5. Complex Multiple Wiener Integral.

By

Kiyosi Itô.

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Introduction. It is the aim of our present paper to discuss the properties of multiple Wiener integral on a complex normal random measure and its applications to normal stochastic processes.

We have already treated in our previous paper [3] (abbreviated hereafter as M. W. I.) an integral of the same kind on a real normal random measure, and remarked that it would be more adequate and convenient to consider it in the complex field as is often seen in the theory of Fourier transforms. This is indeed the case, as will be shown in this paper.

Chapter I is mainly concerned with Complex Normal Random Measure on which our integral will be discussed. Roughly speaking, it is an additive set function whose value for any set is a complex-valued random variable subject to an isotropic Gaussian distribution on the complex plane and whose values for disjoint sets are mutually independent. We shall here treat also a more general concept "Complex-normal System" for the later use.

In Chapter II we shall define an integral named as Complex Multiple Wiener Integral:

\[ \int \cdots \int f(t_1, \ldots, t_p; s_1, \ldots, s_q) dM(t_1) \cdots dM(t_p) d\overline{M(s_1)} \cdots d\overline{M(s_q)}, \]

\(M\) being a normal random measure, and in Chapter III we shall establish fundamental properties of this integral. The idea is the same as in M. W. I. but we shall here make the interpretations clearer and more precise.

In M.W.I. we have shown that real multiple Wiener integrals are closely related to Hermite polynomials. To establish such relations on our present complex case, we shall define Hermite polynomials of complex variables in a natural way; this definition seems new as far as we know.

In Chapter IV we shall make use of the results obtained to generalize and derive systematically various known facts on the ergodicity and spectral structure of temporally homogeneous processes; our method will make clearer the essential points of the problems.

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Notations

\begin{align*}
K & \quad \text{complex number space} \\
\Omega(B_\Omega, P) & \quad \text{basic probability space} \\
\omega & \quad \text{probability parameter} \\
\mathbb{E}X = \int_\Omega X(\omega) dP(\omega) & \quad \text{expectation} \\
e(x) = e^x & \quad \text{exponential function} \\
\Re z & \quad \text{real part of } z \\
\end{align*}

Chapter I. Complex Normal Random Measure.

§ 1. Complex-normal Random Variables. Let $Z = Z(\omega) = X(\omega) + iY(\omega)$ be a complex random variable. If the joint distribution of $X$ and $Y$ is a two-dimensional isotropic normal distribution (including degenerate cases), $Z$ will be called a complex-normal variable. This condition is expressible by

\begin{equation}
\mathbb{E}e^{i(Xu + Yv)} = e\left(-\frac{a}{4}(u^2 + v^2)\right) (u, v \in \mathbb{R})
\end{equation}

or equivalently by

\begin{equation}
\mathbb{E}e^{i|w|Zw} = e\left(-\frac{a}{4}|w|^2\right) (w \in K)
\end{equation}

for some $a > 0$. By (1.1) we see that $X$ and $Y$ are independent real random variables and subject to the same distribution $N(0, a/2)$.

Theorem 1.1. Let $Z_1, Z_2, \ldots, Z_n$ be independent complex normal variables. Then $Z = \sum c_i Z_i$, $c_i$'s being complex numbers, is also complex-normal.

Proof. Assume that
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By the independence of \( \{Z_\lambda\} \) we have

\[
\varepsilon e^{iR(Z_\lambda, w)} = \Pi_{\lambda} \varepsilon e^{iR(Z_\lambda c_\lambda, w)} = \Pi_{\lambda} e^{-\frac{a_\lambda}{4} |c_\lambda| w^2},
\]

that is

\[
\varepsilon e^{iR(Z_\mu)} = e^{-\frac{1}{4} \sum_{\lambda} a_\lambda |c_\lambda|^2 |w|^2}.
\]

\textbf{Theorem 1.2.} Suppose that \( Z_n \rightarrow Z \) and that \( Z_n, n=1, 2, \ldots, \) are all complex-normal. Then \( Z \) is also complex-normal.

\textbf{Proof.} It follows immediately from

\[
\varepsilon e^{iR(Z_\mu w)} = \lim_{n \rightarrow \infty} \varepsilon e^{iR(Z_n w)}.
\]

\textbf{§ 2. Complex-normal System.} A system of complex random variables \( Z = \{Z_\lambda(\omega), \lambda \in \Lambda\} \) is called complex-normal, if \( \sum_{n=1}^N c_n Z_{\lambda(i)} \) is complex-normal for any \( n \), for any \( c_i \in K \) and for any \( \lambda(i) \in \Lambda \). By the definition each \( Z_\lambda \) is clearly complex-normal. By Theorem 1.1 we have

\textbf{Theorem 2.1.} If \( Z_\lambda, \lambda \in \Lambda, \) are independent, each being complex-normal, then the system \( Z = \{Z_\lambda, \lambda \in \Lambda\} \) is complex-normal.

By the definition and Theorem 1.2 we get

\textbf{Theorem 2.2.} Let \( Z = \{Z_\lambda, \lambda \in \Lambda\} \) be complex-normal and \( Z' = \{Z'_\mu, \mu \in \mathcal{M}\} \) be any system of complex random variables. If each \( Z'_\mu \) is of the form \( \sum_{i=1}^n c_i Z_{\lambda(i)} \) or more generally \( \lim_{n} \sum_{i=1}^N c_i Z_{\lambda(i)} \) (limit in probability), then \( Z' \) is also complex-normal.

Let \( Z = \{Z_\lambda, \lambda \in \Lambda\} \) be any system of complex random variables such that \( |Z_\lambda|^2 < \infty, \lambda \in \Lambda \). We put

\[
V_{\lambda\mu} = \varepsilon(Z_\lambda \overline{Z_\mu}) = (Z_\lambda, Z_\mu), \lambda, \mu \in \Lambda.
\]

Then \( V_{\lambda\mu} \) is positive-definite in \( \lambda, \mu \in \Lambda \) in the sense that we have

\[
\sum_{i,j=1}^n V_{\lambda(i)\mu(j)} \xi_i \overline{\xi_j} \geq 0
\]

for any \( n \), for any \( \lambda(i) \) and for any \( \xi_i \in K \). Conversely, if \( V_{\lambda\mu} \) is positive-definite in \( \lambda, \mu \), then there exists a system \( Z = \{Z_\lambda, \lambda \in \Lambda\} \) satisfying (2.1). We shall further establish

\textbf{Theorem 2.3 (Existence Theorem).} If \( V_{\lambda\mu} \) is positive-definite, then there exists a complex-normal system satisfying (2.1), and the probability distribution of this system is uniquely determined by \( V_{\lambda\mu} \).

\textbf{Proof of uniqueness.} We shall first show a preliminary lemma.

\textbf{Lemma 2.1.} If \( \{Z_1, Z_2, \ldots, Z_n\} \) is a complex-normal system, then

\[
\varepsilon e^{iR(\sum_{\nu=1}^{n} c_\nu Z_\nu)} = e^{-\frac{1}{4} \sum_{\nu} c_\nu \overline{c_\nu} (Z_\nu, Z_\nu)}.
\]

\textbf{Proof.} \( \{Z_\nu\} \) being complex-normal, \( \sum_{\nu=1}^{n} c_\nu Z_\nu \) is a complex-normal variable, and so we have

\[
\varepsilon e^{iR(\sum_{\nu} c_\nu Z_\nu w)} = e^{-\frac{a}{4} |w|^2}.
\]

where
Putting \( w=1 \) in (2.4), we shall get (2.3).

Now, let \( \{Z_\lambda=\chi_\lambda+i\eta_\lambda\} \) be any complex-normal system for which (2.1) holds. By the above lemma we have

\[
\varepsilon e(i\sum_{\nu} (u_\nu \chi_{\lambda(\nu)} + v_\nu \eta_{\lambda(\nu)})) = \varepsilon e(i\sum_{\nu=1}^{n} (u_{\nu} - iv_{\nu})Z_{\lambda(\nu)}) = e\left(-\frac{1}{4} \sum_{\nu=1}^{n} (u_{\nu} - iv_{\nu})V_{\lambda(\nu)}\right),
\]

which implies that \( \{V_{\lambda\mu}\} \) determines the joint characteristic function and so the joint distribution of \( \chi_{\lambda(\nu)}, \eta_{\lambda(\nu)}, \nu=1, 2, \ldots, n \). Therefore \( V_{\lambda\mu} \) determines the probability distribution of the system \( \{Z_\lambda\} \).

**Proof of the existence.** We shall begin with a lemma.

**Lemma 2.2.** If \( V_{\lambda\mu} \) is positive-definite in \( \lambda, \mu \in A \), then there exists a system of (possibly infinite-dimensional) complex vectors

\[
c_{\lambda} = (c_{\lambda\alpha}, \alpha \in A)
\]

such that

\[
(2.5) \quad \sum_{\alpha} |c_{\lambda\alpha}|^2 < \infty, \lambda \in A,
\]

\[
(2.6) \quad V_{\lambda\mu} = \sum_{\alpha} c_{\alpha}^* c_{\alpha}, \lambda, \mu \in A.
\]

**Proof.** Let \( K_{A0} \) denote the totality of the complex vector (of finite or infinite dimension) \( x=(x_\lambda, \lambda \in A) \in K_a \) such that \( x_\lambda=0 \) except for a finite number of exceptional values of \( \lambda \). \( K_{A0} \) is considered a linear space associated with the inner product:

\[
(x, y) = \sum_{\alpha} x_\alpha y_\alpha,
\]

which is bilinear, symmetric and non-negative:

\[
(\sum_{\alpha} a_\alpha x_\alpha, \sum_{\beta} b_\beta y_\beta) = \sum_{\alpha, \beta} a_\alpha b_\beta (x_\alpha, y_\beta),
\]

\[(x, y) = (y, x),
\]

\[(x, x) > 0.
\]

Let \( N_{A0} \) be the set of all elements \( x \in K_{A0} \) for which \( (x, x)=0 \). Then \( N \) proves to be a linear subspace of \( K_{A0} \). The factor space \( H=K_{A0}/N_{A0} \) may be considered a linear space with the inner product if we define the operations in the usual way. Then the inner product satisfies the positivity:

\[(x, x) > 0 \text{ for } x \neq 0
\]
in addition to the above properties.

