5. On Free Modular Lattices.

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Introduction.

The present investigation is an attempt to solve the word-problem of the free modular lattice. (Birkhoff [1], Whitman [2]). Without solving it completely, it will clarify some facts on the free modular lattices, which may be, as the author hopes, of interest in themselves.

Let G be an arbitrary set, with an arbitrary cardinal number $|G| = g$, and S be the set of all lattice-polynomials generated by the elements of G . We shall introduce in S the following three equivalence relations:

$a \equiv b$ means that a and b have the same form as lattice-polynomials under the commutative and the associative laws;

$a = b$ means that a and b represent the same element of the free lattice under the absorptive law in addition to the laws above mentioned;

$a \sim b$ means that a and b represent the same element of the free modular lattice under the further assumption of the modular law.

In identifying the elements of S , which are equivalent under the above three definitions, we obtain three systems P , L and M . L and M are the free lattice and the free modular lattice generated by G respectively.

Denote with $a, b, \dots$ the elements of P . Let $\bar{a}$ represent the element of L , which is also the ' $=$ -class' of P , containing a ; and $\bar{\bar{a}}$ the element of M , which is also the ' $\sim$ -class' of L , containing $\bar{a}$.

We shall write further $\lambda a \equiv b$ for $a = b$, denoting by λ the operation 'transforming' a to b . Let Λ be the set of all these λ 's, transforming an element a of P to elements $b = a$, a, b running over all ' $=$ -elements' in P . From $\lambda, \lambda_1, \lambda_2 \in \Lambda$ follows clearly $\lambda^0, \lambda^{-1}, \lambda_2 \lambda_1 \in \Lambda$, if λ_1, λ_2 are 'composable', namely when $\lambda_1 a \equiv b$ and $\lambda_2 b \equiv c$. In case where no misunderstanding is to be feared, we shall denote generally and indistinctly with λ any element of Λ . Thus $\lambda^{-1} = \lambda, \lambda \lambda = \lambda$.

Now, let a be an element of P with a subpolynomial $\alpha \smile (\beta \frown \gamma)$:

$$a \equiv \dots \smile \{ \alpha \smile (\beta \frown \gamma) \} \smile \dots$$

and c with a subpolynomial $(\alpha \cup \beta) \cap \gamma$:

$$c \equiv \cdots \smile \{(\alpha \cup \beta) \cap \gamma\} \smile \cdots,$$

α and c being distinct only in the subpolynomials indicated here. If $\alpha \leq \gamma$, $\leq$ denoting the lattice-order in L , we have $a \sim c$ by the modular law. We shall denote with μ the transformation which brings a to c , write

$$a \equiv \cdots \alpha \cup (\beta \cap \gamma) \cdots \overset{\mu}{\sim} \cdots (\alpha \cup \beta) \cap \gamma \cdots \equiv c$$

and call $\alpha \cup (\beta \cap \gamma)$ the *substance*, $\beta \cap \gamma$ the *kernel* and β the *medium* of μ . If we have to distinguish the 'direction' of the transformation μ , we shall write $c \equiv +\mu a$, $a \equiv -\mu c$. Otherwise we shall deal with the symbol μ in the same way as with the symbol λ , namely as a general and indistinct symbol for any of these transformations.

$a \sim b$ means then that $b \equiv \lambda \mu \cdot \lambda \mu \cdots \lambda \mu \cdot \lambda a$, or, rewriting μ^* for $\lambda \mu \lambda$, $b \equiv \mu^* \cdot \mu^* \cdots \mu^* a$. For brevity, we shall also write

$$a \cdot \pi \mu^* \cdot b$$

(π being the symbol of the product of the operations), and say that a and b are connected by a "bond" $\pi \mu^*$. The number of μ 's will be called the *length* of the bond. If all the μ 's have the same 'sign' (+ or -), a and b are said to be connected by a "monotonous bond".

The word-problem of the free modular lattice requires to give finite steps to decide whether $a \sim b$ or not. Thereby we can restrict ourselves in the case $a \leq b$ without loss of generality, as the relation $\sim$ in L is a congruence relation.

The outline of this investigation is as follows. In § 1 we shall show that two lattice-polynomials a, b satisfying $a \leq b$ and $a \sim b$ can be connected by a "monotonous bond." § 2 is devoted to the structure of the class $\bar{a}$. A necessary and sufficient condition will be given for $\bar{a}$ to be the least or the greatest element of $\bar{a}$. In investigating the structure of the class $\bar{a}$, we shall also obtain a theorem on the structure of the free lattice (Theorem 8) and one on the modular and the absorptive laws (Theorem 4).

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§ 0. Preliminaries.

1. Throughout this paper, the join operation is denoted by $\cup$ or $\sum$, the meet

operation by $\cap$ or $\prod$. And the logical symbols $\forall, \exists$ and $\rightarrow$ will be made use of, which should be read 'for all', 'for some' and 'implies' respectively. We shall write

$$a \equiv \sum_{i=0}^s a^i \quad \text{or} \quad a \equiv \prod_{i=0}^s a_i \quad (*)$$

with the 'upper indices' i for the summation and with the 'lower indices' i for the product.

