2. On Dirichlet- and Neumann-problem with integrable boundary values.

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In this paper, we shall treat Dirichlet- and Neumann-problem for two-dimensional and three-dimensional domains, where the boundary value $f(Q)$ is integrable in Lebesgue's sense and a harmonic function or its normal derivative takes the value $f(Q)$ almost everywhere on the boundary. By means of integral equations, we shall prove in §5 and §6, that Dirichlet problem for a bounded simply connected domain and Neumann-problem for a bounded domain of finite connectivity are solvable. By means of Green's function, we shall prove in §8, that Dirichlet problem for a bounded domain of finite connectivity is solvable and shall prove an analogue of Fatou's theorem for a bounded harmonic function in space.

1. Existence of boundary values of logarithmic potentials.

Let D be bounded domain on a plane, whose boundary Γ consists of a finite number of analytic Jordan curves. Let Q be a point of Γ , ds_Q be the arc element and n_Q be the inner normal of Γ at Q . Let $\kappa(Q) \subset L$ on Γ , such that

$$\int_{\Gamma} |\kappa(Q)| ds_Q < \infty. \quad (1)$$

We put for $P \in D$,

$$u(P) = \int_{\Gamma} \kappa(Q) \frac{\cos(r, n_Q)}{r} ds_Q, \quad r = r_{QP}, \quad (2)$$

where (r, n_Q) is the angle subtained by two vectors $\vec{QP}$ and n_Q .

Let $Q \in \Gamma$ and $\Delta(Q)$ be a Stolz domain, whose vertex is at Q and is bounded by two lines, each of which makes an angle $\varphi_0 < \frac{\pi}{2}$ with n_Q . Similarly we define a Stolz domain $\Delta_1(Q)$, which lies outside of D . If $\lim_{P \rightarrow Q} u(P)$ exists, when P tends to Q from the inside of $\Delta(Q)$, then we denote the limiting value by

$u_i(Q)$. Similarly we define $u_e(Q)$, when P tends to Q from the inside of $\Delta_1(Q)$.

Theorem 1. Let $\kappa(Q) \subset L$ on Γ and

$$u(P) = \int_{\Gamma} \kappa(Q) \frac{\cos(r, n_Q)}{r} ds_Q, \quad r = r_{QP}.$$

Then for almost all $Q_0 \in \Gamma$,

$$\begin{aligned} u_i(Q_0) &= \pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} ds_Q, \\ u_e(Q_0) &= -\pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} ds_Q, \quad r_0 = r_{Q_0 Q}. \end{aligned}$$

Proof. By Lebesgue's theorem, for almost all $Q_0 \in \Gamma$,

$$\lim_{Q \rightarrow Q_0} \frac{1}{QQ_0} \int_{Q_0}^Q |f(Q) - f(Q_0)| ds_Q = 0. \quad (1)$$

Let Q_0 be such a point. We take Q_0 as the origin and n_{Q_0} as the positive y -axis, so that a neighbourhood $U(Q_0)$ of Q_0 on Γ is of the form:

$$y = ax^2 + \dots, \quad (|x| \leq \delta).$$

Since for $P \in \Delta(Q_0)$

$$\int_{\Gamma} \frac{\cos(r, n_Q)}{r} ds_Q = 2\pi,$$

we have

$$\begin{aligned} u(P) - 2\pi \kappa(Q_0) &= \int_{U(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r, n_Q)}{r} ds_Q + \int_{\Gamma - U(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r, n_Q)}{r} ds_Q \\ &= \text{I} + \text{II}, \end{aligned} \quad (2)$$

where

$$\lim_{P \rightarrow Q_0} \text{II} = \int_{\Gamma - U(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_Q)}{r_0} ds_Q. \quad (3)$$

In I, let $Q = (x, y)$, we put $\kappa_1(Q) = \kappa(Q) - \kappa(Q_0) = k_1(x)$ and

$$K_1(x) = \int_0^x |k_1(x)| dx. \quad (4)$$

Then by (1) for $|x| \leq \delta$,

$$|K_1(x)| \leq \varepsilon(\delta) |x|, \quad (5)$$

where $\varepsilon(\delta) \rightarrow 0$ with $\delta \rightarrow 0$.

Let $P = (\xi, \eta) \in \Delta(Q_0)$ and $Q = (x, y) \in U(Q_0)$, then

$$\xi = \tau \sin \varphi, \quad \eta = \tau \cos \varphi \quad \left(|\varphi| \leq \varphi_0 < \frac{\pi}{2} \right),$$

$$r^2 = \overline{PQ^2} = (\xi - x)^2 + (\eta - y)^2 = x^2 + \tau^2 - 2x\tau \sin \varphi + O(\tau x^2) + O(x^4).$$

Since $2\tau|x| \leq x^2 + \tau^2$, we have $O(\tau x^2) = O(x(x^2 + \tau^2))$, $O(x^4) = O(x(x^2 + \tau^2))$, so that

$$r^2 \geq x^2 + \tau^2 - \sin \varphi_0 (x^2 + \tau^2) + O(x(x^2 + \tau^2)) = (1 - \sin \varphi_0) (x^2 + \tau^2) (1 + O(x)).$$

Let (l, m) be the direction cosines of n_Q and (λ, μ) be those of $\vec{QP}$, then

$$l = O(x), \quad m = 1 + O(x^2),$$

$$\lambda = \frac{\xi - x}{r} = \frac{O(\tau) + O(x)}{r}, \quad \mu = \frac{\eta - y}{r} = \frac{O(\tau) + O(x^2)}{r},$$

so that

$$\frac{\cos(r, n_Q)}{r} = \frac{l\lambda + m\mu}{r} = \frac{O(\tau) + O(x^2)}{r^2} = \frac{O(\tau) + O(x^2)}{x^2 + \tau^2} = \frac{O(\tau)}{x^2 + \tau^2} + O(1). \quad (7)$$

Hence

$$\begin{aligned} |I| &\leq \int_{v(Q_0)} |\kappa_1(Q)| \left| \frac{\cos(r, n_Q)}{r} \right| ds_Q \leq \int_{-\delta}^{\delta} \frac{|k_1(x)| O(\tau)}{x^2 + \tau^2} dx + O\left(\int_{-\delta}^{\delta} |k_1(x)| dx\right) \\ &= \left[\frac{K_1(x) O(\tau)}{x^2 + \tau^2} \right]_{-\delta}^{\delta} + 2 \int_{-\delta}^{\delta} \frac{K_1(x) O(\tau) x dx}{(x^2 + \tau^2)^2} + O(\varepsilon(\delta)), \end{aligned}$$

where

$$\begin{aligned} \lim_{\tau \rightarrow 0} \left[\frac{K_1(x) O(\tau)}{x^2 + \tau^2} \right]_{-\delta}^{\delta} &= 0, \\ \int_{-\delta}^{\delta} \frac{K_1(x) |O(\tau) x dx|}{(x^2 + \tau^2)^2} &\leq \varepsilon(\delta) \int_{-\delta}^{\delta} \frac{O(\tau) x^2 dx}{(x^2 + \tau^2)^2} \leq O(\varepsilon(\delta)) \int_{-\infty}^{\infty} \frac{t^2 dt}{(1+t^2)^2} = O(\varepsilon(\delta)), \quad (x = \tau t). \end{aligned}$$

Hence $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = 0$, so that from (2), (3),

$$\begin{aligned} u_i(Q_0) - 2\pi\kappa(Q_0) &= \int_{\Gamma} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_Q)}{r_0} ds_Q \\ &= \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} ds_Q - \kappa(Q_0) \int_{\Gamma} \frac{\cos(r_0, n_Q)}{r_0} ds_Q \\ &= \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} ds_Q - \pi\kappa(Q_0), \end{aligned}$$

Hence

$$u_i(Q_0) = \pi\kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} ds_Q.$$

Similarly we can prove the second part of the theorem.

Theorem 2. Let $\kappa(Q) \in L$ on Γ and

$$u(P) = \int_{\Gamma} \kappa(Q) \log \frac{1}{r} ds_Q, \quad r = r_{PQ}.$$

Then for almost all $Q_0 \in \Gamma$, when P tends to Q_0 on n_{Q_0} ,

$$\frac{\partial u(P)}{\partial v} \rightarrow -\pi\kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q, \quad r_0 = r_{Q_0Q},$$

where $\frac{\partial u(P)}{\partial v}$ is the derivative of $u(P)$ in the direction of n_{Q_0} and (r_0, n_{Q_0}) is the angle subtained by two vectors $\overrightarrow{Q_0Q}$ and n_{Q_0} .

Proof. Let $Q_0 = (0, 0)$ satisfy the condition of the proof of Theorem 1 and $P = (0, \tau)$ ($\tau > 0$) be a point on n_{Q_0} .