Consider the completion of \( H: \bar{H}=\bar{H} \). Then \( H \) is imbedded in \( \bar{H} \). Let \( x^\lambda \) be the element in \( K_{A0} \) whose \( \lambda \)-component is equal to 1 and whose other components are all equal to 0. By the natural mapping from \( K_{A0} \) onto \( H \subseteq \bar{H} \), \( x^\lambda \) will be mapped to an element of \( \bar{H} \), say \( \bar{x}^\lambda \). We have clearly

\[
(\bar{x}^\lambda, \bar{x}^\mu) = V_{\lambda\mu}.
\]

We shall take a complete orthonormal system in \( \bar{H} : \phi_\alpha, \alpha \in A \). \( x^\lambda \) may be expressible as

\[
x^\lambda = \sum_{\alpha \in A} c_{\lambda\alpha} \phi_\alpha,
\]

where
Thus we obtain a system of vectors \( c^\alpha = (c^\alpha_a, \alpha \in A), \lambda \in A \), satisfying (2.5) and (2.6).

Now we shall proceed to the proof of existence. We shall define a normal distribution \( \mu \) on \( K \) by

\[
d\mu(z) = \frac{1}{\sqrt{\pi}} e^{-x^2 - y^2} dx dy (z = x + iy)
\]

and consider the direct product measure \( \mu^A \), \( A \) being the parameter set mentioned in the above lemma. The existence of \( \mu^A \) is assured by the extension theorem of A. Kolmogorov [1] or more directly by a theorem of S. Kakutani [2]. Now we take \( K^0(\mu^A) \) as the underlying probability space \( \Omega(P) \) and define \( W_\alpha(\omega) = \) the \( \alpha \)-component of \( \omega, \omega \in \Omega = K \).

Then \( W_\alpha, \alpha \in A \), are independent complex random variables, each being complex-normal. Therefore they constitute a complex-normal system as well as an orthogonal system in \( L^2(\Omega, P) \).

Making use of the \( c^\alpha \)'s in the above lemma, we shall define \( Z_\lambda \) in \( L^2(\Omega, P) \) by

\[
Z_\lambda = \sum a c^\lambda_a W_a
\]

for each \( \lambda \), where the infinite sum is understood in the sense of convergence in \( L^2(\Omega, P) \) and so may be considered in the sense of convergence in probability. Therefore we see, by Th. 2.2, that \( Z = \{Z_\lambda, \lambda \in A\} \) is a complete-normal system. \( \{W_\alpha\} \) being orthonormal, we get

\[
(Z_\lambda, Z_\mu) = \sum a c^\lambda_a c^\mu_a = V_{\lambda\mu}.
\]

**Theorem 2.4.** Given a complex-normal system \( \{Z_\lambda, \lambda \in A\} \). It is necessary and sufficient for \( \{Z_\lambda\} \) to be independent that \( Z_\lambda, \lambda \in A \), are mutually orthogonal in the Hilbert space \( L^2(\Omega, P) \).

**Proof.** The necessity is clear since the independence of \( \{Z_\lambda\} \) implies

\[
(Z_\lambda, Z_\mu) = \varepsilon Z_\lambda \bar{Z}_\mu = \varepsilon Z_\lambda Z_\mu = 0 \quad (\lambda \neq \mu).
\]

Conversely assume that \( \{Z_\lambda, \lambda \in A\} \) be mutually orthogonal. Let \( \lambda(\nu), 1 \leq \nu \leq n, \) be any finite subsystem of \( A \). Then we have

\[
\varepsilon E[\bar{\varepsilon} (\sum \nu Z_{\lambda(\nu)} w_\nu)] = \varepsilon \left[ -\frac{1}{4} \sum \nu |Z_{\lambda(\nu)} w_\nu|^2 \right] = \varepsilon \left[ -\frac{1}{4} \sum \nu |Z_{\lambda(\nu)}|^2 |w_\nu|^2 \right]
\]

by Lemma 2.1. Put \( Z_{\lambda(\nu)} = X_{\lambda(\nu)} + i Y_{\lambda(\nu)} \) and \( w_\nu = u_\nu - i v_\nu \). Then we have

\[
\varepsilon E[i \sum \nu (X_{\lambda(\nu)} u_\nu + Y_{\lambda(\nu)} v_\nu)] = \varepsilon \left[ -\frac{1}{4} \sum \nu |Z_{\lambda(\nu)}|^2 (|u_\nu|^2 + |v_\nu|^2) \right]
\]

\[
= \Pi_v \varepsilon \left( -\frac{1}{4} ||Z_{\lambda(\nu)}||^2 (u_\nu^2 + v_\nu^2) \right) \Pi_u \varepsilon \left( -\frac{1}{4} ||Z_{\lambda(\nu)}||^2 (u_\nu^2 + v_\nu^2) \right)
\]

\[
= \Pi_v \varepsilon \left( -\frac{1}{2} ||X_{\lambda(\nu)}||^2 (u_\nu^2 + v_\nu^2) \right) \Pi_u \varepsilon \left( -\frac{1}{2} ||X_{\lambda(\nu)}||^2 (u_\nu^2 + v_\nu^2) \right)
\]

which implies, by virtue of a theorem of Kac-Steinhaus, the independence of
§ 3. Complex Normal Random Measure. Let $T(B_{\tau}, m)$ be any abstract space, and $B^*_{\tau}$ be the totality of the elements in $B_{\tau}$ with finite $m$-measure. $m(E \cap F)$ is positive-definite in $E, F \in B^*_{\tau}$ since we have
\[
\sum_{i,j=1}^{n} \xi_i \xi_j m(E_i \cap E_j) = \int_{\tau} |\sum_{i} \xi_i \chi(t; E_i)|^2 dm(t) \geq 0
\]
for any $E_i \in B^*_{\tau}$, $\chi(t; E)$ denoting the characteristic function of the set $E_i$. Thus we get, by the existence theorem 2.3.

**Theorem 3.1.** There exists a complex normal system $M = \{M(E), E \in B^*_{\tau}\}$ such that
\[
(M(E), M(F)) = m(E \cap F).
\]
The probability distribution of $M$ is uniquely determined by $m$.

The above $M$ is defined as a complex normal random measure on $T(B_{\tau}, m)$; the nomenclature “random measure” will be justified by the following theorem.

**Theorem 3.2.** Let $M = \{M(E), E \in B^*_{\tau}\}$ be a complex normal random measure. If $E_1, E_2, \ldots$ are disjoint, then $M(E_i), i=1,2,\ldots$, are independent, and further if $E = \sum_{i=1}^{n} E_i \in B^*_{\tau}$, then
\[
(M(E)) = \sum_{i=1}^{n} M(E_i) \quad (\text{Convergence in } L^2(\Omega, P))
\]

**Proof.** $\{M(E_1), \ldots, M(E_n)\}$ is complex-normal for any $n$. Since we have
\[
(M(E_i), M(E_j)) = m(E_i \cap E_j) = 0, \quad i \neq j,
\]
$M(E_1), \ldots, M(E_n)$ are independent by Theorem 2.4. $n$ being arbitrary, $M(E_i), i=1,2,\ldots$ are independent.

\[
||M(E) - \sum_{i=1}^{n} M(E_i)||^2 = ||M(E)||^2 + \sum_{i=1}^{n} ||M(E_i)||^2 - 2\sum_{i=1}^{n} (M(E), M(E_i))
\]
\[
+ 2\sum_{i<j} (M(E_i), M(E_j)) = m(E) + \sum_{i=1}^{n} m(E_i) - 2\sum_{i=1}^{n} m(E_i) = m(E) - \sum_{i=1}^{n} m(E_i) \to 0 \quad (n \to \infty),
\]
which proves (3.1).

**Remark.** By a theorem of P. Lévy [4] the above convergence in $L^2(\Omega, P)$ (=mean convergence) implies the almost certain convergence on account of the independence of $\{M(E_i)\}$.

The following facts will be proved by simple computations and will become useful in the next §:

\[
(3.2) \quad \varepsilon M(E)^{\frac{3}{2}} = 0, \quad \varepsilon |M(E)|^2 = m(E), \quad \varepsilon |M(E)|^4 = 2m(E).
\]

§ 4. Continuity. Let $T(B_{\tau}, m)$ be any measure space. A decomposition of a set $E \in B^*_{\tau}$ into disjoint parts:
\[
E = \sum_{i=1}^{n} E_i
\]
is called an $\varepsilon$-decomposition, if $m(E_i) < \varepsilon, i=1,2,\ldots, n$. If, for any $\varepsilon > 0$ and $E \in B^*_{\tau}$, there exists an $\varepsilon$-decomposition of $E$, the measure $m$ is called to be continuous.

Let $M = \{M(E), E \in B^*_{\tau}\}$ be a complex normal random measure on $T(B_{\tau}, m)$. $M$ is said to be continuous if the corresponding measure $m$ is continuous.
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in the above sense.

Hereafter we shall often assume the continuity to make use of the following theorem.

**Theorem 4.** Let $M$ be any continuous complex normal random measure on $T(B_t, m)$ and

$$
\Delta : E = \sum_{i=1}^{n} E_i
$$

be any $\varepsilon$-decomposition of a set $E \in B^*_r$. For this decomposition we define

$$
S_{\Delta} = \sum_{i=1}^{n} |M(E_i)|^2, \quad S'_{\Delta} = \sum_{i=1}^{n} M(E_i)^2.
$$

Then we have, as $\varepsilon \to 0$,

$$
||S_{\Delta} - m(E)|| \to 0, \quad ||S'_{\Delta}|| \to 0
$$

**Proof.** By virtue of the independence of $\{M(E_i)\}$ and the identities (3.2) we get

$$
||S_{\Delta} - m(E)||^2 = \sum_{i} |M(E_i)|^2 - m(E_i)|^2
$$

$$
= \sum_{i} (|M(E_i)|^2 - m(E_i))^2
$$

$$
= \sum_{i} m(E_i)^2 < \varepsilon \sum_{i} m(E_i) = \varepsilon m(E),
$$

$$
||S'_{\Delta}||^2 = \sum_{i} |M(E_i)|^2 = \sum_{i} 2m(E_i)^2 < \varepsilon 2m(E),
$$

which prove our theorem.