Let us define inductively, the type, the rank etc. of the lattice-polynomial as follows:

Definition : The type of $a \in P$:

- (i) The generator $x \in G$ has the $\sum$ - and the $\prod$ -types.
- (ii) $a \equiv \sum_{i=0}^s a^i$ ($s \geq 1$) has the $\sum$ -type, when a^i has the $\prod$ -type for all i . And its dual.

When we hereafter write as $(*)$, we shall always assume a^i to be of $\prod$ -type and a_i to be of $\sum$ -type.

Definition : The rank $R(a)$ of $a \in P$:

- (i) $R(x) = 0$ for $x \in G$.
- (ii) $R(a) = r$ where $a \equiv \sum_{i=0}^s a^i$ and $\max R(a^i) = r - 1$. And its dual.

Definition : The n th-component of $a \in P$:

- (i) a^i is a first-component of a where $a \equiv \sum a^i$. And its dual.
- (ii) A first-component of an n th-component of a is called an $(n+1)$ th-component of a .

Then we have clearly:

"The kernel of μ is a first-component of the substance of μ ."

Definition : The n th-subpolynomial of a :

- (i) The join of more than one a^i 's is called the 0-subpolynomial of a , when $a \equiv \sum a^i$. And its dual.
- (ii) The 0-subpolynomial of the n th-component of a is called the n th-subpolynomial of a .

Then it is clear that:

"If a has the $\sum$ -type, then the $2n$ th-subpolynomial of a has the $\sum$ -type, and the $(2n+1)$ th-subpolynomial of a has the $\prod$ -type." "If a has the $\prod$ -type, then the $2n$ th-subpolynomial of a has the $\prod$ -type, and the $(2n+1)$ th-subpolynomial of a has the $\sum$ -type."

A generator $x \in G$ is, for itself, of $\sum$ -type and of $\prod$ -type at the same time.

But if it is contained in a with $R(a) \geq 1$, the type of x will be determined by these two rules. We shall always interpret thus throughout this paper.

Definition : The n th-modularity:

When the substance of μ is an n th-subpolynomial, μ is called an n th-modularity.

Then we have:

"If a has the $\sum$ -type, then the $2n$ th-modularity is a $+\mu$, the $(2n+1)$ th-modularity is a $-\mu$. And dually."

2. We shall base ourselves on the following theorem on the free lattice. (Whitman [1], [2]).

Theorem : $a \leq b$ if and only if

$$\begin{array}{ll} a = \sum a^i & \& \forall i \quad a^i \leq b \\ \text{or} & a = \prod a_i & \& \exists i \quad a_i \leq b \\ \text{or} & b = \sum b^i & \& \exists i \quad a \leq b^i \\ \text{or} & b = \prod b_i & \& \forall i \quad a \leq b_i. \end{array}$$

Definition : The number of generators contained in a (repetition being allowed) is called the *length* of a and denoted by $S(a)$.

Theorem : Each $\equiv$ -class $\bar{a}$ contains one and only one lattice-polynomial with the shortest length. This is called the *canonical form* of a .

Theorem : $a \equiv \sum a^i \equiv \sum \prod a^i_j$ is canonical if and only if

(0) $\forall i \quad a^i$ is canonical.

(1)* $\forall i, k \quad (a^i \leq a^k \rightarrow i = k)$ or (1) $\forall i \quad a^i \leq \sum_{k \neq i} a^k$.

(2) $\forall i, j \quad (a^i_j \neq a^i \rightarrow a^i_j \leq a)$.

(Note: $a^i_j \neq a^i$ means that a^i is not a generator. (1)* and (1) are equivalent under (2).)

3. Theorems on the rank.

Denote the canonical form of a by κa . Operation κ is the one of λ 's.

Lemma : (Whitman [2]) We can obtain κa from a by the finite applications of the following three kinds of substitutions.

Let $b \equiv \sum b^i \equiv \sum \prod b^i_j$.

(0) If $b^i \neq \kappa b^i$, then substitute b by $\kappa b^i \cup \sum_{k \neq i} b^k$.

(1) If $b^i \leq \sum_{k \neq i} b^k$, then substitute b by $\sum_{k \neq i} b^k$.

(2) If $b^i_j \neq b^i$ and $b^i_j \leq b$, then substitute b by $b^i_j \cup \sum_{k \neq i} b^k$.

Theorem : $R(\kappa a) \leq R(a)$.

Proof: Let $R(a) = r$. (i) It is evident for $r = 0$.

(ii) Suppose that the theorem is true up to $R(a) = r-1$.

In the lemma above, the substitution (0) does not raise the rank after the hypothesis of the induction, as well as the substitutions (1) and (2). Q.E.D.