First we consider a special case:

$$v(P) = \int_{\Gamma} \log \frac{1}{r} ds_Q \quad (1)$$

and shall prove that when $P \rightarrow Q_0$ on n_{Q_0} ,

$$\frac{\partial v(P)}{\partial \nu} \rightarrow -\pi + \int_{\Gamma} \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q. \quad (2)$$

Since $\frac{\partial}{\partial \nu} \log \frac{1}{r} = \frac{\cos(r, n_{Q_0})}{r}$ ($r = r_{PQ}$), we have

$$\frac{\partial v(P)}{\partial \nu} = \int_{\Gamma(Q_0)} \frac{\partial}{\partial \nu} \log \frac{1}{r} ds_Q + \int_{\Gamma - \Gamma(Q_0)} \frac{\cos(r, n_{Q_0})}{r} ds_Q = \text{I} + \text{II}, \quad (3)$$

where

$$\lim_{P \rightarrow Q_0} \text{II} = \int_{\Gamma - \Gamma(Q_0)} \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q. \quad (4)$$

In I, let $P = (0, \tau)$, $Q = (x, y)$, then $r^2 = x^2 + (y - \tau)^2$,

$$\begin{aligned} \frac{\partial}{\partial \nu} \log \frac{1}{r} &= \frac{\partial}{\partial \tau} \log \frac{1}{r} = \frac{-\tau + y}{x^2 + (y - \tau)^2} = \frac{-\tau + O(x^2)}{(x^2 + \tau^2)(1 + O(x))} \\ &= \frac{(-\tau + O(x^2))(1 + O(x))}{x^2 + \tau^2} = \frac{-\tau + O(x\tau) + O(x^2)}{x^2 + \tau^2} = \frac{-\tau}{x^2 + \tau^2} + O(1) \end{aligned} \quad (5)$$

and $d\sigma_Q = (1 + O(x^2))dx$, so that

$$\text{I} = \int_{-\delta}^{\delta} \frac{-\tau}{x^2 + \tau^2} dx + O(\delta) \rightarrow -\pi + O(\delta) \quad (\tau \rightarrow 0),$$

hence $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} \text{I}) = -\pi$, so that from (3), (4),

$$\frac{\partial v(P)}{\partial \nu} \rightarrow -\pi + \int_{\Gamma} \frac{\cos(r_0, n_{Q_0})}{r_0} d\sigma_Q \quad \text{as } P \rightarrow Q_0. \quad (2)$$

From

$$u(P) - \kappa(Q_0)v(P) = \int_{\Gamma} (\kappa(Q) - \kappa(Q_0)) \log \frac{1}{r} ds_Q,$$

we have

$$\begin{aligned} \frac{\partial u(P)}{\partial \nu} - \kappa(Q_0) \frac{\partial v(P)}{\partial \nu} &= \int_{\Gamma(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\partial}{\partial \nu} \log \frac{1}{r} ds_Q + \int_{\Gamma - \Gamma(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r, n_{Q_0})}{r} ds_Q \\ &= \text{I} + \text{II}, \end{aligned} \quad (6)$$

where

$$\lim_{P \rightarrow Q_0} \text{II} = \int_{\Gamma - \Gamma(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q. \quad (7)$$

In I, we put $\kappa_1(Q) = \kappa(Q) - \kappa(Q_0)$, then

$$|\text{I}| \leq \int_{-\delta}^{\delta} |\kappa_1(Q)| \left(\frac{\tau}{x^2 + \tau^2} + O(1) \right) dx = \int_{-\delta}^{\delta} \frac{|\kappa_1(Q)| \tau dx}{x^2 + \tau^2} + O(\varepsilon(\delta)).$$

We have as the proof of Theorem 1, $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} \text{I}) = 0$, so that from (6), (7),

$$\frac{\partial u(P)}{\partial \nu} - \kappa(Q_0) \frac{\partial v(P)}{\partial \nu} \rightarrow \int_{\Gamma} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q,$$

hence from (2), we have

$$\frac{\partial u(P)}{\partial \nu} \rightarrow -\pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q, \text{ as } P \rightarrow Q_0.$$

2. Existence of boundary values of Newtonian potentials.

Let D be a bounded domain in space, whose boundary S consists of a finite number of analytic closed surfaces. Let Q be a point of S , $d\omega_Q$ be the surface element, n_Q be the inner normal of S at Q . Let $\kappa(Q) \in L$ on S , such that

$$\iint_S |\kappa(Q)| d\omega_Q < \infty. \quad (1)$$

We put for $P \in D$,

$$u(P) = \iint_S \kappa(Q) \frac{\cos(r, n_Q)}{r^2} d\omega_Q, \quad r = r_{QP}, \quad (2)$$

where (r, n_Q) is the angle subtained by two vectors $\vec{QP}$ and n_Q . Let $Q \in S$ and $\Delta(Q)$ be a Stolz domain, whose vertex is at Q and is bounded by a cone, whose generator makes an angle $\theta_0 < \frac{\pi}{2}$ with n_Q . Similarly we define a Stolz domain $\Delta_1(Q)$, which lies outside of D . If $\lim_{P \rightarrow Q} u(P)$ exists, when P tends to Q from the inside of $\Delta(Q)$, then we denote the limiting value by $u_i(Q)$. Similarly we define $u_e(Q)$, when P tends to Q from the inside of $\Delta_1(Q)$.

Theorem 3. Let $\kappa(Q) \in L$ on S and

$$u(P) = \iint_S \kappa(Q) \frac{\cos(r, n_Q)}{r^2} d\omega_Q, \quad r = r_{QP}.$$

Then for almost all Q_0 on S ,

$$\begin{aligned} u_i(Q_0) &= 2\pi \kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q, \\ u_e(Q_0) &= -2\pi \kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q, \quad r_0 = r_{Q_0Q}. \end{aligned}$$

Proof. Let $Q_0 \in S$ and $U(Q_0)$ be the part of S , which lies in a sphere of radius ρ about Q_0 . Then by Lebesgue's theorem, for almost all $Q_0 \in S$,

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^2} \iint_{U(Q_0)} |\kappa(Q) - \kappa(Q_0)| d\omega_Q = 0. \quad (1)$$

Let Q_0 be such a point. We take Q_0 as the origin and n_{Q_0} as the positive z -axis, so that a neighbourhood $U(Q_0)$ of Q_0 on S is of the form:

$$z = ax^2 + by^2 + \dots, \quad x^2 + y^2 = \rho^2 \leq \delta^2.$$

Since for $P \in \Delta(Q_0)$, $\iint_S \frac{\cos(r, n_Q)}{r^2} d\omega_Q = 4\pi$, we have

$$\begin{aligned} u(P) - 4\pi \kappa(Q_0) &= \iint_{U(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r, n_Q)}{r^2} d\omega_Q + \iint_{S - U(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r, n_Q)}{r^2} d\omega_Q \\ &= I + II, \end{aligned} \quad (2)$$

where

$$\lim_{P \rightarrow Q_0} II = \iint_{S - U(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q. \quad (3)$$

In I, let $Q = (x, y, z)$, $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, we put $\kappa_1(Q) = \kappa(Q) - \kappa(Q_0) = k_1(\rho, \varphi)$ and

$$k_1(\rho) = \int_0^{2\pi} \rho |k_1(\rho, \varphi)| d\varphi, \quad K_1(\rho) = \int_0^\rho k_1(\rho) d\rho. \quad (4)$$

Then by (1)

$$K_1(\rho) \leq \varepsilon(\delta) \rho^2 \quad (\rho \leq \delta), \quad (5)$$

where $\varepsilon(\delta) \rightarrow 0$ with $\delta \rightarrow 0$.

Let $P = (\xi, \eta, \zeta) \in \Delta(Q_0)$ and $Q = (x, y, z) \in U(Q_0)$, then

$$\xi = \tau \sin \theta \cos \psi, \quad \eta = \tau \sin \theta \sin \psi, \quad \zeta = \tau \cos \theta \quad \left(0 \leq \theta \leq \theta_0 < \frac{\pi}{2}, \quad 0 \leq \psi \leq 2\pi\right),$$

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = O(\rho^2), \quad \rho \leq \delta,$$

so that as before

$$\begin{aligned} r^2 &= (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = \rho^2 + \tau^2 - 2\rho\tau \sin \theta \cos(\varphi - \psi) \\ &\quad + O(\tau\rho^2) + O(\rho^4) \geq (1 - \sin \theta_0)(\rho^2 + \tau^2)(1 + O(\rho)). \end{aligned}$$

Let (l, m, n) be the direction cosines of n_Q and (λ, μ, ν) be those of $\vec{QP}$, then

$$l = O(\rho), \quad m = O(\rho), \quad n = 1 + O(\rho^2),$$

$$\lambda = \frac{\xi - x}{r} = \frac{O(\tau) + O(\rho)}{r}, \quad \mu = \frac{\eta - y}{r} = \frac{O(\tau) + O(\rho)}{r}, \quad \nu = \frac{\zeta - z}{r} = \frac{O(\tau) + O(\rho^2)}{r},$$

hence

$$\frac{\cos(r, n_Q)}{r^2} = \frac{l\lambda + m\mu + n\nu}{r^2} = \frac{O(\tau) + O(\rho^2)}{r^3} = \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}}. \quad (7)$$

