1. On the Number of Prime Factors of Integers.

By Minoru Tanaka
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1. Introduction.

We shall denote by $\omega(n)$ the number of distinct prime factors of a positive integer $n$, that is,

$$\omega(n) = \sum_{p \mid n} 1,$$

where $p$ runs through the prime factors of $n$.

We shall call a (C)-set any finite set of polynomials in $\xi$, $\{f_1(\xi), \ldots, f_k(\xi)\}$, satisfying the following conditions:

(C1) Each $f_i(\xi)$, $1 \leq i \leq k$, has integral coefficients;
(C2) Each $f_i(\xi)$, $1 \leq i \leq k$, is of positive degree;
(C3) Each $f_i(\xi)$, $1 \leq i \leq k$, is positive for $\xi \geq 1$;
(C4) $f_1(\xi), \ldots, f_k(\xi)$ are relatively prime in pairs.

In virtue of the conditions (C1) and (C2), we can factorize $f_i(\xi)$, $1 \leq i \leq k$, uniquely in the forms

$$f_i(\xi) = a_i \prod_{j=1}^{r_i} (q_{ij}(\xi))^{a_{ij}} \quad \text{for } 1 \leq i \leq k,$$

where $a_i > 0$, $1 \leq i \leq k$, and the polynomials $q_{ij}(\xi)$, $1 \leq i \leq k$, $1 \leq j \leq r_i$, have the following properties:

Each $q_{ij}(\xi)$ has the positive leading coefficient and is primitive and irreducible;
For each $i$, $1 \leq i \leq k$, the $r_i$ polynomials $q_{ij}(\xi)$, $1 \leq j \leq r_i$, are different from each other.

By the way we mention that, according to the condition (C4), the $r_1 + \ldots + r_k$ polynomials $q_{ij}(\xi)$, $1 \leq i \leq k$, $1 \leq j \leq r_i$, taken all together are different from each other.

We put for brevity

$$r = r_1 + \ldots + r_k.$$

Until section 3, we shall consider a fixed (C)-set $\{f_1(\xi), \ldots, f_k(\xi)\}$ as given once for all.

We shall put, for integers $n \geq 3$ and for $1 \leq i \leq k$,

$$\omega(f_i(n)) - r_i \log \log n \quad \text{for } \sqrt{r_i \log \log n} = u_i(n).$$

To each integer $n \geq 3$, there corresponds a point $(u_1(n), \ldots, u_k(n))$ in a $k$-dimensional space $R^k$. Let $E$ be a Jordan-measurable set, bounded or unbounded, in $R^k$.

Now let $A(x; E)$ denote the number of integers $n$, $3 \leq n \leq x$, for which the points $(u_1(n), \ldots, u_k(n))$ belong to the set $E$. Then we have the following Main Theorem:

**Theorem A.**

$$\lim_{x \to \infty} \frac{A(x; E)}{x} = (2\pi)^{-\frac{k}{2}} \int_{E} \exp \left(-\frac{1}{2} \sum_{i=1}^{k} u_i^2 \right) du_1 \ldots du_k.$$

The integral is in the sense of Riemann.
Some special cases of this theorem were proved by Hardy and Ramanujan [5], Turán [11], Erdős [2], Delange [1] and others. Turán [12], Erdős and Kac [3], LeVeque [8] and Halberstam [4] proved allied theorems on additive number-theoretic functions. Their proofs depend on the central limit theorem or some other theorems of the theory of probability. We shall show in this paper how we can prove our Theorem A without the help of the probability theory.

In section 2, we shall expose the sieve method of Brun, following the model of Landau [7], in the form to be used in section 3, where we prove the main theorem. In section 4, we shall mention some special cases of our main theorem, explain the connections with the existing literatures, and indicate finally some related results.

The author is indebted to Erdős [2] and Turán [12] as to the idea of this work. He wishes to express his thanks to Prof. S. Iyanaga for his encouragement during the preparation of this paper.