**Chapter II. Definition of Complex Multiple Wiener Integral.**

§ 5. $L^2_{pq}, S_{pq}$. Let $T(B_t, m)$ be any measure space. Then the product measure spaces $T^p$ and $T^* \times T^q$ may be usually defined and $T^* \times T^q$ is clearly isomorphic to $T^{p+q}$. Let $L^2_{pq}$ be the $L^2$-space over $T^* \times T^q$ which is clearly isomorphic to $L^2(T^{p+q})$. For $f \in L^2_{pq}$ we define $\tilde{f} \in L^2_{pq}$ by the identity:

$$
\tilde{f}(t_1, \ldots, t_p; s_1, \ldots, s_q) = \frac{1}{|p||q|} \sum_{\pi} f(t_{i_1}, \ldots, t_{i_p}; s_{j_1}, \ldots, s_{j_q}),
$$

where $(i) = (i_1, \ldots, i_p)$ and $(j) = (j_1, \ldots, j_q)$ run over all permutations of $(1, 2, \ldots, p)$ and $(1, 2, \ldots, q)$ respectively. We have clearly

$$
||\tilde{f}|| \leq ||f||.
$$

Next we shall a subsystem $S_{pq}$ of $L^2_{pq}$ which consists of all functions of the following type. Let $E_1, \ldots, E_n$ be any system of disjoint sets in $B^*_r$ and $a_{i_1, \ldots, i_p, j_1, \ldots, j_q}$ be a complex-valued function defined for $i_1, \ldots, i_p, j_1, \ldots, j_q = 1, 2, \ldots, n$ such that $a_{i_1, \ldots, i_p, j_1, \ldots, j_q} = 0$ unless $i_1, \ldots, i_p, j_1, \ldots, j_q$ are all different. $f$ is defined by

$$
f(t_1, \ldots, t_p; s_1, \ldots, s_q) = a_{i_1, \ldots, i_p, j_1, \ldots, j_q}
$$

for

$$
(t_1, \ldots, t_p; s_1, \ldots, s_q) \in E_{i_1} \times \cdots \times E_{i_p} \times E_{j_1} \times \cdots \times E_{j_q},
$$

while

$$
f(t_1, \ldots, t_p; s_1, \ldots, s_q) = 0
$$

if some of $t_1, \ldots, t_p, s_1, \ldots, s_q$ lies outside of $E = \sum_{\gamma=1}^{n} E_{\gamma}$. We shall call the above system $\{E_{\gamma}\}$ the base of expression of $f \in S_{pq}$.

From Theorem 2.1 in M.W.I. follows at once
THEOREM 5. If $m$ is continuous, then $S_{pq}$ is a linear manifold dense in $L^2_{pq}$.

§ 6. Definition of $I_{pq}(f)$ for $f \in S_{pq}$. Let $f$ be any element of $S_{pq}$ and $f$ be expressed by (5.4) and (5.5). We shall define $I_{pq}(f)$ by

$$I_{pq}(f) = \sum_{i,j} a_{i_1, \ldots, i_p, j_1, \ldots, j_q} M(E_{i_1}) \cdots M(E_{i_p}) \overline{M(E_{j_1})} \cdots \overline{M(E_{j_q})}.$$ 

$I_{pq}(f)$ is clearly independent of the expression of $f$ and is well determined by $f$ itself.

We shall here establish the following properties of $I_{pq}$ :

(I*1) $I_{pq}(f) = I_{pq}(\tilde{f})$,

(I*2) $I_{pq}(\alpha f + \beta g) = \alpha I_{pq}(f) + \beta I_{pq}(g)$,

(I*3) $E I_{pq}(f) = (I_{pq}(f), 1) = 0$,

(I*4) $(I_{pq}(f), I_{pq}(g)) = |p| q(f, g)$,

(I*5) $I_{pq}(f) \leq |p| q \leq \sum_{i_1 < \cdots < i_p, j_1 < \cdots < j_q} a_{i_1, \ldots, i_p, j_1, \ldots, j_q}$

where $a_{i_1, \ldots, i_p, j_1, \ldots, j_q}$ is the value of $f$ and $\Sigma'$ is the summation over the suffices $(i), (j)$ such that $i_1, \ldots, i_p, j_1, \ldots, j_q$ are all different and $i_1 \ldots < i_p, j_1 < \cdots < j_q$. It is clear that the terms in $\Sigma'$ are mutually orthogonal. Expressing $f$ and $g$ on the same base $\{E_{\nu}\}$ we shall obtain (I*4) and (I*6) at once if we remark that

$$|I_{pq}(\tilde{f})|^2 = \sum_{p, q} |p| q |\tilde{f}|^2 \leq |p| q |f|^2$$

Since $M(E_{i_1}) \cdots M(E_{i_p}) M(E_{j_1}) \cdots M(E_{j_q})$ is symmetric in $(i_1, \ldots, i_p)$ as well as in $(j_1, \ldots, j_q)$, (I*1) is clear. (I*2) will be easily shown if we express $f$ and $g$ on the same base of expression. Since $M(E_{i_1}), \ldots, M(E_{i_p}), M(E_{j_1}), \ldots, M(E_{j_q})$ are independent when $i_1, \ldots, i_p, j_1, \ldots, j_q$ are all different, (I*3) is evident. By the definition $I_{pq}(f)$ is expressible as

$$I_{pq}(f) = \sum' |p| q \tilde{a}_{i_1, \ldots, i_p, j_1, \ldots, j_q} M(E_{i_1}) \cdots M(E_{i_p}) \overline{M(E_{j_1})} \cdots \overline{M(E_{j_q})},$$

where $\tilde{a}_{i_1, \ldots, i_p, j_1, \ldots, j_q}$ is the value of $f$ and $\Sigma'$ is the summation over the suffices $(i), (j)$ such that $i_1, \ldots, i_p, j_1, \ldots, j_q$ are all different and $i_1 < \cdots < i_p, j_1 < \cdots < j_q$. It is clear that the terms in $\Sigma'$ are mutually orthogonal. Expressing $f$ and $g$ on the same base $\{E_{\nu}\}$ we shall obtain (I*4) and (I*6) at once if we remark that

$$||M(E_{i_1}) \cdots M(E_{i_p}) M(E_{j_1}) \cdots M(E_{j_q})||^2 = m(E_{i_1}) \cdots m(E_{i_p}) m(E_{j_1}) \cdots m(E_{j_q})$$

and

$$0 = (M(E_{i_1}) \cdots M(E_{i_p}) \overline{M(E_{j_1})} \cdots \overline{M(E_{j_q})}, M(E_{k_1}) \cdots M(E_{k_q}) \overline{M(E_{l_1})} \cdots \overline{M(E_{l_q})})$$

except for the special case that $p=r$, $q=s$ and $i_\sigma = k_\sigma$, $j_\sigma = l_\sigma$ for all $\pi, \sigma$. Putting $f=g$ in (I*4) we shall get (I*5).

For the unity of arguments we shall consider also the trivial case $p=q=0$ which was excluded in the above discussions. We put $L^2_{pq} = K(=\text{the complex number space})$, $I_{pq}(c) = c$. Then the above properties (I*1)～(I*6) except (I*3) will hold also in case $p$ or $q$ is equal to 0; (I*3) will be included in (I*6) as the special case of $r=s=0$.

§ 7 Definition of $I_{pq}(f)$ for $f \in L^2_{pq}$. In the preceding § we have defined $I_{pq}(f)$ for $f \in S_{pq}$. We shall extend this definition by taking the limit. By virtue of Theorem 5 we may define, for $f \in L^2_{pq}$, a sequence $(f_n)$ in $S_{pq}$ such that

$$||f_n - f|| \to 0.$$
Then we have by (I*2) and (I*5), as \( m, n \to \infty \),
\[
||I_{pq}(f_m) - I_{pq}(f_n)|| = ||I_{pq}(f_m - f_n)|| \leq ||f_m - f_n||
\]
so that we may define
\[
I_{pq}(f) = \lim_{n \to \infty} I_{pq}(f_n);
\]
this definition is independent of the choice of the sequence \( \{f_n\} \).

Now we shall prove

**Theorem 7.**

(I.1) \( I_{pq}(f) = I_{pq}(\tilde{f}) \)

(I.2) \( I_{pq}(\alpha f + \beta g) = \alpha I_{pq}(f) + \beta I_{pq}(g) \)

(I.3) \( \epsilon(I_{pq}(f)) = (I_{pq}(f), 1) = 0 \), \( p+q = 0 \)

(I.4) \( (I_{pq}(f), I_{pq}(g)) = \frac{p}{q} \left\langle \tilde{f}, \tilde{g} \right\rangle \)

(I.5) \( ||I_{pq}(f)||^2 = \frac{p}{q} \left\langle f, f \right\rangle \)

(I.6) \( (I_{pq}(f), I_{rs}(g)) = 0 \) if \( (p, q) \neq (r, s) \)

**Proof.** These properties were verified in \( S_{pq} \) in the preceding §. To prove them in \( L^2_{\alpha \beta} \) we need only remark the continuity of the inner product and the following inequalities:

\[
||\tilde{f} - \tilde{g}|| \leq ||f - g||.
\]

We shall call \( I_{pq}(f) \) the complex multiple Wiener integral of \( f \) and denote it often by the following integral form:

\[
\int_{\cdots} f(t_1, \ldots, t_p; s_1, \ldots, s_q) dM(t_1)dM(t_2)dM(s_1)dM(s_2)\cdots dM(s_q).
\]

**§ 8. Transformation of Measure.** Let \( M = (M(E), E \in B^*_\tau) \) be any complex normal random measure on \( T(B\tau, m) \) and \( \lambda(t) \) be any complex-valued measurable function defined on \( T(B\tau, m) \) satisfying

\[
|\lambda(t)| \equiv 1.
\]

If we define \( M_\lambda(E) \) by

\[
M_\lambda(E) \equiv \int E(t) \lambda(t) dM(t),
\]

\( \chi_{E}(t) \) being the characteristic function of the set \( E \in B^*_\tau \), then \( M_\lambda = (M_\lambda(E), E \in B^*_\tau) \) will be also a complex normal random measure on \( T(B\tau, m) \). Then we obtain immediately