Hence

$$|I| \leq \iint_{U(Q_0)} |\kappa_1(Q)| \left| \frac{\cos(r, n_Q)}{r^2} \right| d\omega_Q \leq \iint_{U(Q_0)} |\kappa_1(Q)| \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}} \rho d\rho d\varphi, \quad (8)$$

where

$$\iint_{U(Q_0)} \frac{|\kappa_1(Q)| O(\tau) \rho d\rho d\varphi}{(\rho^2 + \tau^2)^{3/2}} = \int_0^\delta \frac{k_1(\rho) O(\tau) d\rho}{(\rho^2 + \tau^2)^{3/2}} = \left[\frac{K_1(\rho) O(\tau)}{(\rho^2 + \tau^2)^{3/2}} \right]_0^\delta + 3 \int_0^\delta \frac{K_1(\rho) O(\tau) \rho d\rho}{(\rho^2 + \tau^2)^{5/2}},$$

$$\lim_{\tau \rightarrow 0} \left[\frac{K_1(\rho) O(\tau)}{(\rho^2 + \tau^2)^{3/2}} \right]_0^\delta = 0,$$

$$\int_0^\delta \frac{K_1(\rho) O(\tau) \rho d\rho}{(\rho^2 + \tau^2)^{5/2}} \leq \varepsilon(\delta) \int_0^\delta \frac{O(\tau) \rho^3 d\rho}{(\rho^2 + \tau^2)^{5/2}} \leq O(\varepsilon(\delta)) \int_0^\infty \frac{t^3 dt}{(1 + t^2)^{5/2}} = O(\varepsilon(\delta)), \quad (\rho = \tau t),$$

$$\begin{aligned} \iint_{U(Q_0)} \frac{|\kappa_1(Q)| O(\rho^2) \rho d\rho d\varphi}{(\rho^2 + \tau^2)^{3/2}} &\leq O(1) \int_0^\delta \frac{|k_1(\rho)|}{\rho} d\rho \\ &= O(1) \left[\left(\frac{K_1(\rho)}{\rho} \right)_0^\delta + \int_0^\delta \frac{K_1(\rho)}{\rho^2} d\rho \right] = O(\delta \varepsilon(\delta)). \end{aligned}$$

Hence $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = 0$, so that from (2), (3),

$$u_i(Q_0) - 4\pi\kappa(Q_0) = \iint_S (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q.$$

Since $\iint_S \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q = 2\pi$, we have

$$u_i(Q_0) = 2\pi \kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q.$$

Similarly we can prove the second part of the theorem.

Theorem 4. Let $\kappa(Q) \subset L$ on S and

$$u(P) = \iint_S \kappa(Q) \frac{1}{r} d\omega_Q, \quad r = r_{PQ}.$$

Then for almost all $Q_0 \in S$, when P tends to Q_0 on n_{Q_0} ,

$$\frac{\partial u(P)}{\partial \nu} \rightarrow -2\pi \kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q, \quad r_0 = r,$$

where $\frac{\partial u(P)}{\partial \nu}$ is the derivative of $u(P)$ in the direction of n_{Q_0} and (r_0, n_{Q_0}) is the angle subtained by two vectors $\vec{Q_0Q}$ and n_{Q_0} .

Proof. Let $Q_0 = (0, 0, 0)$ satisfy the condition of the proof of Theorem 3 and $P = (0, 0, \tau)$ ($\tau > 0$) be a point on n_{Q_0} .

First we consider a special case:

$$v(P) = \iint_S \frac{1}{r} d\omega_Q \quad (1)$$

and shall prove that, when $P \rightarrow Q_0$ on n_{Q_0} ,

$$\frac{\partial v(P)}{\partial \nu} \rightarrow -2\pi + \iint_S \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q. \quad (2)$$

Since $\frac{\partial}{\partial \nu} \left(\frac{1}{r} \right) = \frac{\cos(r, n_{Q_0})}{r^2}$ ($r = r_{PQ}$), we have

$$\frac{\partial v(P)}{\partial \nu} = \iint_{U(Q_0)} \frac{\partial}{\partial \nu} \left(\frac{1}{r} \right) d\omega_Q + \iint_{S-U(Q_0)} \frac{\cos(r, n_{Q_0})}{r^2} d\omega_Q = I + II,$$

where

$$\lim_{P \rightarrow Q_0} II = \iint_{S-U(Q_0)} \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q. \quad (4)$$

In I, let $Q = (x, y, z)$, $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, $z = O(\rho^2)$, then $\rho^2 = x^2 + y^2 \leq \delta^2$, $r = \sqrt{\rho^2 + (z - \tau)^2}$, so that

$$\begin{aligned} \frac{\partial}{\partial \nu} \left(\frac{1}{r} \right) &= \frac{\partial}{\partial \tau} \left(\frac{1}{r} \right) = \frac{-\tau + z}{(\rho^2 + (z - \tau)^2)^{3/2}} = \frac{-\tau + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}} (1 + O(\rho)) \\ &= \frac{-\tau + O(\tau\rho) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}} = \frac{-\tau + O(\rho^2 + \tau^2)}{(\rho^2 + \tau^2)^{3/2}} \end{aligned} \quad (5)$$

Since $d\omega_Q = (1 + O(\rho^2)) \rho d\rho d\varphi$,

$$\begin{aligned} I &= \iint_{U(Q_0)} \frac{-\tau + O(\rho^2 + \tau^2)}{(\rho^2 + \tau^2)^{3/2}} \rho d\rho d\varphi = \iint_{U(Q_0)} \frac{-\tau \rho d\rho d\varphi}{(\rho^2 + \tau^2)^{3/2}} + O(\delta) \\ &= -2\pi \int_0^\delta \frac{\tau \rho d\rho}{(\rho^2 + \tau^2)^{3/2}} + O(\delta) \rightarrow -2\pi \int_0^\infty \frac{t dt}{(1 + t^2)^{3/2}} + O(\delta) \\ &= -2\pi + O(\delta), \quad (\tau \rightarrow 0), \quad (\rho = \tau t). \end{aligned}$$

Hence $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = -2\pi$, so that from (3), (4),

$$\frac{\partial v(P)}{\partial \nu} \rightarrow -2\pi + \iint_S \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q, \text{ as } P \rightarrow Q_0. \quad (2)$$

From

$$u(P) - \kappa(Q_0) v(P) = \iint_S (\kappa(Q) - \kappa(Q_0)) \frac{1}{r} d\omega_Q,$$

we have

$$\begin{aligned} \frac{\partial u(P)}{\partial \nu} - \kappa(Q_0) \frac{\partial v(P)}{\partial \nu} &= \iint_{\bar{U}(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\partial}{\partial \nu} \left(\frac{1}{r} \right) d\omega_Q + \iint_{S - \bar{U}(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r, n_{Q_0})}{r^2} d\omega_Q \\ &= I + II, \end{aligned} \quad (6)$$

where

$$\lim_{P \rightarrow Q_0} II = \iint_{S - \bar{U}(Q_0)} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q. \quad (7)$$

In I, we put $\kappa_1(Q) = \kappa(Q) - \kappa(Q_0)$, then since by (5)

$$\frac{\partial}{\partial \nu} \left(\frac{1}{r} \right) = \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}},$$

we have

$$|I| \leq \iint_{\bar{U}(Q_0)} |\kappa_1(Q)| \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}} \rho d\rho d\tau.$$

Hence as the proof of Theorem 3, we have $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = 0$, so that from (6), (7),

$$\frac{\partial u(P)}{\partial \nu} - \kappa(Q_0) \frac{\partial v(P)}{\partial \nu} \rightarrow \iint_S (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q.$$

Hence by (2)

$$\frac{\partial u(P)}{\partial \nu} \rightarrow -2\pi \kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q, \text{ as } P \rightarrow Q_0.$$

3. Theorems on the normal derivatives of potentials.

We shall use the following theorems in the sequel.

Theorem 5. *Let D be a bounded domain on a plane, whose boundary Γ consists of a finite number of analytic Jordan curves. Let $\kappa(Q)$ be a bounded measurable function on Γ and*

$$u(P) = \int_{\Gamma} \kappa(Q) \frac{\cos(r, n_Q)}{r} ds_Q, \quad r = r_{QP}.$$

Let $Q_0 \in \Gamma$. We take two points P_i, P_e on the both sides of Q_0 on n_{Q_0} , such that $Q_0 \bar{P}_i = Q_0 \bar{P}_e = \tau > 0$, where P_i lies in D and P_e lies outside of D . Then

$$(i) \quad \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu}$$

is uniformly bounded in a neighbourhood of Γ , such that if $\tau \leq \tau_0$, then for all Q_0 on Γ

$$\left| \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right| \leq M \quad (= \text{const.}).$$

(ii) For almost all $Q_0 \in \Gamma$,

$$\lim_{\tau \rightarrow 0} \left(\frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right) = 0.$$

Proof. We take Q_0 as the origin and n_{Q_0} as the positive y -axis, so that a neighbourhood $U(Q_0)$ of Q_0 on Γ is of the form:

$$y = ax^2 + \dots, \quad |x| \leq \delta.$$

Let $P_i = (0, \tau)$, $P_e = (0, -\tau)$ ($\tau > 0$). For $Q \in \Gamma$, we put $r_i = \overline{QP_i}$, $r_e = \overline{QP_e}$. Since

$$\begin{aligned} \int_{\Gamma} \frac{\cos(r_i, n_Q)}{r_i} ds_Q &= 2\pi, \quad \int_{\Gamma} \frac{\cos(r_e, n_Q)}{r_e} ds_Q = 0, \\ u(P_i) - 2\pi\kappa(Q_0) &= \int_{\Gamma} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_i, n_Q)}{r_i} ds_Q, \\ u(P_e) &= \int_{\Gamma} (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_e, n_Q)}{r_e} ds_Q. \end{aligned}$$