Definition : Let C denote a class of the elements of L with respect to the equivalence relation $\sim$. From the theorem above we can define $R(C)$, the rank of C , as

$$R(C) = \min_{a \in C} \{R(a)\} = \min_{a \in C} \{R(\kappa a)\}.$$

Theorem : If $g < \aleph_0$ then the set $T_r = \{\kappa a \mid R(a) \leq r\}$ is finite.

Proof: Let us estimate this value under the idempotent law which is much weaker than the absorptive law.

Put $|T_r| = t_r$, then

(i) $t_0 = g$.

(ii) $t_r \leq t_{r-1} + 2 \sum_{\tau=2}^{t_{r-1}} \binom{t_{r-1}}{\tau} \leq 2^{t_{r-1}+1}$

Therefore,

Q. E. D.

Corollary : If $g \geq 4$, then the set of integers $R(C)$ is unbounded.

Proof: If $g \geq 4$, M has infinitely many elements. (Birkhoff [1]). Thus the corollary is proved for $4 \leq g < \aleph_0$. To remove the condition $g < \aleph_0$, we have only to prove that "If $G \subset G^*$ and $a \sim b$ in G^* , then $a \sim b$ in G ."¹⁾ We shall omit the proof, which is very simple and always hold for any abstract algebra. (Substitute the elements of $G^* - G$ by an arbitrary element of G). Q.E.D.

(Note: Later on, we shall discuss the case $R(C) \geq 3$, the existence of such C being established by this corollary.)

4. Application of the notion of the rank.

Whitman proposed the problem: "Is there any order-convergent series in a

1) This proposition means that if the word-problem is solved for finite g , the problem is solved for arbitrary g .

free lattice?" (Whitman [2], Unsolved Problem 3). We are going to show that if $g \geq \aleph_0$, there exists an order-convergent series in $L(g)$.

Let $g \geq \aleph_0$ and $a = x \bigwedge_{\substack{S \\ 0}} \sum_{x \in S} x$, where S denotes an element of the stack (in the sense of Tukey [1]) of $G - x$. Since a is canonical, $S_1 \subset S_2 \rightarrow a_{s_1} < a_{s_2}$.

Proposition : $a \uparrow_{\substack{S \\ 0}} x$, i.e. $c < x \rightarrow \mathcal{H}S c < a_{\substack{S \\ 0}}$.

Proof: Without loss of generality, suppose that c is canonical.

(i) The proposition is evidently true for $R(c) \leq 1$.

(ii) Suppose that if $R(c) < r$ the proposition is true, and let us consider the case $R(c) = r$.

The case $c \equiv \sum c^i$:

$$c < x \rightarrow \bigvee i \quad c^i < x \rightarrow \bigvee i \quad \mathcal{H}S_i \quad c^i < a_{s_i}$$

$$\mathcal{H}S \quad c \leq a_{s'} < a_{\substack{S \\ 0}} \quad \text{where} \quad S' = \sum_i S_i.$$

The case $c \equiv \prod c_i$

$$c < x \xrightarrow{\substack{c \equiv \kappa c \\ 0}} \mathcal{H}i \quad \bigvee j \quad c_i^j < x_{\substack{0 \\ 0}}$$

$$\mathcal{H}i \quad \mathcal{H}S \quad c_i < a_{\substack{S \\ 0}} \rightarrow \mathcal{H}S \quad c < a_{\substack{S \\ 0}} \quad \text{Q.E.D.}$$

Remark: The problem is not solved for finite g , although $L(g)$ (g being finite) contains $L(\aleph_0)$ as a sublattice.

§ 1. Theorems concerning the direction of the modularity.

Theorem 1 : $a \leq b$ & $a \sim b \rightarrow a \sim b$ i.e. two lattice-polynomials a, b satisfying $a \leq b$ and $a \sim b$ can be connected by a "monotonous bond".

Proof: We divide the proof into four steps.

$$(I) \quad a = c \wedge a^* \quad \& \quad c \cup a^* = c^* \quad \& \quad a \sim c \rightarrow a^* \sim c^* \quad (\varepsilon_i = \pm).$$

$$(Proof) \quad a^* = a \cup a^* \sim c \cup a^* = c^*.$$

(II) If a and a are connected by a "monotonous bond" $a_0 \leq a_1 \leq \dots \leq a_n$ with the length n , then there exists a "monotonous bond" with the length at most $2n$ connecting a_0 and a_n through any b satisfying $a_0 \leq b \leq a_n$.

$$(Proof) \quad a_0 = a_0 \wedge b \sim a_1 \wedge b \sim \dots \sim a_n \wedge b = b$$

$$= a_0 \cup b \sim a_1 \cup b \sim \dots \sim a_n \cup b = a_n.$$

(III) A necessary and sufficient condition for a and d with $a \leq d$ to be connected by a "monotonous bond" is that

$$\mathcal{A}b, c \quad \begin{matrix} \pi + \mu^* \\ a \sim b \end{matrix} \geq \begin{matrix} \pi + \mu^* \\ c \sim d \end{matrix}.$$

(Proof) Necessity: Take d, a for b, c respectively.