**Theorem 8.**

\[
\int_{\cdots} f(t_1, \ldots, t_p; s_1, \ldots, s_q) dM_\lambda(t_1) \cdots dM_\lambda(t_p) dM_\lambda(s_1) \cdots dM_\lambda(s_q)
\]

\[
= \int_{\cdots} f(t_1, \ldots, t_p; s_1, \ldots, s_q) \lambda(t_1) \cdots \lambda(t_p) \lambda(s_1) \cdots \lambda(s_q)
\]

\[
\times dM(t_1) \cdots dM(t_p) dM(s_1) \cdots dM(s_q).
\]
§9. Recurrence Formulae. For \( f \in L^2_{p,q} \) and \( g \in L^2(T) \) we shall define the following four functions:

\[
\begin{align*}
(f \cdot g)(t_1, \ldots, t_{p+1}; s_1, \ldots, s_q) &= f(t_1, \ldots, t_p; s_1, \ldots, s_q)g(t_{p+1}), \\
(f \circ g)(t_1, \ldots, t_p; s_1, \ldots, s_{q+1}) &= f(t_1, \ldots, t_p; s_1, \ldots, s_q)g(s_{q+1}), \\
(f_\circ g)(t_1, \ldots, t_{p-1}; s_1, \ldots, s_q) &= \sum_{k=1}^q f(t_1, \ldots, t_{k-1}, t, t_k, \ldots, t_{p-1}; s_1, \ldots, s_q)g(t)dm(t), \\
(f_\circ g)(t_1, \ldots, t_p; s_1, \ldots, s_{q-1}) &= \sum_{k=1}^q f(t_1, \ldots, t_p; s_1, \ldots, s_{k-1}, s, s_k, \ldots, s_q)g(s)dm(s).
\end{align*}
\]

We shall easily show

\[
\begin{align*}
(f \cdot g) &\in L^2_{p+1,q}, \quad \|f \cdot g\| = \|f\| \cdot \|g\|, \\
(f \circ g) &\in L^2_{p,q+1}, \quad \|f \circ g\| = \|f\| \cdot \|g\|, \\
(f_\circ g) &\in L^2_{p-1,q}, \quad \|f_\circ g\| \leq \|f\| \cdot \|g\|, \\
(f_\circ g) &\in L^2_{p,q-1}, \quad \|f_\circ g\| \leq \|f\| \cdot \|g\|.
\end{align*}
\]

Any element of \( L^2(T) \) may be considered to belong to \( L^2_{10} \) as well as to \( L^2_{01} \).

Now we shall establish recurrence formulae concerning complex multiple Wiener integral.

**Theorem 9.** For any continuous complex normal random measure we have

\[
\begin{align*}
(R.1) \quad I_{p,q}(f)I_{0,1}(g) &= I_{p+1,q}(f \cdot g) + I_{p,q-1}(f_\circ g), \quad p > 0, \quad q > 1, \\
(R.1') \quad I_{p,q}(f)I_{0,1}(g) &= I_{p+1,q}(f \cdot g), \\
(R.2) \quad I_{p,q}(f)I_{1,0}(g) &= I_{p,q+1}(f \cdot g) + I_{p-1,q}(f_\circ g), \quad p > 1, \quad q > 0, \\
(R.2') \quad I_{p,q}(f)I_{1,0}(g) &= I_{p,q+1}(f \cdot g).
\end{align*}
\]

**Proof.** We shall show only the first formula (R.1) since the others may be proved in the same way. We shall first discuss the following special cases.

**Case 1.** \( f(t_1, \ldots, t_p; s_1, \ldots, s_q) \) and \( g(t) \) are the characteristic functions of \( E_1 \times \cdots \times E_p \times F_1 \times \cdots \times F_q \) and \( G \) respectively, where \( G, E_1, \ldots, E_p, F_1, \ldots, F_q \) are mutually disjoint sets \( \in \mathcal{B}_T^* \).

In this case both sides of (R.1) are equal to zero.

**Case 2.** \( f \) and \( g \) are the same as in Case 1 but \( E_1, \ldots, E_p, F_1, \ldots, F_q \) are mutually disjoint sets \( \in \mathcal{B}_T^* \) and \( G \) is coincident with some of \( E_i \), for example \( E_p \).

In this case we get

\[
(f_\circ g) = 0 \quad \text{and so} \quad I_{p,q-1}(f_\circ g) = 0.
\]

On the other hand we have

\[
I_{p,q}(f)I_{1,0}(g) = M(E_1) \cdots M(E_{p-1})M(G)M(F_1) \cdots M(F_q).
\]

By the assumption of continuity \( G \) may be decomposed as follows:

\[
G = G_{n_1} + G_{n_2} + \ldots + G_{n_\nu(n)}, \quad m(G_{n_i}) < 1/n, \quad i = 1, 2, \ldots, \nu(n), \\
(n = 1, 2, \ldots).
\]

Therefore we have

\[
I_{p,q}(f)I_{1,0}(g) = \sum_{i \neq j} M(E_1) \cdots M(E_{p-1})M(G_{n_i})M(G_{n_j})M(F_1) \cdots M(F_q).
\]
Complex Multiple Wiener Integral.

\[+M(E_1) \cdots M(E_{p-1})M(F_1) \cdots M(F_q) \sum_i M(G_{ni})^2\]

\[= A_n + B_n.\]

Let \( h_n(t_1, \ldots, t_{p+1}; s_1, \ldots, s_q) \) be the characteristic function of the set:

\[\sum_{i<j} E_1 \cdots E_{p-1} \times G_{ni} \times G_{nj} \times F_1 \cdots \times F_q.\]

Then we have

\[\|A_n - I_{p+1,q}(f,g)\|^2 = \|h_n - f \cdot g\|^2 = \sum_{i=1}^{m(E_1)} \cdots m(E_{p-1})m(G_{ni})^2m(F_1) \cdots m(F_q)\]

and so

\[A_n \to I_{p+1,q}(f \cdot g),\]

while \( B_n \to 0 \) by Theorem 4. Thus we get

\[(9.5) I_{p,q}(f)I_{1,0}(g) = I_{p+1,q}(f \cdot g),\]

which, combined with (9.3), proves (R.1) in Case 2,

Case 3. \( f \) and \( g \) are the same as in Case 1, but \( E_1, \ldots, E_p, F_1, \ldots, F_q \) are mutually disjoint sets \( \in B \) and \( G \) is coincident with some of \( F_j \), for example \( F_q \).

By making use of the decompositions in Case 2 we obtain

\[I_{p,q}(f)I_{0,0}(g) = \sum_{i=1}^{m(E_1)} \cdots m(E_{p-1})M(F_1) \cdots M(F_{n-1})M(G_{ni})M(G_{nj})\]

\[+ M(E_1) \cdots M(E_p)M(F_1) \cdots M(F_{n-1}) \sum_i M(G_{ni})^2,\]

which, as \( n \to \infty \), tends to.

\[I_{p+1,q}(f \cdot g) + M(E_1) \cdots M(E_p)M(F_1) \cdots M(F_{n-1})m(G)\]

\[= I_{p+1,q}(f \cdot g) + I_{p,q}(f \cdot g)\]

by virtue of Theorem 4.

In considering that both sides of (R.1) are bilinear in \( f, g \) we shall easily deduce the case \( f \in S_{p,q}, g \in S_{0,1} \) from the above three cases and further show the most general case \( f \in L^2_{p,q}, g \in L^2_{0,1} \) by taking the limit.

\( \S 10. \) The Completeness of Complete Multiple Wiener Integral. Let \( Z = (Z_\lambda(\omega), \lambda \in \Lambda) \) be any system of complex random variables. A complex-valued Baire function of \( Z \) is defined in different ways which are equivalent to each other. We shall here adopt the following definition. A complex-random variable \( Z_\omega \) is defined to be a Baire function of \( Z \), if it belongs to the least class \( B \) that satisfies the following two conditions:

(\( B.1 \)) When \( f \) is a complex-valued Baire function of \( n \) complex variables in the usual sense, \( f(Z_{\lambda(1)}(\omega), \ldots, Z_{\lambda(n)}(\omega)) \) belongs to \( B \).

(\( B.2 \)) If \( (Z^{(n)}(\omega)) \) is a sequence in \( B \) which is convergent for every \( \omega \), then the limit belongs to \( B \).

We shall denote with \( L^\omega(Z) \) the totality of the Baire functions of \( Z \) belonging to \( L^\omega(\Omega, P) \). We shall easily obtain

\( \text{Lemma 10.1. For any } Z_\omega(\omega) \in L^\omega(Z) \text{ and any } \epsilon > 0 \text{ there exist } \lambda(1), \ldots, \lambda(n) \in \Lambda \text{ and a Baire function of } n \text{ variables } f \text{ such that }\)

\[(10.1) \|Z - f(Z_{\lambda(1)}, \ldots, Z_{\lambda(n)})\| < \epsilon.\]
PROOF. Let \( \varphi_\theta(x)(0 \leq x < \infty) \) be defined as
\[
\varphi_\theta(x) = 1(0 \leq x < G), = 0(x > G).
\]
we shall denote with \( \mathcal{B} \) the totality of the complex random variables \( Z \) for which \( Z_\theta = \varphi_\theta(|Z|) \cdot Z \) may have the property stated in this lemma for every \( G > 0 \). Then \( \mathcal{B} \) will satisfy (3.1) and (3.2) and so we see that \( \mathcal{B} \supseteq L^2(Z) \), namely that \( Z_\theta \in \mathcal{B} \) for \( Z \in L^2(Z) \). But
\[
\|Z - Z_\theta\| \to 0 \quad \text{as} \quad G \to \infty
\]
and so we obtain \( Z \in \mathcal{B} \).