Hence if we put $\kappa_1(Q) = \kappa(Q) - \kappa(Q_0)$, then

$$\begin{aligned} \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} &= \int_{U(Q_0)} \kappa_1(Q) \frac{\partial}{\partial \tau} \left(\frac{\cos(r_i, n_Q)}{r_i} - \frac{\cos(r_e, n_Q)}{r_e} \right) ds_Q \\ &+ \int_{\Gamma - U(Q_0)} \kappa_1(Q) \frac{\partial}{\partial \tau} \left(\frac{\cos(r_i, n_Q)}{r_i} - \frac{\cos(r_e, n_Q)}{r_e} \right) d\sigma_Q = \text{I} + \text{II}. \end{aligned} \quad (1)$$

If δ is fixed, then

$$\lim_{\tau \rightarrow 0} \text{II} = 0 \quad \text{uniformly for all } Q_0 \text{ on } \Gamma. \quad (2)$$

Since $\frac{\cos(r, n_Q)}{r} = \frac{\partial}{\partial n_Q} \log \frac{1}{r}$, $r^2 = x^2 + (y - \tau)^2$, $Q = (x, y)$, we have

$$\begin{aligned} \text{I} &= \int_{U(Q_0)} \kappa_1(Q) \frac{\partial}{\partial n_Q} \frac{\partial}{\partial \tau} \left(\log \frac{1}{r_i} - \log \frac{1}{r_e} \right) ds_Q \\ &= \int_{U(Q_0)} \kappa_1(Q) \frac{\partial}{\partial n_Q} \left(\frac{y - \tau}{x^2 + (y - \tau)^2} - \frac{y + \tau}{x^2 + (y + \tau)^2} \right) ds_Q. \end{aligned} \quad (3)$$

Now

$$\begin{aligned} \frac{\partial}{\partial n_Q} \left(\frac{y - \tau}{x^2 + (y - \tau)^2} - \frac{y + \tau}{x^2 + (y + \tau)^2} \right) &= \left(\frac{1}{x^2 + (y - \tau)^2} - \frac{1}{x^2 + (y + \tau)^2} \right) \frac{\partial y}{\partial n_Q} \\ &- 2 \left(\frac{y - \tau}{(x^2 + (y - \tau)^2)^2} - \frac{y + \tau}{(x^2 + (y + \tau)^2)^2} \right) x \frac{\partial x}{\partial n_Q} \\ &- 2 \left(\frac{(y - \tau)^2}{(x^2 + (y - \tau)^2)^2} - \frac{(y + \tau)^2}{(x^2 + (y + \tau)^2)^2} \right) \frac{\partial y}{\partial n_Q}. \end{aligned} \quad (4)$$

$$\frac{1}{x^2 + (y - \tau)^2} = \frac{1}{(x^2 + y^2 + \tau^2) \left(1 - \frac{2\tau y}{x^2 + y^2 + \tau^2} \right)}.$$

Since $2\tau|x| \leq x^2 + \tau^2$,

$$\frac{\tau y}{x^2 + y^2 + \tau^2} = \frac{O(\tau x^2)}{x^2 + y^2 + \tau^2} = \frac{O(x(x^2 + \tau^2))}{x^2 + y^2 + \tau^2} = O(x),$$

hence

$$\begin{aligned} \frac{1}{x^2 + (y - \tau)^2} &= \frac{1}{x^2 + y^2 + \tau^2} \left(1 + \frac{2\tau y}{x^2 + y^2 + \tau^2} + O(x^2) \right), \\ \frac{1}{(x^2 + (y - \tau)^2)^2} &= \frac{1}{(x^2 + y^2 + \tau^2)^2} \left(1 + \frac{4\tau y}{x^2 + y^2 + \tau^2} + O(x^2) \right). \end{aligned}$$

Since $\left| \frac{\partial y}{\partial n_Q} \right| \leq 1$, $\frac{\partial x}{\partial n_Q} = O(x)$, $y = O(x^2)$, we have easily from (4)

$$\frac{\partial}{\partial n_Q} \left(\frac{y - \tau}{x^2 + (y - \tau)^2} - \frac{y + \tau}{x^2 + (y + \tau)^2} \right) = \frac{O(\tau) + O(x^2)}{x^2 + \tau^2} = \frac{O(\tau)}{x^2 + \tau^2} + O(1). \quad (5)$$

Let $|\kappa(Q)| \leq K$ on Γ , then $|\kappa_1(Q)| \leq 2K$, so that from (3)

$$|I| \leq \int_{-\delta}^{\delta} \left(\frac{O(\tau)}{x^2 + \tau^2} + O(1) \right) dx = O(1) \int_0^{\infty} \frac{dt}{1 + t^2} + O(\delta) = O(1) \quad (x = \tau t). \quad (6)$$

Hence from (1), (2), (6),

$$\left| \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right| \text{ is uniformly bounded in a neighbourhood of } \Gamma.$$

Since

$$|I| \leq \int_{-\delta}^{\delta} |\kappa_1(Q)| \left(\frac{O(\tau)}{x^2 + \tau^2} + O(1) \right) dx,$$

if Q_0 satisfies the condition of the proof of Theorem 1, then $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = 0$, so that for almost all $Q_0 \in \Gamma$,

$$\lim_{\tau \rightarrow 0} \left(\frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right) = 0.$$

Theorem 6. Let D be a bounded domain in space, whose boundary S consists of a finite number of analytic closed surfaces. Let $\kappa(Q)$ be a bounded measurable function on S and

$$u(P) = \iint_S \kappa(Q) \frac{\cos(r, n_Q)}{r^2} d\omega_Q, \quad r = r_{QP}.$$

Let $Q_0 \in S$. We take two points P_i, P_e on the both sides of Q_0 on n_{Q_0} , such that $\overline{Q_0 P_i} = \overline{Q_0 P_e} = \tau > 0$, where P_i lies in D and P_e lies outside of D . Then

(i) $\frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu}$ is uniformly bounded in a neighbourhood of S , such that if $\tau \leq \tau_0$, then for all Q_0 on S ,

$$\left| \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right| \leq M \quad (= \text{const.}).$$

(ii) For almost all $Q_0 \in S$,

$$\lim_{\tau \rightarrow 0} \left(\frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right) = 0.$$

Proof. We take Q_0 as the origin and n_{Q_0} as the positive z -axis, so that a neighbourhood $U(Q_0)$ of Q_0 on S is of the form:

$$z = ax^2 + by^2 + \dots, \quad x^2 + y^2 = \rho^2 \leq \delta^2.$$

For $Q \in S$, we put $r_i = \overline{QP_i}$, $r_e = \overline{QP_e}$.

$$\text{Since} \quad \iint_S \frac{\cos(r_i, n_q)}{r_i^2} d\omega_q = 4\pi, \quad \iint_S \frac{\cos(r_e, n_q)}{r_e^2} d\omega_q = 0,$$

we have

$$\begin{aligned} u(P_i) - 4\pi\kappa(Q_0) &= \iint_S (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_i, n_q)}{r_i^2} d\omega_q, \\ u(P_e) &= \iint_S (\kappa(Q) - \kappa(Q_0)) \frac{\cos(r_e, n_q)}{r_e^2} d\omega_q. \end{aligned}$$

Hence if we put $\kappa_1(Q) = \kappa(Q) - \kappa(Q_0)$,

$$\begin{aligned} \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} &= \iint_{S \setminus \overline{U(Q_0)}} \kappa_1(Q) \frac{\partial}{\partial \tau} \left(\frac{\cos(r_i, n_q)}{r_i^2} - \frac{\cos(r_e, n_q)}{r_e^2} \right) d\omega_q \\ &+ \iint_{S \setminus \overline{U(Q_0)}} \kappa_1(Q) \frac{\partial}{\partial \tau} \left(\frac{\cos(r_i, n_q)}{r_i^2} - \frac{\cos(r_e, n_q)}{r_e^2} \right) d\omega_q = \text{I} + \text{II}. \quad (1) \end{aligned}$$

If δ is fixed, then

$$\lim_{\tau \rightarrow 0} \text{II} = 0 \quad \text{uniformly for all } Q_0 \text{ on } S. \quad (2)$$

Since $\frac{\cos(r, n_q)}{r^2} = \frac{\partial}{\partial n_q} \left(\frac{1}{r} \right)$, $r = \sqrt{\rho^2 + (z - \tau)^2}$, $\rho^2 = x^2 + y^2$, $Q = (x, y)$, we have

$$\begin{aligned} \text{I} &= \iint_{S \setminus \overline{U(Q_0)}} \kappa_1(Q) \frac{\partial}{\partial n_q} \frac{\partial}{\partial \tau} \left(\frac{1}{r_i} - \frac{1}{r_e} \right) d\omega_q \\ &= \iint_{S \setminus \overline{U(Q_0)}} \kappa_1(Q) \frac{\partial}{\partial n_q} \left(\frac{z - \tau}{(\rho^2 + (z - \tau)^2)^{3/2}} - \frac{z + \tau}{(\rho^2 + (z + \tau)^2)^{3/2}} \right) d\omega_q. \quad (3) \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial}{\partial n_q} \left(\frac{z - \tau}{(\rho^2 + (z - \tau)^2)^{3/2}} - \frac{z + \tau}{(\rho^2 + (z + \tau)^2)^{3/2}} \right) &= \left(\frac{1}{(\rho^2 + (z - \tau)^2)^{3/2}} - \frac{1}{(\rho^2 + (z + \tau)^2)^{3/2}} \right) \frac{\partial z}{\partial n_q} \\ &- \frac{3}{2} \left(\frac{z - \tau}{(\rho^2 + (z - \tau)^2)^{5/2}} - \frac{z + \tau}{(\rho^2 + (z + \tau)^2)^{5/2}} \right) \frac{\partial \rho^2}{\partial n_q} \\ &- 3 \left(\frac{(z - \tau)^2}{(\rho^2 + (z - \tau)^2)^{5/2}} - \frac{(z + \tau)^2}{(\rho^2 + (z + \tau)^2)^{5/2}} \right) \frac{\partial z}{\partial n_q}. \quad (4) \end{aligned}$$