Sufficiency: Since $\begin{matrix} a \\ c \end{matrix} \leq \begin{matrix} b \wedge d \\ d \end{matrix} \leq \begin{matrix} b \\ d \end{matrix}$, we have $\begin{matrix} \pi + \mu^* \\ a \sim b \wedge d \end{matrix} \sim \begin{matrix} \pi + \mu^* \\ d \end{matrix}$ by (II).

(IV) If $\begin{matrix} a \\ 0 \end{matrix} \leq \begin{matrix} a \\ 2n+1 \end{matrix}$ & $\begin{matrix} \pi + \mu^* \\ a \\ 0 \end{matrix} \sim \begin{matrix} \pi + \mu^* \\ a \\ 1 \end{matrix} \geq \begin{matrix} \pi + \mu^* \\ a \\ 2 \end{matrix} \sim \begin{matrix} \pi + \mu^* \\ a \\ 3 \end{matrix} \geq \dots \geq \begin{matrix} \pi + \mu^* \\ a \\ 2n \end{matrix} \sim \begin{matrix} \pi + \mu^* \\ a \\ 2n+1 \end{matrix}$, then

$$\begin{matrix} \pi + \mu^* \\ a \\ 0 \end{matrix} \sim \begin{matrix} a \\ 2n+1 \end{matrix}.$$

(Proof) Since $\begin{matrix} a \\ 2i+1 \end{matrix} \geq (\begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i-1 \end{matrix}) \wedge \begin{matrix} a \\ 2i+1 \end{matrix} \geq \begin{matrix} a \\ 2i \end{matrix}$ for all $i \leq n-1$, we have, by (II),

$$(\begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i-1 \end{matrix}) \wedge \begin{matrix} \pi + \mu^* \\ a \\ 2i+1 \end{matrix} \sim \begin{matrix} \pi + \mu^* \\ a \\ 2i+1 \end{matrix}$$

and further by (I), $\begin{matrix} \pi + \mu^* \\ a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i-1 \end{matrix} \sim \begin{matrix} \pi + \mu^* \\ a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i+1 \end{matrix}$.

Hence, $\begin{matrix} \pi + \mu^* \\ a \\ 0 \end{matrix} \sim \begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2n-1 \end{matrix}$.

By (III), therefore, $\begin{matrix} \pi + \mu^* \\ a \\ 0 \end{matrix} \sim \begin{matrix} a \\ 2n+1 \end{matrix}$.

Q.E.D.

Remark: We can estimate the length of the "monotonous bond" in this theorem, and the maximum length of the lattice-polynomials which appear in this "bond".

Let $N_i (i=0, \dots, n)$ be the length of the bond $\begin{matrix} \pi + \mu^* \\ a \\ 2i \end{matrix} \sim \begin{matrix} a \\ 2i+1 \end{matrix}$, and $N = \sum N_i$, then the required length of the "monotonous bond" is

$$\leq 2(N_0 + 2N_1 + 2N_2 + \dots + 2N_{n-1}) + 2N_n < 4N.$$

Let $\begin{matrix} i \\ b \\ j \end{matrix} \equiv (\begin{matrix} i \\ +\mu \end{matrix}) \begin{matrix} i \\ \lambda \end{matrix} (\begin{matrix} i \\ +\mu \end{matrix}) \dots \begin{matrix} i \\ \lambda \end{matrix} \begin{matrix} i \\ a \\ 0 \end{matrix} \begin{matrix} i \\ 2i \end{matrix}$, $\begin{matrix} i \\ b^* \\ j \end{matrix} \equiv \begin{matrix} i \\ \lambda \end{matrix} (\begin{matrix} i \\ +\mu \end{matrix}) \begin{matrix} i \\ \lambda \end{matrix} \dots \begin{matrix} i \\ \lambda \end{matrix} \begin{matrix} i \\ a \\ j \end{matrix} \begin{matrix} i \\ 0 \end{matrix} \begin{matrix} i \\ 2i \end{matrix}$ where

$$\begin{matrix} a \\ 2i+1 \end{matrix} \equiv \begin{matrix} i \\ \lambda \end{matrix} (\begin{matrix} i \\ +\mu \end{matrix}) \begin{matrix} i \\ \lambda \end{matrix} \dots \begin{matrix} i \\ \lambda \end{matrix} \begin{matrix} i \\ a \\ v \end{matrix} \begin{matrix} i \\ 2i \end{matrix}, \text{ and } \max_{i, j} \{S(\begin{matrix} i \\ b \\ j \end{matrix}), S(\begin{matrix} i \\ b^* \\ j \end{matrix})\} = l.$$

Since $S(\begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i-1 \end{matrix}) \wedge \begin{matrix} a \\ 2i+1 \end{matrix} \leq (i+1)l$, the length of the lattice-polynomials which appear in the "monotonous bond" connecting $\begin{matrix} a \\ 1 \end{matrix}$ and $\begin{matrix} a \\ 2i+1 \end{matrix}$ through

$\begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i-1 \end{matrix}) \wedge \begin{matrix} a \\ 2i+1 \end{matrix}$ is, by (II), $\leq (i+2)l$.