Now let \( M = (M(E), E \in B^s_T) \) be any continuous normal random measure on \( T(B_T, m) \). Then we obtain the following lemma from the above lemma:

**LEMMA 10.2.** For any \( Z \in L^2(M) \) and any \( \varepsilon > 0 \) there exists a disjoint system \( E_1, \ldots, E_n \in B^s_T \) and a complex-valued Baire function of \( n \) complex variables \( f \) such that
\[
\|Z - f(M(E_1), \ldots, M(E_n))\| < \varepsilon.
\]
In making use of the completeness of Hermite polynomials we obtain

**LEMMA 10.3.** If \( f(x_1, \ldots, x_m) \) be a complex-valued Baire function of \( m \) real variables for which we have
\[
\sum \left| f(x_1, \ldots, x_m) \right|^2 e^{-\sum x_i^2} dx_1 \cdots dx_m < \infty,
\]
then there exists a polynomial of \( m \) variables \( P(x_1, \ldots, x_m) \) for any \( \varepsilon > 0 \), such that
\[
\sum \left| f(x_1, \ldots, x_m) - P(x_1, \ldots, x_m) \right|^2 e^{-\sum x_i^2} dx_1 \cdots dx_m < \varepsilon.
\]
From the above two lemmas follows

**LEMMA 10.4.** For any \( Z \in L^2(M) \) and any \( \varepsilon > 0 \) there exists a disjoint system \( E_1, \ldots, E_n \in B^s_T \) and a polynomial of \( 2n \) variables \( Q \) such that
\[
\|Z - Q(M(E_1), \ldots, M(E_n), \bar{M}(E_1), \ldots, \bar{M}(E_n))\| < \varepsilon.
\]

**Proof.** By Lemma 10.2 we need only to discuss the case in which \( Z \) is of the form
\[
Z = f(M(E_1), \ldots, M(E_n)),
\]
where \( f \) is a complex-valued Baire function of \( n \) complex variables and \( E_1, \ldots, E_n \) are disjoint sets \( \in B^s_T \). By denoting the real and imaginary parts of \( M(E_i)/m(E_i) \) with \( X_i \) and \( Y_i \) respectively, we may put
\[
Z = g(X_1, Y_1, \ldots, X_n, Y_n),
\]
g being a complex-valued Baire function of \( 2n \) real variables.

Since \( \{M(E)\} \) is a normal random measure, \( X_1, Y_1, \ldots, X_n, Y_n \) are independent, each having the probability law \( 1/\sqrt{2\pi} e^{-x^2} dx \). By Lemma 10.3 we can find a polynomial \( P(x, y, \ldots, x, y) \) such that
\[
\sum \left| g(x_1, y_1, \ldots, x_n, y_n) - P(x_1, y_1, \ldots, x_n, y_n) \right|^2 \times e^{-\sum x_i^2 - \sum y_i^2} dx_1 dy_1 \cdots dx_n dy_n < \varepsilon^2,
\]
namely that
\[
\|Z - P(X_1, Y_1, \ldots, X_n, Y_n)\|^2 < \varepsilon^2.
\]
If we define $Q$ by

$$Q(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n) = P((z_1 + \bar{z}_1)/2, (z_1 - \bar{z}_1)/2i, \ldots, (z_n + \bar{z}_n)/2, (z_n - \bar{z}_n)/2i),$$

then it follows from the above arguments that $Q$ satisfies (10.6).

**Theorem 10.** (The Completeness of Multiple Wiener Integral) The system:

$$\mathcal{G} = \{I_{pq}(f_{pq}); f_{pq} \in L_{pq}, p, q = 0, 1, \ldots\}$$

is complete in $L^2(M)$.

**Proof.** By the above Lemma 10.4 it is sufficient to show that

$$Z = M(E_1)^{p_1} \cdots M(E_n)^{p_n} M(E_1)^{q_1} \cdots M(E_n)^{q_n}$$

is expressible of the form:

$$Z = \sum_{p+q} \sum_{p_{1} \leq 1} \cdots \sum_{p_n \leq 1} I_{pq}(f_{pq}), p = \sum p_{r}, q = \sum q_{r}.$$

We shall make use of mathematical induction with regard to $p+q$. In case $p+q=0$ our assertion is trivially true. Suppose that it is true for $p+q=r$. If $p+q=r+1$, some of $\{p_{1}\}$ or $\{q_{1}\}$ is positive.

**Case 1.** Some of $\{p_{1}\} > 0$. To fix the idea we shall assume that $p_1 > 0$. By the assumption of induction we have

$$\int Z = \int M(E_1)^{p_1-1} \cdots M(E_n)^{p_n} M(E_1)^{q_1} \cdots M(E_n)^{q_n} = \sum_{p_1 \geq 0} \sum_{q_1 \geq 0} I_{pq}(f_{pq}),$$

and so, denoting with $\chi(t)$ the characteristic function of $E$, we get, by Theorem 9. (R.1),

$$\int Z = \int M(E_1)^{p_{1}} \cdots M(E_n)^{p_n} \int M(E_1)^{q_1} \cdots M(E_n)^{q_n} = \sum_{p_1 \geq 0} \sum_{q_1 \geq 0} I_{pq}(f_{pq}),$$

which proves that $Z$ is also of the above form (10.8).

**Case 2.** Some of $\{q_{1}\} > 0$. We can perform the proof in the same way as above by making use of the recurrence formula (R.2) instead of (R.1).

§ 11. Expansion Theorem by Means of Complex Multiple Wiener Integral.

For any system:

$$\mathcal{F} = \{f_{pq} \in L_{pq}; p, q = 0, 1, 2, \ldots\}$$

we shall consider a series

$$\sum_{pq} I_{pq}(f_{pq}).$$

Since the terms in the summation are mutually orthogonal by Theorem 7 (I.6) it is necessary and sufficient for $\sum_{pq} I_{pq}(f_{pq})$ to be convergent in the norm that

$$\sum_{pq} \|I_{pq}(f_{pq})\|^2 < \infty,$$

which is equivalent to

$$\sum_{pq} |p| \|q\| \|f_{pq}\|^2 < \infty$$

by Theorem 7 (I.5).

If (11.4) is satisfied, $\sum_{pq} I_{pq}(f_{pq})$ will clearly belong to $L^2(M)$ since each term of $\sum_{pq} I_{pq}(f_{pq})$ does so. We shall now establish the converse of this fact.

**Theorem 11.** (Expansion Theorem) Let $M$ be a continuous normal random measure. Every element of $L^2(M)$ is expressible of the form (11.2). Further we may require that each $f_{pq}$ appearing here is symmetric in the sense that

$$f_{pq} = f_{qp}. $$
Under this restriction the expression (11.2) i.e. the system \( \mathcal{F} \) is uniquely determined.

Proof. Let \( \tilde{L}^2_{pq} \) be the totality of the symmetric functions in \( L^2_{pq} \). Then \( \tilde{L}^2_{pq} \) is clearly a closed linear manifold \( L^2_{pq} \) and so a Hilbert space. Let \( L^2_{pq}(M) \) be the totality of the form \( I_{pq}(f_{pq}) \), \( f_{pq} \) running over \( L^2_{pq} \) or (by virtue of Theorem 7 (I.1)) equivalently over \( \tilde{L}^2_{pq} \). \( L^2_{pq}(M) \) is also a closed linear manifold which is unitary equivalent to \( \tilde{L}^2_{pq} \) by the correspondence:

\[
(11.6) \quad \tilde{L}^2_{pq} \ni \sqrt{p} q f_{pq} \mapsto I_{pq}(f_{pq}) \in L^2_{pq}(M).
\]

Let \( Z \) be any element of \( L^2(M) \) and the projection of \( Z \) onto \( L^2_{pq}(M) \) be \( Z_{pq} = I_{pq}(f_{pq}) \). Then \( \{Z_{pq}\} \) are mutually orthogonal and so we have

\[
\sum_{pq} \|Z_{pq}\|^2 \leq \|Z\|^2 \quad \text{i.e.} \quad \sum_{pq} \|p q f_{pq}\|^2 \leq \|Z\|^2 < \infty
\]

by Bessel's inequality. Therefore we see that

\[
Z = \sum_{pq} Z_{pq}, \quad Z' = Z - Z
\]

is well defined and that \( Z' \) is orthogonal to the system:

\[
\mathcal{Z} = \{I_{pq}(f_{pq}); f_{pq} \in L^2_{pq}, \ p, q = 0, 1, 2, \ldots\}.
\]

From the completeness of \( \mathcal{Z} \) (Theorem 10) follows \( Z' = 0 \) and so

\[
Z = \sum_{pq} Z_{pq} = \sum_{pq} I_{pq}(f_{pq}) = \sum_{pq} I_{pq}(\tilde{f}_{pq}).
\]

Thus the possibility of the expression was proved.

\[
\sum_{pq} I_{pq}(f_{pq}) = \sum_{pq} I_{pq}(g_{pq}), \quad f_{pq}, g_{pq} \in \tilde{L}^2_{pq}.
\]

Since \( L^2_{pq}(M), \ p, q = 0, 1, 2, \ldots \) are mutually orthogonal, we see that

\[
I_{pq}(f_{pq}) = I_{pq}(g_{pq}),
\]

from which we deduce \( f_{pq} = g_{pq} \), remembering the unitary equivalence (11.6) between \( L^2_{pq}(M) \) and \( L^2_{pq} \).

§ 12. Definition of Hermite Polynomials of Complex Variables. In M. W.I. we have shown a close relation between real multiple Wiener integral and Hermite polynomials of real variables. We shall here define Hermite polynomials of complex variables to which complex multiple Wiener integrals stand in the similar relation as is shown in the next §.

Definition. Let \( z \) be a complex variable and \( \bar{z} \) denote its conjugate. For \( p, q = 0, 1, 2, \ldots \) we define

\[
(12.1) \quad H_{pq}(z, \bar{z}) = \sum_{n=0}^{p+q} (-1)^n \frac{p! q!}{n! (p-n)! (q-n)!} z^{p-n} \bar{z}^{q-n}, \quad p \land q = \min(p, q).
\]

For the unity of the formulations we shall define trivially

\[
(12.2) \quad H_{pq}(z, \bar{z}) = 0 \quad \text{if} \ p \ or \ q < 0.
\]

The known identities concerning Hermite polynomials of real variables are extended to our complex case as follows.

Theorem 12.

(A) \( e(-\bar{t}z + t\bar{z} + \bar{z}z) = \sum_{p,q=0}^{\infty} \frac{1}{p! q!} H_{pq}(z, \bar{z}) \bar{t}^p t^q, \)
Complex Multiple Wiener Integral.