As before

$$\frac{1}{\rho^2 + (z - \tau)^2} = \frac{1}{\rho^2 + z^2 + \tau^2} \left(1 + \frac{2\tau z}{\rho^2 + z^2 + \tau^2} + O(\rho^2) \right),$$

so that

$$\begin{aligned} \frac{1}{(\rho^2 + (z - \tau)^2)^{3/2}} &= \frac{1}{(\rho^2 + z^2 + \tau^2)^{3/2}} \left(1 + \frac{3\tau z}{\rho^2 + z^2 + \tau^2} + O(\rho^2) \right), \\ \frac{1}{(\rho^2 + (z - \tau)^2)^{5/2}} &= \frac{1}{(\rho^2 + z^2 + \tau^2)^{5/2}} \left(1 + \frac{5\tau z}{\rho^2 + z^2 + \tau^2} + O(\rho^2) \right). \end{aligned}$$

Since $\left| \frac{\partial z}{\partial n_q} \right| \leq 1$, $\frac{\partial \rho^2}{\partial n_q} = O(\rho^3)$, $z = O(\rho^3)$, if we put these values in (4), we have easily

$$\frac{\partial}{\partial n_Q} \left(\frac{z-\tau}{(\rho^2+(z-\tau)^2)^{3/2}} - \frac{z+\tau}{(\rho^2+(z+\tau)^2)^{3/2}} \right) = \frac{O(\tau)+O(\rho^2)}{(\rho^2+\tau^2)^{3/2}}. \quad (5)$$

Let $|\kappa(Q)| \leq K$ on S , then $|\kappa_1(Q)| \leq 2K$, so that

$$\begin{aligned} |I| &\leq \iint_{v(Q_0)} \frac{O(\tau)+O(\rho^2)}{(\rho^2+\tau^2)^{3/2}} \rho d\rho d\tau = \int_0^\delta \frac{O(\tau)+O(\rho^2)}{(\rho^2+\tau^2)^{3/2}} \rho d\rho \\ &= \int_0^\delta \frac{O(\tau) \rho d\rho}{(\rho^2+\tau^2)^{3/2}} + O(\delta) = O(1) \int_0^\infty \frac{t dt}{(1+t^2)^{3/2}} + O(\delta) = O(1), \quad (\rho=\tau t). \end{aligned}$$

Hence from (1), (2), we have

$$\left| \frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right| \text{ is uniformly bounded in a neighbourhood of } S.$$

From

$$|I| \leq \iint_{v(Q_0)} |\kappa_1(Q)| \frac{O(\tau)+O(\rho^2)}{(\rho^2+\tau^2)^{3/2}} \rho d\rho d\tau,$$

if Q_0 satisfies the condition of the proof of Theorem 3, then $\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = 0$, so that for almost all $Q_0 \in S$,

$$\lim_{\tau \rightarrow 0} \left(\frac{\partial u(P_i)}{\partial \nu} - \frac{\partial u(P_e)}{\partial \nu} \right) = 0.$$

4. Integral equations.

We shall use the following theorem¹⁾ in the sequel, where $\sim$ means = almost everywhere in $[0, 1]$.

Theorem 7. Let $K(x, y)$ be measurable and bounded in a square $\Delta: 0 \leq x \leq 1, 0 \leq y \leq 1$, or $K(x, y)$ is of the form:

$$K(x, y) = \frac{H(x, y)}{|x-y|^\alpha}, \quad 0 < \alpha < 1,$$

where $H(x, y)$ is continuous in Δ .

Consider integral equations:

$$\varphi(x) - \lambda \int_0^1 K(x, y) \varphi(y) dy \sim f(x), \quad (I)$$

$$\varphi(x) - \lambda \int_0^1 K(x, y) \varphi(y) dy \sim 0, \quad (II)$$

where $\varphi(x) \in L, f(x) \in L$ in $[0, 1]$.

Then we have the following Fredholm's alternatives:

(i). (II) has only a trivial solution $\varphi(x) \sim 0$. Then (I) has one and only one solution $\varphi(x) \in L$ for any $f(x) \in L$.

(ii). (II) has $r(1 \leq r < \infty)$ linearly independent solutions $\psi_i(x) (i=1, 2, \dots, r)$. Then the conjugate homogeneous equation:

1) M. Tsuji: On an integral equation with a kernel $K(x, y) \in L^p (1 < p \leq 2)$. Journ. Math. Soc. Japan. 5 (1954).

$$\varphi(x) - \lambda \int_0^1 K(y, x) \varphi(y) dy \sim 0 \quad (\text{II}')$$

has r linearly independent solutions $\chi_i(x)$ ($i=1, 2, \dots, r$) and (I) has a solution, when and only when

$$(f, \chi_i) = \int_0^1 f(x) \chi_i(x) dx = 0 \quad (i=1, 2, \dots, r).$$

The same result holds if we consider an integral equation in n variables:

$$\varphi(P) - \lambda \int_D K(P, Q) \varphi(Q) dv_Q \sim f(P),$$

where $P = (x_1, \dots, x_n)$, $Q = (\xi_1, \dots, \xi_n)$ are points of an n -dimensional finite D , $dv_Q = d\xi_1 \dots d\xi_n$, $\varphi(P) \in L$, $f(P) \in L$ in D and $K(P, Q)$ is measurable and bounded on $D \times D$ or $K(P, Q)$ is of the form

$$K(P, Q) = \frac{H(P, Q)}{PQ^\alpha}, \quad 0 < \alpha < n,$$

where $H(P, Q)$ is continuous on $D \times D$.

5. Dirichlet- and Neumann-problem on a plane.

1. Dirichlet problem.

Theorem 8. Let D be a bounded domain on a plane, whose boundary Γ consists of only one analytic Jordan curve. Let $f(Q)$ be a given integrable function on Γ . Then there exists a harmonic function $u(P)$ in D , such that $u_i(Q) = f(Q)$ for almost all $Q \in \Gamma$.

Proof. We assume that $u(P)$ can be expressed by

$$u(P) = \int_{\Gamma} \kappa(Q) \frac{\cos(r, n_Q)}{r} ds_Q, \quad r = r_{QP}, \quad (1)$$

where $\kappa(Q) \in L$ on Γ .

Then by Theorem 1, for almost all $Q_0 \in \Gamma$,

$$u_i(Q_0) = \pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} ds_Q, \quad r_0 = r_{QQ_0}. \quad (2)$$

Hence $\kappa(Q)$ satisfies an integral equation:

$$\kappa(P) - \lambda \int_{\Gamma} K(P, Q) \kappa(Q) ds_Q \sim f(P)/\pi, \quad (3)$$

where $\lambda = -1$ and

$$K(P, Q) = \frac{1}{\pi} \frac{\cos(r_{QP}, n_Q)}{r_{QP}}. \quad (4)$$

Since $\lim_{Q \rightarrow P} \frac{\cos(r_{QP}, n_Q)}{r_{QP}} = \frac{1}{2\rho(P)}$, where $\rho(P)$ is the radius of curvature of Γ at P , $K(P, Q)$ is bounded. Hence by Theorem 7, (3) has a solution $\kappa(Q) \in L$, if the homogeneous equation:

$$\kappa(P) - \lambda \int_{\Gamma} K(P, Q) \kappa(Q) ds_Q \sim 0 \quad (5)$$

has only a trivial solution $\kappa(Q) \sim 0$.

Suppose that (5) has a non-trivial solution $\kappa(Q) \subset L$, such that

$$\kappa(P) - \lambda \int_{\Gamma} K(P, Q) \kappa(Q) d\sigma_Q \sim 0, \quad \int_{\Gamma} |\kappa(Q)| d\sigma_Q = 1, \quad (6)$$

then

$$|\kappa(P)| \leq |\lambda| K \int_{\Gamma} |\kappa(Q)| d\sigma_Q = |\lambda| K,$$

where K is the bound of $|K(P, Q)|$. Hence $\kappa(Q)$ is bounded on Γ . We put

$$v(P) = \int_{\Gamma} \kappa(Q) \frac{\cos(r, n_Q)}{r} d\sigma_Q, \quad (7)$$

then $v(P)$ is a bounded harmonic function in D and by (2) $v_i(Q) = 0$ almost everywhere on Γ , so that $v(P) \equiv 0$ in D . Let D' be the complementary domain of D , which contains ∞ . Then $v(P)$ is a bounded harmonic function in D' . Now by Theorem 5, $\frac{\partial v(P_i)}{\partial \nu} - \frac{\partial v(P_e)}{\partial \nu}$ is bounded in a neighbourhood of Γ and $\lim_{\tau \rightarrow 0} \left(\frac{\partial v(P_i)}{\partial \nu} - \frac{\partial v(P_e)}{\partial \nu} \right) = 0$ almost everywhere on Γ . Since $v(P) \equiv 0$ in D , $\frac{\partial v(P_i)}{\partial \nu} = 0$, so that $\frac{\partial v(P_e)}{\partial \nu}$ is bounded in a neighbourhood of Γ and $\lim_{\tau \rightarrow 0} \frac{\partial v(P_e)}{\partial \nu} = 0$ almost everywhere on Γ , so that $v(P) \equiv \text{const.}$ in D' . Since $v(P)$ vanishes at ∞ , $v(P) \equiv 0$ in D' , so that $v_e(Q_0) = 0$ on Γ .