2. Brun's Sieve Method.

Throughout the paper, the letter \( p \) will be reserved for prime numbers. We shall first prove three lemmas concerning algebraic congruences, the first two of which are elementary, but the third is based on algebraic number theory.

**Lemma 2.1.** Let \( \varphi(\xi) \) and \( \psi(\xi) \) be relatively prime polynomials with integral coefficients, then, for any prime \( p \), except possibly for a finite number of primes, the two congruences \( \varphi(\xi) \equiv 0 \pmod{p} \) and \( \psi(\xi) \equiv 0 \pmod{p} \) have no common solution.

**Proof.** On account of the conditions imposed upon \( \varphi(\xi) \) and \( \psi(\xi) \), their resultant \( R \) is a non-zero integer. Let us suppose that the two congruences \( \varphi(\xi) \equiv 0 \pmod{p} \) and \( \psi(\xi) \equiv 0 \pmod{p} \) have a common solution \( \xi \). Then we obtain the identical congruences \( \varphi(\xi) \equiv \psi(\xi) \equiv (\xi-\alpha)\varphi_1(\xi) \pmod{p} \). But \( \varphi(\xi) \) and \( \psi(\xi) \) are polynomials with integral coefficients, and hence the resultant \( R \) is congruent modulo \( p \) to the resultant \( R' \) of the two polynomials \( (\xi-\alpha)\varphi_1(\xi) \) and \( (\xi-\alpha)\varphi_1(\xi) \). But \( R' \) is obviously zero, and consequently \( p \) is a prime factor of \( R \). Thus we see that the exceptional primes divide \( R \), and hence they are finite in number.

**Lemma 2.2.** Let \( \varphi(\xi) \) be a polynomial with integral coefficients, having no square factor of positive degree, then, for any prime \( p \), except possibly for a finite number of primes, the number of solutions, incongruent modulo \( p \), of the congruence \( \varphi(\xi) \equiv 0 \pmod{p} \) is equal to the number of solutions, incongruent modulo \( p^2 \), of the congruence \( \varphi(\xi) \equiv 0 \pmod{p^2} \).

**Proof.** The discriminant \( D \) of the polynomial \( \varphi(\xi) \) is a non-zero integer. If a prime \( p \) does not divide \( D \), then, for any solution \( \alpha \) of the congruence \( \varphi(\xi) \equiv 0 \pmod{p} \), we have \( \varphi'(\alpha) \equiv 0 \pmod{p} \). It follows from this, as is well-known, that there is only one solution \( \beta \) modulo \( p^2 \) of the congruence \( \varphi(\xi) \equiv 0 \pmod{p^2} \) such that \( \beta \equiv \alpha \pmod{p} \). From this fact the lemma follows easily, the exceptional primes being the factors of \( D \).

**Lemma 2.3.** Let \( \varphi(\xi) \) be an irreducible polynomial with integral coefficients,
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and for any prime $p$, let $\nu_p$ denote the number of incongruent solutions of the congruence $\varphi(\xi) \equiv 0 \pmod{p}$, then we have

$$\sum_{p \leq x} \frac{\nu_p}{p} = \log \log x + O(1).$$

Cf. Turán [12], p. 127. Originally this formula was derived by Rademacher [10] from a result of Nagel [9]. Nagel proved the formula

$$\sum_{p \leq x} \frac{\nu_p \log p}{p} = \log x + O(1),$$

by the aid of algebraic number theory.

Now we return to the (C)-set $\{f_1(\xi), \ldots, f_k(\xi)\}$. We shall denote by $l_i$ the degree of the polynomial $f_i(\xi)$, and for any prime $p$, we denote by $\nu_i(p)$ the number of incongruent solutions of the congruence $f_i(\xi) \equiv 0 \pmod{p}$. We put for convenience

$$l = l_1 + \cdots + l_k,$$

$$\nu(p) = \nu_1(p) + \cdots + \nu_k(p).$$

It is plain that $\nu_i(p) \leq l_i$ for $1 \leq i \leq k$ and $\nu(p) \leq l$.

