P_{ikmi}^j , R_{ikl}^j and a torsion tensor $C_{it,mn}^{jt}$ of the n -dimensional space. But, since the p^{mn} are not independent of each other, the derivative of any function of p^{mn} is indeterminate, followingly these tensors are undetermined also. In order to avoid this indefiniteness, we shall introduce a new operation that will be written down in the following lines.

Let us consider any two functions of p^{mn} Φ and Φ^* that have equal functional value, then the relation between Φ and Φ^* is given by

$$(3.30) \quad \Phi^* = \Phi + \lambda_{ijkl} p^{[ij} p^{kl]},$$

where λ_{ijkl} denote arbitrary functions of the p^{mn} and are skew-symmetric with respect to their indices. Differentiating the both sides of (3.30) with respect to the p^{mn} , we get

$$(3.31) \quad \frac{\partial \Phi^*}{\partial p^{mn}} = \frac{\partial \Phi}{\partial p^{mn}} + \frac{\partial \lambda_{ijkl}}{\partial p^{mn}} p^{[ij} p^{kl]} + \psi_{mn} = \frac{\partial \Phi}{\partial p^{mn}} + \psi_{mn},$$

where $\psi_{mn} = 2\lambda_{ijmn} p^{ij}$, from which follows easily that

$$(3.32) \quad \psi_{mn} p^{ml} = 2\lambda_{ijmn} p^{ij} p^{ml} = 2\lambda_{ijmn} p^{[ij} p^{m]l} = 0.$$

Now we can determine uniquely the functions ψ_{mn} from the condition

$$(3.33) \quad \frac{\partial \Phi^*}{\partial p^{mn}} G_{ij} = 0$$

with help of G_{ij} ([20]) which gives the transversal two-direction of p^{ij} . Because (3.31) and (3.33) lead us to

$$(3.34) \quad G_{[ij} \frac{\partial \Phi}{\partial p^{mn]} + G_{[ij} \psi_{mn]} = 0.$$

Multiplying the both sides of (3.34) by p^{ij} and summing with respect to i and j , we obtain in use of (3.32)

$$(3.35) \quad G_{[ij} \frac{\partial \Phi}{\partial p^{mn]} p^{ij} + \frac{1}{3} f^2 \psi_{mn} = 0 \quad (G_{ij} p^{ij} = f^2),$$

consequently

$$(3.36) \quad \psi_{mn} = \frac{-3}{f^2} G_{[ij} \frac{\partial \Phi}{\partial p^{mn]} p^{ij}.$$

In such the way, $\frac{\partial \Phi^*}{\partial p^{mn}}$ are determined uniquely.

Let us adopt the uniquely determined derivatives of the above meaning for the quantities $A_{ij,kl}$, or Γ_{il}^{*j} , $C_{il;m}^{jt}$ $\frac{\partial G_{ab}^m}{\partial p_b^k}$ which are all functions of p^{mn} , then three curvature tensors and torsion tensor can be determined uniquely. The geometrical meaning of these tensors may be given in the same way as that of Cartan in a Finsler space ([18]).

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Mathematical Institute,

Hokkaido University, Sapporo.

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