Since by Theorem 1, for almost all $Q_0 \in \Gamma$,

$$0 = v_i(Q_0) = \pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} d\sigma_Q,$$

$$0 = v_e(Q_0) = -\pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} d\sigma_Q,$$

we have $\kappa(Q) = 0$ almost everywhere on Γ , which contradicts (6). Hence (5) has only a trivial solution $\kappa(Q) \sim 0$, so that (3) is solvable for any $f(Q) \subset L$.

Remark. If D is bounded by more than one curve: $C_0, C_1, \dots, C_n$, where C_0 lies outermost and $C_i (1 \leq i \leq n)$ lies inside of C_0 , then if we put $\kappa(Q) = 0$ on $C_0, C_1, \dots, C_{n-1}$, $\kappa(Q) = 1$ on C_n , then $u(P) \equiv 0$, so that (5) has a non-trivial solution, hence (3) is not solvable in general.

2. Neumann problem.

Theorem 9. Let D be a bounded domain on a plane, whose boundary Γ consists of a finite number of analytic Jordan curves. Let $f(Q)$ be a given integrable function on Γ , such that

$$\int_{\Gamma} f(Q) d\sigma_Q = 0.$$

Then there exists a harmonic function $u(P)$ in D , such that for almost all $Q \in \Gamma$, when P tends to Q on n_Q ,

$$\frac{\partial u(P)}{\partial \nu} \rightarrow f(Q).$$

Proof. We assume that

$$u(P) = \int_{\Gamma} \kappa(Q) \log \frac{1}{r} d\sigma_Q, \quad r = r_{PQ}. \quad (1)$$

where $\kappa(Q) \in L$ on Γ . Then by Theorem 2, for almost all $Q_0 \in \Gamma$

$$\frac{\partial u(P)}{\partial \nu} \rightarrow -\pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_{Q_0})}{r_0} ds_Q, \quad r_0 = r_{Q_0 Q}.$$

Hence $\kappa(Q)$ satisfies an integral equation:

$$\kappa(P) - \lambda \int_{\Gamma} K(P, Q) \kappa(Q) d\sigma_Q \sim -f(P)/\pi, \quad (2)$$

where $\lambda=1$ and

$$K(P, Q) = \frac{1}{\pi} \frac{\cos(r_{PQ}, n_P)}{r_{PQ}}. \quad (3)$$

As before $K(P, Q)$ is bounded for any P, Q on Γ .

Consider the conjugate homogeneous equation:

$$\kappa(P) - \lambda \int_{\Gamma} K(Q, P) \kappa(Q) d\sigma_Q \sim 0, \quad (4)$$

or

$$\kappa(P) - \frac{1}{\pi} \int_{\Gamma} \frac{\cos(r_{QP}, n_Q)}{r_{QP}} \kappa(Q) d\sigma_Q \sim 0. \quad (5)$$

(5) has a solution $\kappa(Q)=1$. We shall prove that if $\kappa(Q)$ is a solution of (5), then $\kappa(Q) \sim \text{const.}$

Let $\kappa(Q) \in L$ be a solution of (5), such that $\int_{\Gamma} |\kappa(Q)| d\sigma_Q = 1$. Then as before, $\kappa(Q)$ is bounded on Γ .

We put

$$v(P) = \int_{\Gamma} \kappa(Q) \frac{\cos(r, n_Q)}{r} d\sigma_Q, \quad r = r_{QP}, \quad (6)$$

then by Theorem 1, for almost all $Q_0 \in \Gamma$,

$$\begin{aligned} v_i(Q_0) &= \pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} d\sigma_Q, \quad r_0 = r_{Q_0 Q}, \\ v_e(Q_0) &= -\pi \kappa(Q_0) + \int_{\Gamma} \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0} d\sigma_Q = 0, \end{aligned}$$

so that

$$v_i(Q_0) = 2\pi \kappa(Q_0) \quad \text{almost everywhere on } \Gamma. \quad (7)$$

Since $v_e(Q_0)=0$ almost everywhere on Γ and $v(P)$ is bounded in the complementary domain D' of D , $v(P) \equiv 0$ in D' , so that $\frac{\partial v(P_e)}{\partial \nu} = 0$. Hence by Theorem 5, $\frac{\partial v(P_i)}{\partial \nu}$ is bounded in a neighbourhood of Γ and $\lim_{\tau \rightarrow 0} \frac{\partial v(P_i)}{\partial \nu} = 0$ almost everywhere on Γ , so that $v(P) \equiv \text{const.}$ in D . Hence by (7), $\kappa(Q) \sim \text{const.}$ Hence by Theorem 7, (2) is solvable, if $\int_{\Gamma} f(Q) d\sigma_Q = 0$.

3. Neumann problem for a Riemannian domain.

Let D be a domain on a closed Riemann surface F , spread over the z -sphere. The boundary Γ of D consists of a finite number of analytic Jordan curves. Since Dirichlet problem for such a domain D can be solved by mapping the

universal covering surface of D on $|\zeta| < 1$ and by means of Piosson integral on $|\zeta| = 1$, we consider only Neumann problem for D .

Theorem 10. *Let $f(\zeta)$ be a given integrable function on Γ , such that*

$$\int_{\Gamma} f(\zeta) ds_{\zeta} = 0.$$

Then there exists a harmonic function $u(z)$ in D , such that for almost all ζ on Γ , when z tends to ζ on n_{ζ} ,

$$\frac{\partial u(z)}{\partial \nu} \rightarrow f(\zeta).$$

Proof. The complement of D with respect to F consists of a finite number of connected domains D_i ($i=1, 2, \dots, N$). We take a fixed point ζ_0 in D_N and ζ ($\neq \zeta_0$) be a point of F . Then there exists a potential function $v(z; \zeta, \zeta_0)$ on F , which satisfies the following condition²⁾:

$v(z; \zeta, \zeta_0)$ is harmonic on F , except at ζ, ζ_0 , where

$$v(z; \zeta, \zeta_0) - \log \frac{1}{|z - \zeta|} \text{ is harmonic at } z = \zeta,$$

$$v(z; \zeta, \zeta_0) + \log \frac{1}{|z - \zeta_0|} \text{ is harmonic at } z = \zeta_0.$$

If we take $v(z; \zeta, \zeta_0)$ instead of $\log \frac{1}{r}$ in the proof of Theorem 9 and put

$$u(z) = \int_{\Gamma} \kappa(\zeta) v(z; \zeta, \zeta_0) ds_{\zeta}, \quad \kappa(\zeta) \in L \text{ on } \Gamma,$$

then we can prove our theorem similarly as Theorem 9.

6. Dirichlet- and Neumann-problem in space.

1. Dirichlet problem.

Theorem 11. *Let D be a bounded domain in space, whose boundary S consists of only one analytic closed surface.*

Let $f(Q)$ be a given integrable function on S . Then there exists a harmonic function $u(P)$ in D , such that for almost all $Q \in S$,

$$u_i(Q) = f(Q).$$

Proof. We assume that

$$u(P) = \iint_S \kappa(Q) \frac{\cos(r, n_Q)}{r^2} d\omega_Q, \quad r = r_{QP}, \quad (1)$$

2) M. Tsuji: Existence of a potential function with a prescribed singularity on any Riemann surface. Tôhoku Math. Journ. 4 (1951).

where $\kappa(Q) \in L$ on S . Then by Theorem 3, for almost all $Q_0 \in S$,

$$u_i(Q_0) = 2\pi\kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_Q)}{r_0^2} d\omega_Q, \quad r_0 = r_{Q_0},$$

hence $\kappa(Q)$ satisfies an integral equation:

$$\kappa(P) - \lambda \iint_S K(P, Q) \kappa(Q) d\omega_Q \sim f(P)/2\pi, \quad (2)$$

where $\lambda = -1$ and

$$K(P, Q) = \frac{1}{2\pi} \frac{\cos(r_{QP}, n_Q)}{r_{PQ}^2}. \quad (3)$$

$K(P, Q)$ is of the form $\kappa(P, Q) = \frac{H(P, Q)}{PQ}$, where $H(P, Q)$ is continuous on $S \times S$. Since for such a kernel, Theorem 7 holds, (2) has a solution, if

$$\kappa(P) - \lambda \iint_S K(P, Q) \kappa(Q) d\omega_Q \sim 0 \quad (4)$$

has only a trivial solution $\kappa(Q) \sim 0$. By means of Theorem 6, we can prove as Theorem 8, that (4) has only a trivial solution $\kappa(Q) \sim 0$, hence (2) is solvable for any $f(Q) \in L$.

Remark. As remarked in Theorem 8, if D is bounded by more than one closed surface, than (2) is not solvable in general.