In virtue of the condition (C4) and Lemma 2.1, we can take a positive number $\gamma_1$ such that, for any prime $p > \gamma_1$, no two of the congruences $f_i(\xi) \equiv 0 \pmod{p}$, $1 \leq i \leq k$, have common solution, and therefore the congruence

$$f_1(\xi) \ldots f_k(\xi) \equiv 0 \pmod{p}$$

has $\nu(p)$ incongruent solutions.

Now we put

$$g_i(\xi) = a_i \prod_{j=1}^{l_i} q_{ij}(\xi)$$

for $1 \leq i \leq k$, replacing all the exponents $\alpha_{ij}$ in (1.1) by 1. Since $\omega(n)$ counts the prime factors of $n$ regardless of their multiplicity, we have obviously

$$\omega(f_i(n)) = \omega(g_i(n))$$

for $1 \leq i \leq k$, for all integer $n$ for which $g_i(n)$, $1 \leq i \leq k$, are all positive. Therefore we may and shall presuppose without loss of generality that the exponents $\alpha_{ij}$ in (1.1) are all equal to 1, that is, each of the polynomials $f_i(\xi)$, $1 \leq i \leq k$, constituting our (C)-set, has no square factor of positive degree. Now we can apply Lemma 2.2 to our $f_i(\xi)$, $1 \leq i \leq k$, and consequently we can take a positive number $\gamma_2$ such that, for any prime $p > \gamma_2$, each of the congruence $f_i(\xi) \equiv 0 \pmod{p^2}$, $1 \leq i \leq k$, has just $\nu_i(p)$ incongruent solutions modulo $p^2$.

We shall put $\gamma = \max(\gamma_1, \gamma_2)$.

Next we shall mention here that, from Lemmas 2.1 and 2.3, we have

$$\sum_{p \leq x} \frac{\nu_i(p)}{p} = r_i \log \log x + O(1)$$

for $1 \leq i \leq k$, and

$$\sum_{p \leq x} \frac{\nu(p)}{p} = r \log \log x + O(1).$$

Now, to deal with Brun's method, we use the following notations:

$m$ is a positive integer;

$h_1$, $\ldots$, $h_{m+1}$ are integers such that $h_1 \geq \cdots \geq h_{m+1}$, $h_1 > h_{m+1}$;

$\epsilon_j$, $h_1 > h_{m+1}$ are arbitrarily given numbers;
Empty sum is as usual interpreted as zero.

With these notations we first have

**Lemma 2.4.** For $1 \leq \mu \leq m$,

$$R^{(\mu-1)} = \prod_{h_{\mu} \geq j > h_{\mu+1}} (1-c_j) - R^{(\mu)} = \sum_{j=0}^{\infty} (-1)^j R^{(\mu-1)}_{j} \sum_{\rho=0}^{\infty} (-1)^\rho S^{(\rho)}_{\rho}.$$  

The proof is similar to the proofs of Landau [7], Satz 77 and 78. Notice that $R^{(\mu-1)}=0$ as an empty sum.

**Lemma 2.5.** If we impose the conditions that

$$0 < \varepsilon < 0.1,$$

$$0 < c_j < 1 \quad \text{for} \quad h_{1} \geq j > h_{m+1},$$

$$P_\mu = \prod_{h_{\mu} \geq j > h_{\mu+1}} (1-c_j) \geq \frac{1}{1+\varepsilon} \quad \text{for} \quad 1 \leq \mu \leq m,$$

then we have

$$\left| \frac{R^{(\mu)}}{P_1 \cdots P_{\mu-1} - 1} \right| < \frac{\varepsilon}{2}.$$  