Since $S(\begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i-1 \end{matrix}) < S(\begin{matrix} a \\ 1 \end{matrix} \cup \begin{matrix} a \\ 3 \end{matrix} \cup \dots \cup \begin{matrix} a \\ 2i+1 \end{matrix}) \leq (i+1)l$, the length of the lattice-polynomials which appear in the "monotonous bond" connecting

$a \underset{1}{\cup} a \underset{3}{\cup} \dots \underset{2i-1}{\cup} a$ and $a \underset{1}{\cup} a \underset{3}{\cup} \dots \underset{2i+1}{\cup} a$ is, by (I), $\leq (2i+3)l$. Hence, the length of the lattice-polynomials which appear in the "monotonous bond" connecting a and $a \underset{0}{\cup} a \underset{1}{\cup} \dots \underset{2n-1}{\cup} a$ is $\leq (2n+1)l$.

Therefore, the required length is, by (III),

$$\leq \max((2n+3)l, 3l) = (2n+3)l.$$

Remark: From this theorem we can obtain the following statement. "The word-problem of the free modular lattice means no more than to give the finite steps to decide whether two lattice-polynomials can be connected by a "monotonous bond" or not."

Corollary: " $\bar{c}$ is not the least element of C " $\rightarrow \mathcal{A} - \mu^* - \mu^*c < c$. " $\bar{c}$ is not the greatest element of C " $\rightarrow \mathcal{A} + \mu^* + \mu^*c > c$.

Proof: As $\sim$ is a congruence relation, c is not a minimal element unless it is the least element. The rest is an immediate consequence of the theorem above.

Q.E.D.

Conjecture: In the above estimation on the length of the "monotonous bond", can $4N$ be replaced more precisely by N ?

The following theorem, although not used later, may be of importance for this conjecture.

Theorem 2: Let $\mu(\beta)$ denote a μ having β as a medium. If $a \leq b \leq c$, then

$$\mathcal{A}\mu^* a \underset{\mu^*(\beta)}{\sim} c \not\geq \mathcal{A}\mu^*, \mu^* a \underset{\mu^*(\beta)}{\sim} b \underset{\mu^*(\beta)}{\sim} c.$$

Proof: First we prove that

$$a \leq a^* \leq c^* \leq c \quad \& \quad a \underset{\mu^*(\beta)}{\sim} c \rightarrow a^* \underset{\mu^*(\beta)}{\sim} c^*$$

$$\text{If } a = a \underset{\lambda}{\cup} (\beta \cap \gamma) \underset{\mu(\beta)}{\sim} (a \cup \beta) \cap \gamma = c$$

$$\text{then } a^* \leq a^* \cup (\beta \cap c^*) \leq a^* \cup (\beta \cap c) \leq a^* \cup (\beta \cap \gamma) = a^*.$$

$$\text{Therefore, } a^* = a^* \underset{\lambda}{\cup} (\beta \cap c^*) \underset{\mu(\beta)}{\sim} (a^* \cup \beta) \cap c^* = c^*.$$

(I) Proof of $\rightarrow$:

Put first $a^* \equiv a$, $c^* \equiv b$, and next $a^* \equiv b$, $c^* \equiv c$, we get then

$$a = a \underset{\lambda}{\cup} (\beta \cap b) \underset{\mu_0(\beta)}{\sim} (a \cup \beta) \cap b = b = b \underset{\lambda}{\cup} (\beta \cap c) \underset{\mu_1(\beta)}{\sim} (b \cup \beta) \cap c = c.$$

(II) Proof of $\leftarrow$:

From $(a \cup \beta) \cap b = b = b \cup (\beta \cap c)$ we get $a \cup \beta \geq \frac{b}{\beta} \geq \beta \cap c$,

and whence $a \cup \beta = b \cup \beta \geq \beta \cap b = \beta \cap c$.

Therefore, $a = a \cup (\beta \cap b) = a \cup (\beta \cap c) \stackrel{\mu(\beta)}{\sim} (a \cup \beta) \cap c = (b \cup \beta) \cap c = c$

Q. E. D.

§ 2. Theorems concerning the structure of the classes.

1. For brevity, we write $a \underset{s}{\frown} a \underset{s-1}{\frown} \dots \frown a \underset{2}{\frown} a \underset{1}{\frown} a \underset{0}{\frown}$ for $a \underset{s}{\frown} (a \underset{s-1}{\frown} \dots \frown (a \underset{2}{\frown} (a \underset{1}{\frown} a) \dots))$. ($\frown$ stand for $\cup$ or $\cap$ appearing in any combination.)

Lemma 1: Let $R(c) \leq 1$.

If c has the Π -type, then $c \leq a \cup b \rightarrow c \leq a$ or $c \leq b$.