2. Neumann problem.

Theorem 12. Let D be a bounded domain in space, whose boundary S consist of a finite number of analytic closed surfaces. Let $f(Q)$ be a given integrable function on S , such that

$$\iint_S f(Q) d\omega_Q = 0.$$

Then there exists a harmonic function $u(P)$ in D , such that for almost all $Q \in S$, when P tends to Q on n_Q ,

$$\frac{\partial u(P)}{\partial \nu} \rightarrow f(Q).$$

Proof. We assume that

$$u(P) = \iint_S \kappa(Q) \frac{1}{r} d\omega_Q, \quad r = r_{PQ}, \quad (1)$$

where $\kappa(Q) \in L$ on S . Then by Theorem 4, for almost all $Q_0 \in S$, when P tends to Q_0 on n_{Q_0} ,

$$\frac{\partial u(P)}{\partial \nu} \rightarrow -2\pi\kappa(Q_0) + \iint_S \kappa(Q) \frac{\cos(r_0, n_{Q_0})}{r_0^2} d\omega_Q, \quad r_0 = r_{Q_0},$$

hence $\kappa(Q)$ satisfies an integral equation:

$$\kappa(P) - \lambda \iint_S K(P, Q) \kappa(Q) d\omega_Q \sim -f(P)/2\pi, \quad (2)$$

where $\lambda = 1$ and

$$K(P, Q) = \frac{1}{2\pi} \frac{\cos(r_{PQ}, n_P)}{r_{PQ}^2}.$$

Since $K(P, Q)$ is of the form $K(P, Q) = \frac{H(P, Q)}{PQ}$, where $H(P, Q)$ is continuous on $S \times S$, Theorem 7 holds for such a kernel. Now we consider the conjugate homogeneous equation:

$$\kappa(P) - \lambda \iint_S K(Q, P) \kappa(Q) d\omega_Q \sim 0, \quad (3)$$

or

$$\kappa(P) - \frac{1}{2\pi} \iint_S \frac{\cos(r_{QP}, n_Q)}{r_{QP}^2} \kappa(Q) d\omega_Q \sim 0. \quad (4)$$

Then similarly as Theorem 9, we can prove that (4) has only solutions of the form $\kappa(Q) \sim \text{const.}$. Hence (2) is solvable, if $\iint_S f(Q) d\omega_Q = 0$.

7. Theorems on the normal derivatives of Green's functions.

In Theorem 11, we had to assume that D is bounded by only one closed surface. To prove that Dirichlet problem is solvable by means of Green's function, when D is bounded by more than one closed surface, we shall first prove theorems on the normal derivatives of Green's functions.

Lemma 1. *Let $u(P)$ be harmonic in a sphere S and continuous on S . Let O be the center and R be the radius of S . We take O as the origin and (r, θ, φ) be the polar coordinates of a point (x, y, z) , such that*

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta \quad (0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi).$$

Let $u(r, \theta, \varphi)$ be the value of $u(P)$ at (r, θ, φ) . We put

$$F(\theta) = \frac{1}{2R} \int_0^{2\pi} u(R, \theta, \varphi) d\varphi$$

and suppose that

$$F(\theta) = O(\theta^2).$$

Let Q be the point of intersection of S with the positive z -axis. Then $\lim_{P \rightarrow Q} \frac{\partial u(P)}{\partial \nu}$ exists, when P tends to Q on the positive z -axis and

$$\lim_{P \rightarrow Q} \frac{\partial u(P)}{\partial \nu} = \frac{\partial u(Q)}{\partial \nu} = \frac{1}{4R} \int_0^\pi F(\theta) \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\theta,$$

where ν is the inner normal of S at Q .

Proof. For $0 < r < R$, we have by Poisson integral

$$\begin{aligned} u(r, 0, 0) &= \frac{R}{4\pi} \int_0^\pi \int_0^{2\pi} u(R, \theta, \varphi) \frac{(R^2 - r^2) \sin \theta}{(R^2 - 2Rr \cos \theta + r^2)^{3/2}} d\theta d\varphi \\ &= \frac{R}{2} \int_0^\pi F(\theta) \frac{(R^2 - r^2) \sin \theta}{(R^2 - 2Rr \cos \theta + r^2)^{3/2}} d\theta. \end{aligned}$$

Since $F(\theta) = O(\theta^2)$, we can prove easily³⁾

$$\lim_{r \rightarrow R} \frac{\partial u}{\partial r} = -\frac{1}{4R} \int_0^\pi F(\theta) \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\theta,$$

or

$$\frac{\partial u(Q)}{\partial \nu} = -\frac{1}{4R} \int_0^\pi F(\theta) \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\theta.$$

Lemma 2. Let D be a bounded domain in space, whose boundary S consists of a finite number of analytic closed surfaces. Let Q_0 be a point of S and σ_i be a sphere, which touches S at Q_0 internally and is contained in D and O be its center. Let P be a point of D and $G(P, M)$ be the Green's function of D , with P as its pole. Let M be a point on σ_i and θ be the angle subtained by two vectors $\overrightarrow{OQ_0}$ and $\overrightarrow{OM}$. Then

$$G(P, M) = O(\theta^2), \quad M \in \sigma_i.$$

Proof. We take Q_0 as the origin and n_{Q_0} as the positive z -axis. Let σ_e be a sphere, which touches S at Q_0 externally and lies outside of D and R_1 be its radius and O_1 be its center, so that $O_1 = (0, 0, -R_1)$.

Let P_1 be the inverse point of P with respect to σ_e , such that

$$\overline{O_1P} \cdot \overline{O_1P_1} = R_1^2.$$

Let $a = \overline{O_1P}$ and (l, m, n) be the direction cosines of $\overrightarrow{O_1P}$, then

$$\begin{aligned} P = (\xi, \eta, \zeta): \quad & \xi = al, \quad \eta = am, \quad \zeta = -R_1 + an, \\ P_1 = (\xi_1, \eta_1, \zeta_1): \quad & \xi_1 = \frac{R_1^2}{a}l, \quad \eta_1 = \frac{R_1^2}{a}m, \quad \zeta_1 = -R_1 + \frac{R_1^2}{a}n. \end{aligned} \quad (1)$$

Now

$$G_0(P, M) = \frac{1}{r} - \frac{R_1}{ar_1}, \quad r = \overline{MP}, \quad r_1 = \overline{MP_1} \quad (2)$$

is the Green's function of the outside of σ_e , with P as its pole. Let R be the radius of σ_i and $M = (x, y, z)$ be a point of σ_i , then

$$x = R \sin \theta \cos \varphi, \quad y = R \sin \theta \sin \varphi, \quad z = R - R \cos \theta = O(\theta^2),$$

so that since $x^2 + y^2 + z^2 = O(\theta^2)$, $z = O(\theta^2)$,

$$\begin{aligned} r^2 &= (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = \xi^2 + \eta^2 + \zeta^2 - 2(x\xi + y\eta) + O(\theta^2) \\ &= a^2 + R_1^2 - 2aR_1n - 2aR \sin \theta (l \cos \varphi + m \sin \varphi) + O(\theta^2). \end{aligned}$$

Similarly

$$r_1^2 = \frac{R_1^4}{a^2} + R_1^2 - \frac{2R_1^3n}{a} - \frac{2R_1^2R}{a} \sin \theta (l \cos \varphi + m \sin \varphi) + O(\theta^2).$$

3) O. D. Kellogg: On the derivatives of harmonic functions on the boundary. Trans. Amer. Math. Soc. **33** (1931).

Hence $a^2r_1^2 - R_1^2r^2 = O(\theta^2)$, $ar_1 - R_1r = O(\theta^2)$, so that $G_0(P, M) = O(\theta^2)$, hence by the maximum principle,

$$G(P, M) \leq G_0(P, M) = O(\theta^2).$$

Theorem 13. *Let D satisfy the same condition as Lemma 2 and $G(P, Q)$ be the Green's function of D , with P as its pole. Then $\frac{\partial G(P, Q)}{\partial \nu} > 0$ exists at every point Q of S and is continuous on S , where ν is the inner normal of S at Q .*

Let $Q_0 \in S$. We take Q_0 as the origin and n_{Q_0} as the positive z -axis, so that a neighbourhood $U(Q_0)$ of Q_0 on S is of the form: $z = ax^2 + by^2 + \dots$, $x^2 + y^2 = \rho^2 \leq \delta^2$.

(i) If P tends to Q_0 , then

$$\lim_{P \rightarrow Q_0} \frac{\partial G(P, Q)}{\partial \nu} = 0 \quad \text{uniformly for } Q \in S - U(Q_0).$$

(ii) For $Q = (x, y, z) \in U(Q_0)$ and P in a Stolz domain $\Delta(Q_0)$,

$$\frac{\partial G(P, Q)}{\partial \nu} = \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}}, \quad \tau = \overline{PQ_0}, \quad \rho^2 = x^2 + y^2.$$

Proof. By Lemma 1 and 2, $\frac{\partial G(P, Q)}{\partial \nu}$ exists at every point Q of S and can be proved easily that it is continuous on S .

Let $Q \in S - U(Q_0)$ and $\sigma_i(Q)$ be a sphere, which touches S at Q internally and is contained in D . Let R be its radius and O be its center.

We define θ for $\sigma_i(Q)$ as Lemma 2, then $G(P, M) = O(\theta^2)$ for $M \in \sigma_i(Q)$, so that if we put

$$F(\theta) = \frac{1}{2\pi} \int_0^{2\pi} G(P, M) d\varphi_M, \quad (1)$$

then $F(\theta) = O(\theta^2)$, where φ_M is the angle between a fixed plane through n_Q and a plane through n_Q and M .