**Proof.** Cf. Landau [7], Satz 79. We have successively

$$0 \leq \sum_{h_{\mu} \geq j > h_{\mu+1}} c_j \leq -\sum_{h_{\mu} \geq j > h_{\mu+1}} \log(1-c_j) = -\log P_\mu \leq \log(1+\varepsilon) < \varepsilon \quad \text{for} \quad 1 \leq \mu \leq m,$$

$$\lambda! R^{(\mu-1)} \leq \left( \sum_{h_{\mu} \geq j > h_{\mu+1}} c_j \right)^\lambda \leq (\mu-1)^{\lambda \varepsilon^k} \quad \text{for} \quad \lambda \geq 1,$$

$$\rho! S^{(\rho)}_{\rho} \leq \left( \sum_{h_{\mu} \geq j > h_{\mu+1}} c_j \right)^\rho < \varepsilon^\rho \quad \text{for} \quad 1 \leq \mu \leq m, \quad \rho \geq 1,$$

$$\sum_{\rho=1}^{\infty} (-1)^\rho S^{(\rho)}_{\rho} \leq \sum_{\rho=1}^{\infty} \frac{\varepsilon^\rho}{\rho!} < \frac{\varepsilon^{\lambda+1}}{(\mu-\lambda)!} \quad \text{for} \quad 0 \leq \lambda \leq \mu-1, \quad 1 \leq \mu \leq m.$$  

Combining the inequalities thus obtained with Lemma 2.4, we deduce for $1 \leq \mu \leq m$,

$$\left| R^{(\mu-1)} P_\mu - R^{(\mu)} \right| = \left| \sum_{j=0}^{\infty} (-1)^j R^{(\mu-1)}_j \sum_{\rho=0}^{\infty} (-1)^\rho S^{(\rho)}_{\rho} \right|$$

$$\leq \sum_{j=0}^{\infty} \frac{(\mu-1)^{j \varepsilon^k}}{\lambda!} \cdot \frac{\varepsilon^{\lambda+1} e^\varepsilon}{(\mu-\lambda)!} \leq \varepsilon^{\lambda+1} e^\varepsilon \sum_{j=0}^{\infty} \frac{1}{\lambda!} \frac{(\mu-1)^{j}}{(\mu-\lambda)!} = \varepsilon^{\lambda+1} e^\varepsilon \frac{\mu^{\lambda}}{\mu!} < \varepsilon^{\lambda+1} e^\varepsilon,$$

and hence

$$\left| \frac{R^{(\mu-1)}}{P_1 \cdots P_{\mu-1}} - \frac{R^{(\mu)}}{P_1 \cdots P_{\mu}} \right| < \varepsilon^{\lambda+1} (1+\varepsilon)^{e^{\lambda+1} e^\varepsilon}.$$  

Here letting $\mu$ run through the numbers 1, ..., $m$, we obtain

$$\left| \frac{R^{(a)}}{P_1 \cdots P_{m}} \right| < \sum_{\rho=1}^{\infty} \varepsilon^{\rho+1} (1+\varepsilon)^{e^{\rho+1} e^\varepsilon} < \frac{\varepsilon^2 (1+\varepsilon)}{1-\varepsilon (1+\varepsilon)} e < \frac{\varepsilon}{2},$$

the last step by the condition $0 < \varepsilon < 0.1$. Thus the lemma is proved.

In what follows, we use the following notations:

- $x$ is a positive number;
- $a$ is an integer.
$d$ is a positive integer;
$p_1, \ldots, p_h$ are primes greater than $\gamma_1$ and do not divide $d$;
$F(x; a, d)$ is the number of positive integers $n \leq x$ for which
$$n \equiv a \pmod{d},$$
$F(x; a, d; p_1, \ldots, p_h)$ is the number of positive integers $n \leq x$ for which
$$n \equiv a \pmod{d},$$
$$f_1(n) \cdots f_h(n) \equiv 0 \pmod{p_j} \quad \text{for } 1 \leq j \leq h.$$
Now we have

**Lemma 2.6.** Let $t$ be a positive integer, and let $h_1, \ldots, h_{2t}$ be integers such that
$$h=h_1=h_2>h_3=h_4>\cdots>h_{2t-1}=h_{2t}>0,$$
then we have
$$F(x; a, d; p_1, \ldots, p_h) \leq F(x; a, d)$$
with suitable $a_{i_1\ldots i_l}$.