If c has the Σ -type, then $c \geq a \cap b \rightarrow c \geq a$ or $c \geq b$.

Proof: It is evident for $R(c) = 0$.

Let $R(c) = 1$ and c have the Π -type. Then

$$\begin{aligned} \Pi c_i \leq a \cup b &\rightarrow (\exists i \ c_i \leq a \cup b) \quad \text{or} \quad c \leq a \quad \text{or} \quad c \leq b \\ &\rightarrow \exists i \ (c_i \leq a \text{ or } c_i \leq b) \quad \text{or} \quad c \leq a \quad \text{or} \quad c \leq b \\ &\rightarrow c \leq a \quad \text{or} \quad c \leq b \end{aligned}$$

Q. E. D.

Remark: The next statement, although having an analogous appearance as above, is not always true.

If c has the Σ -type, then $c \leq a \cup b \rightarrow c \leq a$ or $c \leq b$.

If c has the Π -type, then $c \geq a \cap b \rightarrow c \geq a$ or $c \geq b$.

Theorem 3:

1. Let $R(c) \leq 1$.

(1) If c has the Σ -type, then $a \leq c \rightarrow \mu a \leq c$.

(2) If c has the Π -type, then $a \geq c \rightarrow \mu a \geq c$.

2. Let $R(c) \leq 2$.

(1) If c has the Σ -type, then $a \geq c \rightarrow \mu a \geq c$.

(2) If c has the Π -type, then $a \leq c \rightarrow \mu a \leq c$.

Proof: 1 (1)

(I) The case $a \equiv \sum a^i$: We have only to prove for $+$ μ , i.e. $2n$ -th modularity.

$$\begin{aligned} a \leq c &\rightarrow \forall i \ a^i \leq c \rightarrow \forall i \ (\exists k \ a^i \leq c^k \text{ or } \exists j \ a_j^i \leq c) \\ &\rightarrow \forall i \ (\exists j \ \exists k \ a_j^i \leq c^k \text{ or } \exists j \ a_j^i \leq c) \rightarrow \forall i \ \exists j \ a_j^i \leq c. \quad (*) \end{aligned}$$

(i) Let μ be a 0-modularity, $\alpha \cup \beta \cap \gamma$ be the substance of μ , $a \equiv a^* \cup \alpha \cup \beta \cap \gamma$, and $\beta \cap \gamma$ be the kernel of μ . Then

$$\alpha \equiv \alpha^* \cup \alpha \cap \beta \cap \gamma \quad . + \mu. \quad \alpha^* \cup \gamma \cap \beta \cup \alpha \equiv \mu \alpha.$$

Since $\alpha^*, \alpha \leq \alpha \leq c$ and β or $\gamma \leq c$ (by $(*)$), we have $\mu \alpha \leq c$.

(ii) Suppose that the statement is true for $2n$ -th modularity for all a .

By $(*)$ α_0^0 or $\prod_{j \neq 0} \alpha_j^0 \leq c$.

Since the substance of μ is a $2n$ -subpolynomial of a suitable α_0^0 , we have

$$\mu \alpha_0^0 \quad \text{or} \quad \prod_{j \neq 0} \alpha_j^0 \leq c.$$

Therefore, $\mu \alpha \equiv \sum_{i \neq 0} \alpha^i \cup \prod_{j \neq 0} \alpha_j^0 \cap \mu \alpha_0^0 \leq c$.

(II) The case $a \equiv \prod \alpha_j$: We have only to prove for $+\mu$, i.e. $(2n+1)$ -th modularity.

Since $\alpha \leq c \xrightarrow{\text{(Lemma 1)}} \prod_i \alpha_i \leq c$, we have α_0 or $\prod_{i \neq 0} \alpha_i \leq c$.

Since the substance of μ is a $2n$ -subpolynomial of a suitable α_0 , we have

$$\mu \alpha_0 \quad \text{or} \quad \prod_{i \neq 0} \alpha_i \leq c \quad \text{by (I).}$$

Therefore, $\mu \alpha \equiv \prod_{i \neq 0} \alpha_i \cap \mu \alpha_0 \leq c$.

1(2) is dual to 1(1) and is true as well.

2(1)

$$a \geq c \rightarrow \forall k \quad a \geq c^k \xrightarrow{1(2)} \forall k \quad \mu a \geq c^k \rightarrow \mu a \geq c.$$

2(2) is dual to 2(1) and is true as well.

Q. E. D.

Corollary 1: $R(C) \leq 1 \rightarrow |C| = 1$.

(If $g < \aleph_0$, the number of c satisfying $R(C) \leq 1$ is $2^{g+1} - g - 2$.)

Corollary 2: Let $R(\kappa a) = 2$.

If κa has the $\sum$ -type ($\prod$ -type), $\bar{a}$ is the least (greatest) element of $\bar{a}$. (Hence if $R(C)=2$, the least or the greatest element of C surely exists.)

Proof: Use the Corollary of Theorem 1.