(B) \[ H_{pq}(z, \bar{z}) = e(zz) \frac{(-1)^{p+q}}{2^{p+q}p!q!} \frac{\partial^{p+q}}{\partial z^p \partial \bar{z}^q} e(-z\bar{z}), \quad (p, q \geq 0), \]

(C) \[
\begin{align*}
H_{p+1, q}(z, \bar{z}) - H_{pq}(z, \bar{z})z + qH_{p, q-1}(z, \bar{z}) &= 0, \\
H_{p, q+1}(z, \bar{z}) - H_{pq}(z, \bar{z})\bar{z} + pH_{p-1, q}(z, \bar{z}) &= 0,
\end{align*}
\]

(D) \[
\begin{align*}
\frac{\partial}{\partial z} H_{pq}(z, \bar{z}) &= pH_{p-1, q}(z, \bar{z}) \\
\frac{\partial}{\partial \bar{z}} H_{pq}(z, \bar{z}) &= qH_{p, q-1}(z, \bar{z})
\end{align*}
\]

(E) \[
\begin{align*}
\frac{\partial^2}{\partial z \partial \bar{z}} H_{pq}(z, \bar{z}) - z \frac{\partial}{\partial z} H_{pq}(z, \bar{z}) + qH_{pq}(z, \bar{z}) &= 0, \\
\frac{\partial^2}{\partial z \partial \bar{z}} H_{pq}(z, \bar{z}) - \bar{z} \frac{\partial}{\partial \bar{z}} H_{pq}(z, \bar{z}) + pH_{pq}(z, \bar{z}) &= 0.
\end{align*}
\]

PROOF.

(A) \[
e(-t\bar{t} + t\bar{z} + \bar{t}z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \sum_{q=0}^{\infty} \frac{\bar{z}^q}{q!} \sum_{p=0}^{\infty} \frac{\bar{t}^p}{p!} z^p
\]

\[= \sum_{n,p,q=0} \frac{(-1)^n}{n!} \frac{1}{p!} \frac{1}{q!} \frac{1}{z^n} \frac{1}{z^p} \frac{1}{z^q} \frac{1}{n} \frac{1}{p} \frac{1}{q} \right) z^{n-p-q}.
\]

(B) In remarking that
\[e(-t\bar{t} + t\bar{z} + \bar{t}z) = e(-z\bar{z})e(-(t-z)(\bar{t} - \bar{z}))\]
we have
\[\frac{\partial}{\partial t} e(-t\bar{t} + t\bar{z} + \bar{t}z) = e(-z\bar{z})\frac{\partial}{\partial t} e(-(t-z)(\bar{t} - \bar{z}))
\]
\[= -e(-z\bar{z}) \frac{\partial}{\partial \bar{t}} e(-(t-z)(\bar{t} - \bar{z})).
\]

By repeating similar computations we get
\[-\frac{\partial^p \partial q}{\partial t^p \partial \bar{t}^q} e(-t\bar{t} + t\bar{z} + \bar{t}z) = (-1)^{p+q} e(-z\bar{z}) \frac{\partial^{p+q}}{\partial \bar{z}^p \partial \bar{z}^q} e(-(t-z)(\bar{t} - \bar{z})),
\]
from which we deduce (B) by putting \(t = \bar{t} = 0\).

(C) We have
\[\frac{\partial}{\partial t} e(-t\bar{t} + t\bar{z} + \bar{t}z) = e(-t\bar{t} + t\bar{z} + \bar{t}z)(z-t).
\]

By comparing the coefficients of \(\bar{t}^{p-1}t^q\) in both sides we obtain the first identity. The second will be reduced by the same way.

(D) By differentiating both sides of (A) partially with regards to \(z\) or \(\bar{z}\) we obtain these identities at once.

(E) will be reduced at once from (C) and (D).

§ 13. A Relation Between Complex Multiple Wiener Integral and Hermite Polynomials of Complex Variables.

Theorem 13.1. Let \(M = (M(E), E \in E^{\omega_1})\) be any continuous complex normal random measure on \(T(B, m)\) and \(f(t)\) be any element of \(L^2(T)\) with the norm \(1\). Then we have

\[\int \ldots \int f(t_1) \cdots f(t_p)f(s_1) \cdots f(s_q) dM(t_1) \cdots dM(t_p) dM(s_1) \cdots dM(s_q)
= H_{pq}(Z, \bar{Z}),\]
where

$$Z = \int f(t) dM(t).$$

**Proof.** We shall denote the left side of (13.1) by $K_{pq}$. By the recurrence formulae of §9 we shall easily verify

$$K_{pq} Z = K_{p+1, q+1} + qK_{p, q-1} \quad (p > 0, \ q > 1),$$

$$K_{pq} Z = K_{p+1, q+1} + pK_{p-1, q} \quad (p > 1, \ q > 0);$$

if we define $K_{pq} = 0$ in case $p$ or $q < 0$, the above identities will be also true for all integral values of $p$, $q$. In comparing these identities and (C) in Theorem 12 we shall easily see

$$K_{pq} = H_{pq}(Z, \tilde{Z})$$

by means of mathematical induction with respect to $p + q$, since (13.4) is trivially true for $p + q = 0$.

By the same idea we shall generalize this theorem as follows.

**Theorem 13.2.** Let $M$ be as above and $\{f_1(t), \ldots, f_n(t)\}$ be any orthogonal system in $L^2(T)$. Then we have

$$K_{pq} Z = \sum_{\nu=1}^{\infty} H_{pq}^{(\nu)}(Z, \tilde{Z}),$$

where $p_\nu$ and $q_\nu$ are the number of $\nu$ appearing in $\{\alpha_i\}$ and $\{\beta_j\}$ respectively ($\nu = 1, 2, \ldots, n$) and

$$Z_\nu = \int f_\nu(t) dM(t), \ \nu = 1, 2, \ldots, n.$$
Complex Multiple Wiener Integral.

\[ I_{pq}(f_{\alpha_0}) = \prod_{\lambda \in A} H_{p(\lambda), q(\lambda)}(Z_{\lambda}, \bar{Z}_{\lambda}); \]

this is a finite product in essential and well defined, since we have \( \sum_{\lambda} p(\lambda) = p \) and \( \sum_{\lambda} q(\lambda) = q \).

Since both completeness and orthonormality are unitary invariants, the system:

\[ (14.2) \quad \prod_{\lambda \in A} H_{p(\lambda), q(\lambda)}(Z_{\lambda}, \bar{Z}_{\lambda}), \quad \sum_{\lambda} p(\lambda) = p, \quad \sum_{\lambda} q(\lambda) = q, \]

constitutes a complete orthonormal system in \( L^2_{pq}(M) \). \( L^2(M) \) being a direct sum of \( L^2_{pq}(M) \), \( p, q = 0, 1, 2, \ldots \), by virtue of Theorem 11, we obtain

**Theorem 14.** (Generalized Cameron-Martin Development)

\[ (14.3) \quad \prod_{\lambda \in A} H_{p(\lambda), q(\lambda)}(Z_{\lambda}, \bar{Z}_{\lambda}), \quad \sum_{\lambda} p(\lambda) < \infty, \quad \sum_{\lambda} q(\lambda) < \infty, \]

constitutes a complete orthonormal system in \( L^2(M) \) and so any element of \( L^2(M) \) is expressible in the Fourier series by means of (14.3).

§ 15. Completeness of Hermite Polynomials of Complex Variables. As is well-known, Hermite polynomials of real variables constitute a complete orthonormal system in \( L^2(R^1, e(-x^2)dx) \). We shall here show that our Hermite polynomials have the same property on the complex plane. This may be established by means of the identities in Theorem 12 in the same way as in the real case, but we shall here prove it by making use of Theorem 13.2.

**Theorem 15.** Let \( K \) denote a complex plane and \( N \) be a measure defined by

\[ dN(z) = \pi^{-1} e(-x^2 - y^2) dxdy, \quad z = x + iy. \]

Then

\[ (15.1) \quad dN(z) = \pi^{-1} e(-x^2 - y^2) dxdy, \quad z = x + iy. \]

**Proof.** Let \( T(B, m) \) be a measure space associated with a continuous measure \( m \), for example

\[ T = R^1, \quad m = \text{ordinary Lebesgue measure}. \]

Then there exists a complex normal random measure \( M \) on \( T(B, m) \) by virtue of Theorem 3.1. Let \( \{ f_{\alpha}(t), \alpha \in A \} \) be a complete orthonormal system in \( L^2(T) \). Then we see by § 2, that

\[ Z_{\alpha} = \int f_{\alpha}(t) dM(t), \quad \alpha \in A, \]

constitute a complex normal system and so that \( Z_{\alpha}, \alpha \in A \), are independent, each being subject to the probability law \( N \), since \( \langle Z_{\alpha}, Z_{\beta} \rangle = \langle f_{\alpha}, f_{\beta} \rangle = \delta_{\alpha \beta} \).

We fix an element \( f(t) \) in \( \{ f_{\alpha}(t) \} \) and denote the corresponding \( Z_{\alpha} \) with \( Z \).

Let \( L^2(Z) \) denote the totality of Baire functions of \( Z \) in \( L^2(\Omega) \). \( L^2(Z) \) is a closed linear manifold in \( L^2(M) \). Since \( Z \) is subject to \( N \), we have

\[ \| \varphi(Z) \|^2 = \int_{K} |\varphi(z)|^2 \pi^{-1} e(-x^2 - y^2) dxdy, \quad z = x + iy, \]

and so we see that

\[ L^2(K, N) \ni \varphi(z) \rightarrow \varphi(Z) \in L^2(Z) \]
determines a unitary equivalence. Therefore in order to prove our theorem, it is sufficient to show that \( \langle H_{pq}(Z, \tilde{Z}) \rangle \) constitute a complete orthogonal system in \( L^2(Z) \). It is clear by Theorem 14 that they are mutually orthogonal.