The proof is similar to the proofs of Landau [7], Satze 81, 82, 83, 84 and 85\textsuperscript{a}. Notice that the congruence $f_1(\xi) \cdots f_h(\xi) \equiv 0 \pmod{p_j}$ has $\nu(p_j)$ incongruent solutions since $p_j > \gamma_1$ for $1 \leq j \leq h$.

**Lemma 2.7.** Let $t$ be a positive integer, and let $h_1, \ldots, h_{2t-1}$ be integers such that
$$h=h_1=h_2>h_3=h_4>\cdots>h_{2t-1}=h_{2t}>0,$$
then we have
$$F(x; a, d; p_1, \ldots, p_h) \geq F(x; a, d)$$
with suitable $a_{i_1\ldots i_l}$.

The proof is similar to that of the preceding lemma.

From now on, let $\varepsilon$ be in the interval $0 < \varepsilon < 0.1$, and the primes $p_1, \ldots, p_h$ be such that
$$\max \left( \frac{2l}{\varepsilon}, \frac{1}{\varepsilon} \right) < p_1 < \cdots < p_h.$$

Our course of reasoning will now branch off into two cases for a while.

First case. We shall push Lemma 2.6 further. We define $h_1, \ldots, h_{2t}$, as follows:
First we put $h=h_1=h_2$. If
$$\prod_{h_2=1}^{h_2>0} \left(1-\frac{\nu(p_j)}{p_j} \right) < \frac{1}{1+\varepsilon},$$
then, on account of the condition $p_j > 2l/\varepsilon$ for $1 \leq j \leq h$, we can define $h_3$ as the least positive integer which satisfies
$$\prod_{h_3=1}^{h_3>h_3} \left(1-\frac{\nu(p_j)}{p_j} \right) \geq \frac{1}{1+\varepsilon}.$$
We put
$$P_1=1, \quad P_2=\prod_{h_2=1}^{h_2>h_3} \left(1-\frac{\nu(p_j)}{p_j} \right),$$
$$h_3 = h_4.$$
Next, if
\[ \prod_{h_2^n j > 0} \left( 1 - \frac{\nu(p_j)}{p_j} \right) < \frac{1}{1+\varepsilon}, \]
then we define \( h_0 \) as the least positive integer which satisfies
\[ \prod_{h_2^n j > h_0} \left( 1 - \frac{\nu(p_j)}{p_j} \right) \geq \frac{1}{1+\varepsilon}, \]
and we put
\[ P_3 = 1, \quad P_4 = \prod_{h_2^n j > h_0} \left( 1 - \frac{\nu(p_j)}{p_j} \right), \]
\[ h_0 = h_0. \]

After repeating this process a finite number of times, we can reach a positive integer \( h_{2t-1} = h_{2t} \) such that the other alternative occurs. Then we put
\[ P_{2t-1} = 1, \quad P_{2t} = \prod_{h_2^n j > 0} \left( 1 - \frac{\nu(p_j)}{p_j} \right). \]

Now, with \( \varepsilon; \ p_1, \ldots, p_h; h_1, \ldots, h_{2t} \) thus defined, we have

**Lemma 2.8.** \( F(x; a, d; p_1, \ldots, p_h) < (1 + \frac{\varepsilon}{2}) \prod_{j=1}^{h} \left( 1 - \frac{\nu(p_j)}{p_j} \right) + \prod_{\mu=1}^t (2h_{2\mu-1})^3. \)