We have the following conjecture, which we can neither prove or disprove as yet.

Conjecture: If $R(C) = 2$, either the least or the greatest element of C cannot exist.²⁾

Theorem 4: The modular law and the absorptive law cannot be replaced by one composition-equation (Verknüpfungsgleichung in the sense of Shôda [1], two

2) This conjecture, together with Theorem 5 and Theorem 6, implies that: $R(C) \geq 2$ if and only if C contains an infinite chain.

composition-equations dual to each other being identified) under the associative and the commutative laws.

Proof: Suppose that there exists a composition-equation $\alpha \sim \beta$ (and its dual) having the absorptive and the modular laws as the derived equations (Folgleichungen) under the commutative and the associative laws. Let, without loss of generality, $S(\alpha) \leq S(\beta)$, then, since $\alpha \sim \beta$ has the absorptive law as the derived equation, we have $S(\alpha) = 1$, i.e. $\alpha \sim \beta$ has the form $x \sim \beta$.

Hence, the modular law being its derived equation, we have $\beta \equiv \pi\mu^*x$. By Corollary 1, therefore, $\beta = x$, i.e. $x \sim \beta$ is a derived equation of the absorptive law under the commutative and the associative laws. The modular law, therefore, becomes a derived equation of the absorptive law, which contradicts the existence of non-modular lattices.

Q. E. D.

Remark: This shows the difficulty of the word-problem of the free modular lattice. It suggests also a new problem in the theory of abstract algebras, i.e. to determine the minimum value of the number of the composition-equations necessary to decide an algebra.

2. Let c (having, for example, the $\sum$ -type) be a component of c ($R(c)=r$), then c is expanded uniquely with respect to c in the following form.

$$c \equiv a \underset{s}{\smile} a \cdots \underset{2}{\smile} a \underset{1}{\smile} c \quad (1)$$

(where, $c \underset{\sigma}{\smile} c$, and c means a component of c with the least rank among those containing c).

Here, $s \leq r$, because $R(c) \geq R(c) + 1$.

By the definition of the rank, there exists an expansion satisfying $s = r$. In this expansion, $R(c) = \sigma$ for all σ , and especially c is a generator. Such an expansion is called a *complete expansion* of c .

Replace c in c by c^* , and denote it with c^* .

$$c \equiv \kappa c \ \& \ S(c) = S(c^*) \ \& \ c \not\equiv c^* \rightarrow c \equiv c^*.$$

(The proof is very simple, because $a \equiv \kappa a \ \& \ S(a) = S(b) \ \& \ a = b \rightarrow a \equiv b$).

Notation: For brevity we shall write "contr." and "contr.(1)" for "which is a contradiction." and "which contradicts to (1)" respectively.

Theorem 5:

1. Let $R(\kappa a) = 2$.

If κa has the $\sum$ -type, then $\mathcal{A} + \mu^* + \mu^*a > a$.

If κa has the Π -type, then $\mathcal{A} - \mu^* - \mu^*a < a$.

2. Let $R(\kappa a) \geq 3$ then $\mathcal{H} + \mu^*, -\mu^* - \mu^* a < a < +\mu^* a$.

Proof: Put $\kappa a \equiv c \cdots (2)$, and suppose that $R(C) = 2$ and c has the Σ -type, or that $R(c) \geq 3$. Considering, for example, a complete expansion of c , we conclude that there exists an expansion with the form (1) satisfying that $R(c) \leq 1$, c has the Σ -type and $s \geq 2$.

(I) The case $a \leq c$: Take $+\mu$ as follows.

$$c \equiv a \underset{2}{\cup} a \underset{2}{\cap} c \underset{0}{\cdot} + \mu \cdot c \underset{0}{\cap} a \underset{1}{\cup} a \underset{2}{\cap} c \neq c^*.$$

Since $c \equiv \kappa c$ (by (2)), $S(c) = S(c^*)$ and $c \neq c^*$, we have $c \neq c^*$.

Therefore, $c \neq c^*$, where c^* denotes c with c replaced by c^* .

(II) The case $a \not\leq c \cdots (3)$. Take $+\mu$ as follows.

$$c \equiv a \underset{2}{\cup} a \underset{2}{\cap} c \underset{1}{\cap} a \underset{0}{\cap} c \underset{2}{\cap} a \underset{1}{\cap} (a \underset{2}{\cap} c) \underset{1}{\cup} a \underset{2}{\cap} a \underset{1}{\cup} + \mu \cdot a \underset{2}{\cup} a \underset{2}{\cap} c \underset{1}{\cap} a \underset{0}{\cap} c \underset{2}{\cap} a \underset{1}{\cap} a \equiv c^{**}.$$

Put $c^* \equiv c \underset{0}{\cap} a \underset{0}{\cap} a$, and let c^* be the polynomial obtained from c after substitution of c^* for c , then $c^* \equiv c^{**}$.

Therefore, we have only to prove $c < c^*$ by induction on s .