To show the completeness we need only prove that

\[
\varphi(Z) \perp H_{pq}(Z, \tilde{Z})
\]

implies

\[
\varphi(Z) = 0.
\]

Since \( \varphi(Z) \in L^2(Z) \subseteq L^2(M) \) and the system:

\[
H_{p_1q_1}(Z_{\alpha_1}, \tilde{Z}_{\alpha_1}) \cdots H_{p_nq_n}(Z_{\alpha_n}, \tilde{Z}_{\alpha_n})
\]

is complete in \( L^2(M) \), it remains only to show that \( \varphi(Z) \) is orthogonal to every element of this system. We may assume that

\[
Z_{\alpha_i} = Z \quad \text{and} \quad p_{\alpha_i} + q_{\alpha_i} > 0
\]

for some \( i \), for example \( i = 1 \), because, in the contrary case, our assertion will be true by (15.5). By the independence of \( \{Z_{\alpha}\} \) we have

\[
\varepsilon[H_{p_1q_1}(Z_{\alpha_1}, Z_{\alpha_1}) \cdots H_{p_nq_n}(Z_{\alpha_n}, Z_{\alpha_n}) \varphi(Z)]
\]

\[
= \varepsilon[H_{p_1q_1}(Z_{\alpha_2}, Z_{\alpha_2}) \varphi(Z_{\alpha_2}) \cdots H_{p_nq_n}(Z_{\alpha_n}, Z_{\alpha_n}) \varphi(Z_{\alpha_n})].
\]

Since \( p_{\alpha_i} + q_{\alpha_i} > 0 \), \( H_{p_1q_1}(\tilde{Z}_{\alpha_i}, Z_{\alpha_i}) \in L^2(M) \) is orthogonal to \( L^2(M) = K \), that is \( \varepsilon[H_{p_1q_1}(Z_{\alpha_i}, \tilde{Z}_{\alpha_i})] = 0 \). Thus we see that (15.7) is orthogonal to \( \varphi(Z) \).

IV. Spectral Structure and Ergodicity of Normal Screw Line.

§ 16. Normal Screw Line. A complex-valued stochastic process \( \{Z(t)\} \) is called a normal screw line if it is a complex normal system and satisfies

\[
(16.1) \quad \varepsilon Z(t) = 0,
\]

\[
(16.2) \quad \varepsilon[(Z(t+a) - Z(t+b))(Z(t+c) - Z(t+d))] = \varepsilon[(Z(a) - Z(b))(Z(c) - Z(d))].
\]

If \( Z(t) \) is a normal screw line, then \( Z(t) \) is a screw line in Hilbert space \( L^2(\Omega) \) in the sense of Kolmogorov [5] [6]. Therefore \( Z(t) \) is written in a spectral form:

\[
Z(s) - Z(t) = \int_{-\infty}^{\infty} e(i\lambda s) d\lambda M(\lambda).
\]

In considering that \( M = \{M(A)\} \) is a complex normal system, we easily see that \( M \) is a complex normal random measure on \( R^d(m) \), where \( m(A) = \|M(A)\|^2 \) satisfies

\[
\int_{-\infty}^{\infty} \frac{dm(\lambda)}{1 + \lambda^2} < \infty,
\]

\( m(\lambda) \) is called the spectral measure of \( Z(t) \).

\( M(A) \) is uniquely determined by \( Z(t) \) and expressible in the following form:

\[
M(A) = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}(Z(t_{nk}) - Z(s_{nk}))
\]

where \( a_{nk} \) means the limit in mean i.e. in the norm of \( L^2(\Omega) \).

§ 17. Shift Transformation of Normal Screw Line. Let \( \{Z(t)\} \) be a comp-
lex normal system. We shall derive a random interval function from \( Z(t) \) as follows:

\[
Z\equiv \{Z(a, b) = Z(b) - Z(a), \ -\infty < a < b < \infty \}.
\]

Then we shall easily see that

\[
L^2(Z) = L^2(M),
\]

where \( M \) is the complex normal random measure appearing in the spectral expression (16.3).

We shall now define \( T_t Z = \{(T_t Z)(a, b), \ -\infty < a < b < \infty \} \) by

\[
(T_t Z)(a, b) = Z(a + t, b + t).
\]

\( Z \) being a complex normal system, it is so the case with \( T_t Z \). Therefore by virtue of Theorem 2.3 it follows from (16.1) and (16.2) that \( Z \) and \( T_t Z \) are subject to the same probability distribution of \( Z \). \( T_t \) is called the shift transformation of \( Z \) by definition.

We shall extend this transformation onto \( L^2(Z) \). Let \( L^2_*(Z) \) is the totality of \( Z \)'s in \( L^2(Z) \) which are expressed in the form:

\[
Z = f(Z(a_1, b_1), \ldots, Z(a_n, b_n)),
\]

\( f \) being a Baire function of \( n \) complex variables; \( L^2_*(Z) \) is clearly a linear manifold dense in \( L^2(Z) \). For such \( Z \) we shall define

\[
T_t Z = f(Z(a_1 + t, b_1 + t), \ldots, Z(a_n + t, b_n + t)).
\]

This \( T_t \) becomes a mapping from \( L^2_*(Z) \) onto itself which is isometric, since

\[
\| T_t Z \| = \| Z \|
\]

on account of the fact that \( Z \) and \( T_t Z \) have the same distribution. Therefore we shall further extend \( T_t \) onto \( L^2(Z) \) preserving the isometric property (17.4); thus \( T_t \) constitute a one-parameter group of unitary operators in \( L^2(Z) \) i.e. \( L^2(M) \).

Since the above \( M(\lambda) \) belongs to \( L^2(Z) \), \( T_t M(A) \) and so \( T_t M \) are well defined.

**THEOREM 17.**

\[
(T_t M)(A) = \int_A e(i\lambda t) dM(\lambda),
\]

or symbolically

\[
(T_t M)(\lambda) = e(i\lambda t) dM(\lambda).
\]

**Proof.** Since we have

\[
Z(a, b) = \int_a^b e(i\lambda s) ds dM(\lambda),
\]

we get

\[
T_t Z(a, b) = \int_a^b e(i\lambda (s + t)) ds dM(\lambda)
\]

\[
= \int_a^b e(i\lambda s) ds \cdot e(i\lambda t) dM(\lambda)
\]

\[
= \int_a^b e(i\lambda s) ds dM_\lambda(\lambda),
\]

where

\[
M(\lambda) = \int_A e(i\lambda t) dM(\lambda).
\]
Now suppose that
\[ M(A) = \lim_{\kappa \to \infty} \sum_{k=1}^{\infty} c_{nk} Z(a_n, b_n). \]
In remembering that \( T_t Z \) and \( Z \) have the same distribution and that the random measure \( M \) is uniquely determined by \( Z \), we see that \( M(A) \) is also obtained by the same expression from \( T_t Z \) as follows:
\[ M(A) = \lim_{\kappa \to \infty} \sum_{k=1}^{\infty} c_{nk} T_t Z(a_{nk}, b_{nk}). \]
By the above two identities we obtain \( T_t M(A) = M_t(A) \), which was to be proved.

§ 18. Spectral Structure of \( T_t \). We shall use the same notation as in the preceding §. We define a one-parameter group of unitary operators \( \{ S_t^{(p,q)} \} \) on \( L^2_{pq}(R^1, m) \) by
\[(18.1) \quad S_t^{(p,q)} f(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q) = e(i(\lambda_1 + \cdots + \lambda_p - \mu_1 - \cdots - \mu_q)t) f(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q). \]
Let \( L^2 \) denote the direct sum of Hilbert spaces \( L^2_{pq}(R^1, m) \), \( p, q = 0, 1, 2, \ldots \), and \( S_t \) that of \( S_t^{(p,q)} \), \( p, q = 0, 1, 2, \ldots \). Then we have

**Theorem 18.** If \( m \) is continuous except at \( \lambda = 0 \), then \( S_t \) is unitary-equivalent with \( T_t \).

**Proof.** By the assumption, \( m \) and also \( M \) are continuous in the sense of § 4. Therefore we may use the results obtained concerning multiple Wiener integral. By § 11 \( L^2(Z) \), i.e. \( L^2(M) \) is a direct sum of \( L^2_{pq}(M) \), \( p, q = 0, 1, 2, \ldots \). Making use of the definition of multiple Wiener integral, Theorem 8 and Theorem 17, we see that
\[(18.2) \quad \int L^2_{pq} f(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q) dM(\lambda_1) \cdots dM(\lambda_p) d\overline{M(\mu_1)} \cdots dM(\mu_q), \]
so that \( T_t \) makes each \( L^2_{pq}(M) \) invariant. Therefore we need only prove that the transformation \( T_t \) restricted on \( L^2_{pq}(M) \) is unitary-equivalent with \( S_t^{(p,q)} \). This is evident, since the correspondence:
\[(18.3) \quad L^2_{pq} \ni \psi \mapsto \int L^2_{pq} f \in L^2_{pq}(M) \]
determines a unitary equivalence, as is stated in § 11.

§ 19. Ergodicity of \( T_t \). **Definition.** If \( T_t Z = Z \) implies that \( Z \) is a constant, then \( T_t \) is defined to be ergodic.

**Theorem 19.1.** It is necessary and sufficient for \( T_t \) to be ergodic that \( m \) is continuous.

**Proof of necessity.** Suppose that \( m \) be discontinuous at \( \lambda = \lambda_0 \), then \( Z = |M(\{\lambda_0\})| \) is invariant by \( T_t \), since
\[ T_t Z = |T_t M(\{\lambda_0\})| = |e(i\lambda_0 t) M(\{\lambda_0\})| = |M(\{\lambda_0\})| = Z. \]
\( Z \) is not constant, since it is subject to the distribution:
\[ \frac{2z}{\sqrt{a}} e^{-\left(\frac{z^2}{2a}\right)} dz, a = m(\{\lambda_0\}) > 0. \]
Thus $T_t$ is not ergodic.

**Proof of Sufficiency.** Suppose that $m$ be continuous. Let $Z$ be an invariant element of $T_t$. By Theorem 11, $Z$ and $T_tZ$ are expressed as follows:

$$Z = \sum_{pq} I_{pq}(f_{pq}), \quad f_{pq} \in L^2_{pq}(\mathbb{R}^p, m),$$

$$T_tZ = \sum_{pq} I_{pq}(f_{pq}e(i(\lambda_1 + \cdots + \lambda_p - \mu_1 - \cdots - \mu_q)t)).$$

By the invariance of $Z$ we obtain

$$f_{pq} = f_{pq}e(i(\lambda_1 + \cdots + \lambda_p - \mu_1 - \cdots - \mu_q)t))$$

and so, for $(p, q) \neq (0, 0)$,

$$f_{pq} = 0$$

almost everywhere on $\mathbb{R}^{p+q}(m^{p+q})$ except on the hyperplane $H_{pq} : \lambda_1 + \cdots + \lambda_p - \mu_1 - \cdots - \mu_q = 0$; $m$ being continuous, $H_{pq}$ has the $m^{p+q}$-measure 0. Thus $f_{pq} = 0$ almost everywhere on the entire space $\mathbb{R}^{p+q}(m^{p+q})$ for $(p, q) \neq (0, 0)$, which implies that $Z$ is a constant.