**Proof.** On combining Lemma 2.6 and the fact that
\[ F(x; a; d; p_1, \ldots, p_h) < \frac{x}{d p_1 \ldots p_h} \leq 1, \]
we have
\[ F(x; a; d; p_1, \ldots, p_h) \]
\[ \leq \frac{x}{d} \left( 1 + \sum_{i=1}^{2t} \left( -1 \right)^i \sum_{j=1}^{h} \sum_{\mu=1}^{2t} \nu(p_{j1}) \ldots \nu(p_{jt}) \right) \]
\[ \leq \frac{x}{d} [R^{(2t)} + 1 + \sum_{i=1}^{2t} p_i h_1 \ldots h_2], \]
where \( R^{(2t)} \) is nothing but the \( R^{(m)} \) in Lemma 2.5 with \( m=2t, h_{2t+1}=0 \) and \( c_j=\nu(p_j)/p_j \). Here the conditions imposed upon \( c_j \) and \( P_{\mu} \) in Lemma 2.5 are satisfied, and consequently we have
\[ R^{(2t)} < (1 + \frac{\varepsilon}{2}) P_1 \ldots P_{2t}, \]
or
\[ R^{(2t)} < (1 + \frac{\varepsilon}{2}) \prod_{j=1}^{h} \left( 1 - \frac{\nu(p_j)}{p_j} \right). \]

On the other hand
\[ 1 + \sum_{i=1}^{2t} p_i h_1 \ldots h_2 \leq (2t+1)^2 \prod_{\mu=1}^t (2h_{2\mu-1})^3. \]

The combination of thus obtained inequalities gives the lemma.

Second case. We shall push Lemma 2.7 further. We define \( h_1, \ldots, h_{2t-1} \) as follows: First we put \( h_0 = h_1. \) If
\[ \prod_{h_2^n j > 0} \left( 1 - \frac{\nu(p_j)}{p_j} \right) < \frac{1}{1+\varepsilon}, \]
then we define $h_a$ as the least positive integer which satisfies

$$
\prod_{h_1 \leq j < h_a} \left(1 - \frac{\nu(p_j)}{p_j}\right) \geq \frac{1}{1+\varepsilon},
$$

and we put

$$
P_1 = \prod_{h_1 \leq j < h_a} \left(1 - \frac{\nu(p_j)}{p_j}\right), \quad P_a = 1,
$$

$h_a = h_b$.

Next, if

$$
\prod_{h_1 \leq j < h_a} \left(1 - \frac{\nu(p_j)}{p_j}\right) < \frac{1}{1+\varepsilon},
$$

then we define $h_a$ as the least positive integer which satisfies

$$
\prod_{h_1 \leq j < h_a} \left(1 - \frac{\nu(p_j)}{p_j}\right) \geq \frac{1}{1+\varepsilon},
$$

and we put

$$
P_a = \prod_{h_1 \leq j < h_a} \left(1 - \frac{\nu(p_j)}{p_j}\right), \quad P_a = 1,
$$

$h_a = h_b$.

After repeating this process a finite number of times, we can reach a positive integer $h_{2\ell-2} = h_{2\ell-1}$ such that the other alternative

$$
\prod_{h_1 \leq j < h_{2\ell-1}} \left(1 - \frac{\nu(p_j)}{p_j}\right) > \frac{1}{1+\varepsilon},
$$

occurs. Then we define

$$
P_{2\ell-1} = \prod_{h_1 \leq j < h_{2\ell-1}} \left(1 - \frac{\nu(p_j)}{p_j}\right), \quad P_{2\ell} = 1.
$$

This time, with $\varepsilon, p_1, \ldots, p_h; h_1, \ldots, h_{2\ell-1}$ thus defined, we have

**Lemma 2.9.** $F(x; a, d; p_1, \ldots, p_h) > (1 - \varepsilon) x \prod_{j=1}^h \left(1 - \frac{\nu(p_j)}{p_j}\right) - \frac{1}{2} \prod_{j=1}^h (2h_{2\ell-1})^2$.