(i) The proof of $c < c^*$.

$$c \geq c^* \equiv c \underset{0}{\cup} a \underset{0}{\cap} a \geq a \underset{2}{\cap} a \xrightarrow{(\text{Lemma 1})} c \geq a \text{ or } a \quad \text{contr. (3) or (2)}.$$

The proof of $c < c^*$.

$$c \equiv a \underset{1}{\cap} c \geq a \underset{1}{\cap} c \underset{0}{\cup} a \underset{2}{\cap} a \equiv c^* \rightarrow c \geq a \underset{1}{\cap} c \underset{0}{\cup} a \underset{2}{\cap} a$$

$$\xrightarrow{(\text{Lemma 1})} c \geq a \text{ or } c \geq c \underset{0}{\cup} a \underset{0}{\cap} a \xrightarrow{(\text{Lemma 1})} c \geq a \text{ or } a \quad \text{contr. (3) or (2)}$$

(ii) Suppose that $c < c^* \cdots (4)$.

for even s

$$\begin{aligned} c &= a \underset{s}{\cup} a \underset{s-1}{\cap} c \geq a \underset{s}{\cup} a \underset{s-1}{\cap} c^* = c^* \\ &\rightarrow a \underset{s}{\cup} a \underset{s-1}{\cap} c \geq a \underset{s}{\cup} c^* \\ &\rightarrow \textcircled{1} a \underset{s}{\cup} a \underset{s-1}{\cap} c \geq a \\ &\rightarrow c = a \underset{s}{\cup} a \quad \text{contr. (2)} \\ \text{or } \textcircled{2} a \underset{s}{\cup} a \underset{s-1}{\cap} c &\geq c^* \geq c \\ &\rightarrow c = a \underset{s}{\cup} c \quad \text{contr. (2)} \end{aligned}$$

for odd s

$$\begin{aligned} c &= a \underset{s}{\cap} a \underset{s-1}{\cup} c \geq a \underset{s-1}{\cap} a \underset{s-1}{\cup} c^* = c^* \\ &\rightarrow a \underset{s-1}{\cap} c \geq a \underset{s-1}{\cap} a \underset{s-1}{\cup} c^* \\ &\textcircled{1} a \geq a \underset{s-1}{\cap} a \underset{s-1}{\cup} c^* \geq a \underset{s-1}{\cap} a \underset{s-1}{\cup} c \\ &\rightarrow c = a \underset{s}{\cap} a \quad \text{contr. (2)} \\ \text{or } \textcircled{2} c &\geq a \underset{s-1}{\cap} a \underset{s-1}{\cup} c^* \geq a \underset{s-1}{\cap} a \underset{s-1}{\cup} c \\ &\rightarrow c = a \underset{s}{\cap} c \quad \text{contr. (2)} \end{aligned}$$

or ③ $a \underset{s}{\geq} a \underset{s-1}{\wedge} c^* \underset{s-1}{\geq} a \underset{s-1}{\wedge} c$ $\rightarrow c \underset{s}{=} a$ contr. (2)	or ③ $a \underset{s-1}{\cup} c \underset{s-1}{\geq} a$ $\rightarrow c \underset{s}{=} a$ contr. (2)
or ④ $a \underset{s-1}{\wedge} c \underset{s-1}{\geq} a \underset{s-1}{\wedge} c^* \underset{s-1}{\geq}$ contr. (4)	or ④ $a \underset{s-1}{\cup} c \underset{s-1}{\geq} a \underset{s-1}{\cup} c^* \underset{s-1}{\geq}$ contr. (4) Q. E. D.

Theorem 6 : $R(C) \leq 1$ if and only if C contains only one element.

Proof: By Theorem 5 and Corollary 1 of Theorem 3.

Theorem 7 : Let $R(C) \geq 2$. $\bar{a}$ is the least (greatest) element of C if and only if $R(\kappa a) = 2$ and κa has the Σ -type (Π -type).

Proof: By Theorem 5 and Corollary 2 of Theorem 3.

Theorem 8 : L (with more than two generators) contains an infinite chain with the order-type $(\omega^* + \omega)\omega$. (ω^* means the order-type dual to ω .)

Proof: We employ the following two theorems. (Whitman [2]).

"If $g \geq 4$, $M(g)$ contains an infinite chain."

"If $g \geq 3$, $L(g)$ contains $L(\aleph_0)$ as a sublattice."

Since, in $L(4)$, the set of all C satisfying $R(C) \leq 2$ is finite, $M(4)$ contains an ω -type chain of C satisfying $R(C) \geq 3$. (Use the self-duality of M , if necessary.)

Since, by Theorem 5, every such C contains an infinite chain with the order-type $\omega^* + \omega$, $L(4)$ contains an infinite chain with the order-type $(\omega^* + \omega)\omega$.

If $g \geq 3$, $L(g)$ contains $L(\aleph_0)$ as a sublattice, and $L(\aleph_0)$ surely contains $L(4)$ as a sublattice, so the proof is completed.

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