**Definition.** $T_t$ is defined to be strongly mixing if we have

$$(19.1) \quad (T_tY, Z) \leq (Y, 1)(Z, 1)$$

for any $Y, Z \in L^2(M) = L^2(Z)$.

**Theorem 19.2.** It is necessary and sufficient for $T_t$ to be strongly mixing that it holds

$$(19.2) \quad \int_a^b e(i\lambda t)dm(\lambda) \to 0, \text{ as } t \to \infty,$$

for any $a, b$ such that $-\infty < a < b < \infty$.

**Proof of Necessity.** Suppose that (19.2) does not hold for some $a, b$. Let $f(\lambda)$ denote the characteristic function of the interval $(a, b)$. Put

$$(19.3) \quad Y = \int f(\lambda)dM(\lambda) = I_{10}(f).$$

Then we have

$$(19.4) \quad T_tY = \int f(\lambda)e(i\lambda t)dM(\lambda),$$

and so

$$(19.1) = \int_{-\infty}^{\infty} |f(\lambda)|^2e(i\lambda t)dm(\lambda) = \int_{a}^{b} e(i\lambda t)dm(\lambda) + \cdots + dm(\lambda_1) \to 0 \equiv (Y, 1)(Z, 1),$$

which implies that $T_t$ is not strongly mixing.

**Proof of Sufficiency.** Suppose that (19.2) hold for any $a, b$. Then $m$ is clearly continuous. First we shall show that, for $f(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q) \in L^1(\mathbb{R}^{p+q}, m^{p+q})$, it holds

$$(19.5) \quad \int \cdots \int f(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q)e(i\sum_{\lambda_1} - \sum_{\mu_1} \mu_q t)dm(\lambda_1) \cdots dm(\mu_q) \to 0,$$

as $t \to \infty$. In case $f$ is the characteristic function of a $(p+q)$-dimensional interval (19.5) follows at once from the assumption (19.2). In the general case we shall approximate $f$ by a linear combination of such functions. Then the integral in (19.4) may be approximated uniformly in $t$. Thus we see that it will also tend to 0 as $t \to \infty$. 
For the proof of sufficiency we shall show that, if \( Y \) and \( Z \) be any elements in \( L^2(M) \), then we get (19.1), as \( t \to \infty \).

In case \( Y \) and \( Z \) are written in the form

\[
Y = \sum_{pq=0}^{n} I_{pq}(f_{pq}), \quad Z = \sum_{pq=0}^{n} I_{pq}(g_{pq}),
\]

we have

\[
(TtY, Z) = f_{00}g_{00} + \sum_{pq=0}^{n} \int f_{pq}g_{pq} e(\mu t(\Sigma \lambda - \Sigma \mu)) dm(\lambda_{1}) \ldots dm(\mu_{n}).
\]

By the above remark each term in the summation sign \( \Sigma \) will tend to 0 as \( t \to \infty \), since \( f_{pq}, g_{pq} \in L^{p+q}(R, m) \) by virtue of the fact that \( f_{pq}, g_{pq} \in L^{p+q}(R^{1}, m) \).

Thus we get

\[
(TtY, Z) - f_{00}g_{00} = (Y, 1)(Z, 1).
\]

Assume that \( Y \) and \( Z \) be any arbitrary elements of \( L^2(M) \). Then there exists \( \{Y_{n}\} \) and \( \{Z_{n}\} \) such that \( Y_{n} \) and \( Z_{n} \) are of the above form and that

\[
\|Y_{n} - Y\| \to 0, \quad \|Z_{n} - Z\| \to 0.
\]

Then we have

\[
|(TtY_{n}, Z_{n}) - (TtY, Z)| \to 0 \quad \text{(uniformly in } t),
\]

and

\[
|(Y_{n}, 1)(Z_{n}, 1) - (Y, 1)(Z, 1)| \to 0,
\]

remembering that \( Tt \) is isometric. By the above discussion we see that \( (TtY_{n}, Z_{n}) \to (Y_{n}, 1)(Z_{n}, 1) \) and so that \( (TtY, Z) \to (Y, 1)(Z, 1) \).

**Theorem 19.3.** If \( m(R^{1}) < \infty \), then we may replace (19.1) by the following simple condition:

\[
\int_{-\infty}^{\infty} e(i\lambda t) dm(\lambda) \to 0.
\]

**Proof.** It is sufficient that (19.1) are equivalent with (19.2). The former follows from the latter, since

\[
\left| \int_{-\infty}^{\infty} e(i\lambda t) dm(\lambda) - \int_{-n}^{n} e(i\lambda t) dm(\lambda) \right| \leq \left| \int_{|\lambda| \geq n} dm(\lambda) \right| \to 0, \quad \text{uniformly in } t.
\]

Next we shall deduce (19.2) from (19.1').

If \( f(\lambda) \) be of the form:

\[
f(\lambda) = \int_{-\infty}^{\infty} e(i\lambda s) g(s) ds, \quad \int_{-\infty}^{\infty} |g(s)| ds < \infty,
\]

then we have

\[
\int_{-\infty}^{\infty} f(\lambda) e(i\lambda t) dm(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(i\lambda (t+s)) g(s) ds dm(\lambda)
\]

\[
= \int_{-\infty}^{\infty} g(s) \int_{-\infty}^{\infty} e(i\lambda (t+s)) dm(\lambda) ds
\]

and so, by the assumption (19.1'),

\[
\int_{-\infty}^{\infty} f(\lambda) e(i\lambda t) dm(\lambda) \to 0 \quad \text{(} t \to \infty, \text{)}
\]
Complex Multiple Wiener Integral.

If \( f(\lambda) \) is a continuous function vanishing outside of a certain bounded interval, \( f(\lambda) \) may be approximated uniformly by a function of the form (19.6), as is well-known in the theory of Fourier transforms. Therefore (19.7) will also hold for such \( f(\lambda) \). Since \( m \) is continuous at any value of \( t \) by (19.2) the characteristic function of any interval will be approximated in the norm of \( L^2(R^1, m) \) by a continuous function above mentioned. Thus we see that (19.7) will be also true for such characteristic function \( f \), which implies (19.2).

§ 20. Application to Complex Wiener Process. Let \( B(t) \) be a complex Wiener process. Then \( B(t) \) is a normal screw line. Therefore \( B(t) \) is expressible as follows.

\[
B(t) - B(s) = \int_{-\infty}^{\infty} e(i\lambda t) d\sigma dM(\lambda),
\]

from which follows

\[
(B(t) - B(0), B(s) - B(0)) = \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} \cdot \frac{e^{-i\lambda s} - 1}{-i\lambda} dm(\lambda),
\]

where \( m(\lambda) = ||M(\lambda)||^2 \) is the spectral measure of \( B(t) \). But we have, as a property of a Wiener process,

\[
(B(t) - B(0), B(s) - B(0)) = \frac{1}{2} (|t| + |s| - |t - s|) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} \cdot \frac{e^{-i\lambda s} - 1}{i\lambda} d\lambda.
\]

Thus we get

\[
(20.1) \quad m(E) = |E|/2\pi, \quad |E| = \text{Lebesgue measure of } E.
\]

By a simple calculation we obtain

\[
\int_{a}^{b} e(i\lambda t) d\sigma dM(\lambda) = \frac{1}{2\pi} \int_{a}^{b} e(i\lambda t) d\lambda \rightarrow 0 \quad (t \rightarrow \infty),
\]

and so, by Theorem 19.2, we see that \( T_t \) is strongly mixing; this is a well-known fact.

By making use of Theorem 18 we obtain a theorem of S. Kakutani [7] concerning the spectral structure of \( T \).

§ 21. Application to Complex Normal Stationary Process. Let \( X=\{X(t)\} \) be a complex normal stationary process. Define \( Z=\{Z(t) - Z(s)\} \) by

\[
(21.1) \quad Z(t) - Z(s) = \int_{s}^{t} X(\tau) d\tau.
\]

Then \( Z(t) \) is a normal screw line. As is well-known, \( X(t) \) is expressed in the form:

\[
(21.2) \quad X(t) = \int_{-\infty}^{\infty} e(i\lambda t) dM(\lambda); \quad M(\lambda) \text{ is a complex normal random measure on } R^1(m), \quad \text{where } m(\lambda) = ||M(\lambda)||^2 \text{ is the spectral measure appearing in Khintchine's canonical form of the autocorrelation function of } X(t) \text{ and so } m(R^1) < \infty.
\]

Since we deduce, from (21.2),

\[
(21.3) \quad Z(b) - Z(a) = \int_{a}^{b} \int_{-\infty}^{\infty} e(i\lambda t) dtdM(\lambda),
\]

which is the spectral expression of \( Z(t) \) in the sense of § 16. Thus we see that
the spectral measure $m'$ of $Z(t)$ is the same as that of $X(t)$ i.e. $m$ and so we have $m'(R) < \infty$.

Since $L^2(Z) = L^2(M) = L^2(X)$ and the shift transformation of $X$ has the same effect on $L^2(M)$ with that of $Z$, it is sufficient to study about $Z$ in order to investigate $X$. By Theorems 19.1 and 19.3 we shall obtain the following facts.

Let $T_t$ be the group of shift transformations of complex normal stationary process whose spectral measure is denoted by $m$.

(1) It is necessary and sufficient for $T_t$ to be ergodic that $m$ is continuous.

(2) It is necessary and sufficient for $T_t$ to be strongly mixing that the Fourier transform of $m$ vanishes at $\infty$.

The former has already been obtained by G. Maruyama [8] and by U. Grenander [10] and the latter by the author [9].

REFERENCES


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