The proof is similar to that of the preceding lemma.

Now, combining both Lemmas 2.8 and 2.9, we can deduce the following lemma, which is the object of this section.

**Lemma 2.10.** Given any $\varepsilon$ in the interval $0 < \varepsilon < 0.1$, we can find $\delta = \delta(\varepsilon) > 0$ and $x_1 = x_1(\varepsilon) > 0$ such that the following statement holds.

Let $a$ be an arbitrary integer, $p_1, \ldots, p_h$ be primes such that

$$
\max\left(\gamma_1, \frac{2l}{\varepsilon}\right) < p_1 < \ldots < p_h < x^\delta,
$$

and $d$ be a positive integer not exceeding $\sqrt{x}$, and not divisible by any of the primes $p_1, \ldots, p_h$. Let $F(x; a, d; p_2, \ldots, p_h)$ denote the number of positive integers $n \leq x$ for which

$$
n \equiv a \pmod{d},
$$

$$
f_1(n) \cdots f_{h}(n) \equiv 0 \pmod{p_j} \quad \text{for } 1 \leq j \leq h.
$$

Then we have, for $x > x_1$,

$$
F(x; a, d; p_1, \ldots, p_h) \begin{cases}
(1 + \varepsilon) x \prod_{j=1}^h \left(1 - \frac{\nu(p_j)}{p_j}\right), \\
(1 - \varepsilon) x \prod_{j=1}^h \left(1 - \frac{\nu(p_j)}{p_j}\right),
\end{cases}
$$
Proof. Irrespective of the cases we have
\[ P_{2^{t-1}}^{2^t} = \prod_{h_{2^t-1}=p_{2^t+1}}^{2^t-1} \left(1 - \frac{\nu(p)}{p}\right) \quad \text{for} \ 1 \leq \mu \leq t - 1, \]
\[ P_{2^{t-1}} P_{2^t} = \prod_{h_{2^t-1}=p_{2^t+1}}^{2^t-1} \left(1 - \frac{\nu(p)}{p}\right), \]
\[ P_{2^{t-1}} P_{2^t} \geq \frac{1}{1+\varepsilon} \quad \text{for} \ 1 \leq \mu \leq t, \]
but
\[ P_{2^{t-1}} P_{2^t} \left(1 - \frac{\nu(p_{2^t+1})}{p_{2^t+1}}\right) < \frac{1}{1+\varepsilon} \quad \text{for} \ 1 \leq \mu \leq t - 1. \]

From this and the condition \( p_j > 2l/\varepsilon \) for \( 1 \leq j \leq h \), we have, for \( 1 \leq \mu \leq t - 1, \)

\[ P_{2^{t-1}} P_{2^t} < \frac{1}{1+\varepsilon} \cdot \frac{1}{1 - \frac{\nu(p_{2^t+1})}{p_{2^t+1}}} < \frac{1}{1+\varepsilon} \cdot \frac{1}{1 - \frac{\nu(p_{2^t+1})}{p_{2^t+1}}} < \frac{2}{2+\varepsilon - \varepsilon^2} < 1. \]

Now we put
\[ Q = \prod_{q \in \mathbb{Q}} \left(1 - \frac{\nu(q)}{q}\right), \]
where \( q \) runs through the primes such that

\[ \max(\gamma_1, \frac{2l}{\varepsilon}) < q < x^s, \quad \text{and} \quad q \neq p_j \quad \text{for} \ 1 \leq j \leq h. \]

If such primes as \( q \) do not exist, then \( Q = 1 \) as an empty product.

From (2.2), we derive, by simple calculations\(^7\),

\[ \frac{1}{\alpha_1 \log y} < \prod_{\max(\gamma_1, 2l/\varepsilon) < p < y} \left(1 - \frac{\nu(p)}{p}\right) < \frac{\alpha_1}{\log^2 y}, \]

for \( \max(\gamma_1, 2l/\varepsilon) < y \), where \( \alpha_1 = \alpha_1(\varepsilon) \) is a positive number independent of \( y \).