Hence by Lemma 1,

$$\frac{\partial G(P, Q)}{\partial \nu} = \frac{1}{4R} \int_0^\eta F(\theta) \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\theta + \frac{1}{4R} \int_\eta^\pi F(\theta) \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\theta = I + II \quad (\eta > 0). \quad (2)$$

Since $\lim_{P \rightarrow Q_0} G(P, M) = 0$ uniformly for $M \in \sigma_i(Q)$, such that $\eta \leq \theta \leq \pi$, we have

$$\lim_{P \rightarrow Q_0} II = 0. \quad (3)$$

Since $F(\theta) = O(\theta^2)$,

$$I = O(\eta) \rightarrow 0 \quad (\eta \rightarrow 0), \quad (4)$$

so that

$$\lim_{P \rightarrow Q_0} \frac{\partial G(P, Q)}{\partial \nu} = 0 \quad \text{uniformly for } Q \in S - U(Q_0), \quad (5)$$

Hence (i) is proved. Next we shall prove (ii).

Let $Q = (x, y, z) \in U(Q_0)$, $P \in \Delta(Q_0)$ and $\sigma_e(Q)$ be a sphere, which touches S at Q externally and lies outside of D , R_1 be its radius and O_1 be its center. Then

$$r^2 = \overline{O_1 P^2} = R_1^2 + a^2 + 2R_1 a \cos \psi, \quad a = \overline{Q P}, \quad \psi = (n_Q, \overrightarrow{Q P}). \quad (6)$$

Let (l, m, n) be the direction cosines of n_Q and (λ, μ, ν) be those of $\overrightarrow{Q P}$, then as the proof of Theorem 3,

$$\begin{aligned} l &= O(\rho), \quad m = O(\rho), \quad n = 1 + O(\rho^2), \quad \rho^2 = x^2 + y^2, \\ \lambda &= \frac{O(\tau) + O(\rho)}{a}, \quad \mu = \frac{O(\tau) + O(\rho)}{a}, \quad \nu = \frac{O(\tau) + O(\rho^2)}{a}, \quad \tau = \overline{Q_0 P}, \end{aligned}$$

so that

$$\begin{aligned} a \cos \psi &= a(l\lambda + n\mu + n\nu) = l \cdot a\lambda + m \cdot a\mu + n \cdot a\nu \\ &= O(\rho)(O(\tau) + O(\rho)) + (1 + O(\rho^2))(O(\tau) + O(\rho^2)) = O(\tau) + O(\rho^2). \end{aligned} \quad (7)$$

Since $a^2 = O(\rho^2 + \tau^2)$, we have from (6), (7),

$$r^2 - R_1^2 = O(\tau) + O(\rho^2). \quad (8)$$

As the proof of Theorem 3, we have

$$\overline{P Q^2} \geq (1 - \sin \theta_0)(\rho^2 + \tau^2)(1 + O(\rho)). \quad (9)$$

Let $G_0(P, M)$ be the Green's function of the outside of $\sigma_e(Q)$, then by the maximum principle, $G(P, M) \leq G_0(P, M)$.

Since they vanish at $M = Q$, we have from (8), (9),

$$\frac{\partial G(P, Q)}{\partial \nu} \leq \frac{\partial G_0(P, Q)}{\partial \nu} = \frac{1}{R_1} \cdot \frac{r^2 - R_1^2}{P Q^3} = \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}}.$$

8. Solution of Dirichlet problem by means of Green's function.

Theorem 14.⁴⁾ Let D be a bounded domain in space, whose boundary S consists of a finite number of analytic closed surfaces. Let $f(Q) \in L$ on S and put

$$u(P) = \frac{1}{4\pi} \iint_S f(Q) \frac{\partial G(P, Q)}{\partial \nu} d\omega_Q.$$

Then for almost all $Q \in S$, when P tends to Q from the inside of a Stolz domain $\Delta(Q)$,

$$\lim_{P \rightarrow Q} u(P) = f(Q).$$

Proof. Let $Q_0 = (0, 0, 0)$ be a point of S , which satisfies the condition of the proof of Theorem 3.

Since
$$\frac{1}{4\pi} \iint_S \frac{\partial G(P, Q)}{\partial \nu} d\omega_Q = 1,$$

4) M. Tsuji: On the boundary value of a harmonic function in space. Jap. Journ. Math. **19** (1944).

$$|u(P) - f(Q_0)| \leq \frac{1}{4\pi} \iint_{\overline{U(Q_0)}} |f(Q) - f(Q_0)| \left| \frac{\partial G(P, Q)}{\partial \nu} \right| d\omega_Q \\ + \frac{1}{4\pi} \iint_{S - \overline{U(Q_0)}} |f(Q) - f(Q_0)| \left| \frac{\partial G(P, Q)}{\partial \nu} \right| d\omega_Q = I + II. \quad (1)$$

By Theorem 13 (i)

$$\lim_{P \rightarrow Q_0} II = 0. \quad (2)$$

For $Q \in U(Q_0)$, $P \in \Delta(Q_0)$, we have by Theorem 13 (ii)

$$|I| \leq \frac{1}{4\pi} \iint_{\overline{U(Q_0)}} |f(Q) - f(Q_0)| \left| \frac{O(\tau) + O(\rho^2)}{(\rho^2 + \tau^2)^{3/2}} \right| \rho d\rho d\varphi,$$

where $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, $\tau = \overline{PQ_0}$.

Hence as the proof of Theorem 3,

$$\lim_{\delta \rightarrow 0} (\lim_{\tau \rightarrow 0} I) = 0, \quad (3)$$

so that $\lim_{P \rightarrow Q_0} u(P) = f(Q_0)$ for almost all $Q_0 \in S$.

Theorem 15.⁵⁾ (*Analogue of Fatou's theorem*). Let D satisfy the same condition as Theorem 14 and $u(P)$ be a bounded harmonic function in D . Then for almost all $Q \in S$,

$$\lim_{P \rightarrow Q} u(P) = f(Q)$$

exists, when P tends to Q from the inside of a Stolz domain $\Delta(Q)$ and $u(P)$ can be expressed by

$$u(P) = \frac{1}{4\pi} \iint_S f(Q) \left| \frac{\partial G(P, Q)}{\partial \nu} \right| d\omega_Q.$$

Proof. Let S_ρ be a parallel surface of S , which is generated by end points Q_ρ of segments of length ρ on inner normals n_Q at points Q of S , then the tangent plane of S_ρ at Q_ρ is parallel to that of S at Q . By $Q \longleftrightarrow Q_\rho$, any Borel set e on S is mapped on a Borel set e_ρ on S_ρ . Let $\rho_1 > \rho_2 > \dots > \rho_n \rightarrow 0$, we write $S_{\rho_n} = S_n$, $Q_{\rho_n} = Q_n$, $e_{\rho_n} = e_n$ and put

$$F_n(e) = \iint_{e_n} u(Q_n) d\omega_{Q_n}. \quad (1)$$

Let $|u(P)| \leq K$ in D . then $|F_n(e)| \leq Kme$, where me is the measure of e . Hence $F_n(e)$ are uniformly absolutely continuous additive set functions, so that we can find n_k , which we denote n , such that $F_n(e) \rightarrow F(e)$ ($n \rightarrow \infty$), where $F(e)$ is an absolutely continuous additive set function. Let D_n be the domain, bounded by S_n and $G_n(P, Q)$ be its Green's function, with P as its pole, then $\lim_n G_n(P, Q) = G(P, Q)$, so that if we take the radius of $\sigma_i(Q_n)$ to be equal to a constant α , then we see easily from the proof of Theorem 13 that

$$\frac{\partial G_n(P, Q_n)}{\partial \nu} \rightarrow \frac{\partial G(P, Q)}{\partial \nu} \quad (n \rightarrow \infty).$$

Now

$$\begin{aligned} u(P) &= \frac{1}{4\pi} \iint_{S_n} \frac{\partial G_n(P, Q_n)}{\partial \nu} u(Q_n) d\omega_{Q_n} \\ &= \frac{1}{4\pi} \iint_{S_n} \frac{\partial G_n(P, Q_n)}{\partial \nu} dF_n(Q), \quad P \in D_n, \end{aligned} \quad (2)$$

hence for $n \rightarrow \infty$,

$$u(P) = \frac{1}{4\pi} \iint_S \frac{\partial G(P, Q)}{\partial \nu} dF(Q), \quad P \in D. \quad (3)$$

Since $F(e)$ is absolutely continuous, we can write $dF(Q) = f(Q) d\omega_Q$, where $f(Q) \in L$ on S , so that

$$u(P) = \frac{1}{4\pi} \iint_S f(Q) \frac{\partial G(P, Q)}{\partial \nu} d\omega_Q, \quad (4)$$

hence by Theorem 14, $\lim_{P \rightarrow Q} u(P) = f(Q)$ almost everywhere on S , when $P \rightarrow Q$ from the inside of a Stolz domain $\Delta(Q)$.

Remark. In the former paper,⁶⁾ I have proved Theorem 14 and 15, when D is the inside of a unit sphere.

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6) M. Tsuji: On Fatou's theorems on Poisson integrals. Jap. Journ. Math. **15**(1938).