Now we put, in (2.4), \( y = p_{2^t+1} \), then we obtain, for \( 1 \leq \mu \leq t, \)

\[ Q P_{2^t-1} \ldots P_{2^t} \leq \prod_{\max(\gamma_1, 2l/\varepsilon) < p < p_{2^t+1}} \left(1 - \frac{\nu(p)}{p}\right) < \frac{\alpha_1}{\log^2 p_{2^t+1}}, \]

that is

\[ \log^* p_{2^t+1} < \frac{\alpha_2}{Q P_{2^t-1} \ldots P_{2^t}}, \]

and consequently we have, for \( 1 \leq \mu \leq t, \)

\[ \log^* (2l/p_{2^t+1}) < \frac{\alpha_2}{Q P_{2^t-1} \ldots P_{2^t}}. \]

If we put \( P = P_{1} \ldots P_{2^t} \), then this result can be rewritten as

\[ \log^* (2l/p_{2^t+1}) < \frac{\alpha_2}{P Q} \frac{1}{P^2} \quad \text{for} \ 1 \leq \mu \leq t, \]

and hence, by (2.3),

\[ \log^* (2l/p_{2^t+1}) < \frac{\alpha_2}{P Q} \left(\frac{2}{2+\varepsilon - \varepsilon^2}\right)^{\nu-1} \quad \text{for} \ 1 \leq \mu \leq t. \]

On the other hand, putting \( y = x^s \) in (2.4), we have

\[ P Q > \frac{1}{\alpha_1 \log x} = \frac{1}{\alpha_1 \log x^s}. \]

From (2.5), (2.6) we obtain, for \( 1 \leq \mu \leq t, \)

\[ \log^* (2l/p_{2^t+1}) < \frac{\alpha_2}{P Q} \left(\frac{2}{2+\varepsilon - \varepsilon^2}\right)^{\nu-1} \log^* x, \]
or
\[ \log(2l_{2^{n-1}}) < a_4 \delta \left( \frac{2}{2+2-\varepsilon} \right)^{n-1} \log x, \]
and consequently, noting that \( 2/(2+\varepsilon-\varepsilon^2) < 1 \), we obtain
\[ \log \prod_{\mu=1}^{n} (2l_{2^{n-1}})^{\mu} < 2a_4 \delta \log x \cdot \sum_{\mu=1}^{n} \left( \frac{2}{2+2-\varepsilon} \right)^{n-1} = a_5 \delta \log x, \]
and hence
\[ \prod_{\mu=1}^{n} (2l_{2^{n-1}})^{2} < x^{a_5 \delta}, \]

Now, from this and Lemmas 2.8, 2.9, we have
\[ (2.7) \]
\[ F(x; a, d; p_1, \ldots, p_h) \]
Here we put \( d = 1/3 \alpha \), then, from (2.4) and the condition \( d \leq \sqrt{x} \), we have
\[ \frac{x}{d} \prod_{j=1}^{h} \left( 1 - \frac{\nu(p_j)}{p_j} \right) \geq \sqrt{x} \prod_{x \leq p \leq x^1/2^{l_i-1}} \left( 1 - \frac{\nu(p)}{p} \right) > \frac{(3a_2)^{r}}{a_1} \cdot \frac{\sqrt{x}}{\log^r x}. \]
Therefore we can take \( x_1 = x_1(\varepsilon) \) so large that for \( x > x_1 \), we have
\[ (2.8) \]
Finally (2.7) and (2.8) establish our lemma.

3. The Proof of the Main Theorem.

We consider the set consisting of prime numbers \( p \) which lie in the interval
\[ (3.1) \]
\[ e^{(\log \log x)^a} < p < x^{1/2^{l_i} \log \log x}, \]
and denote this set by the letter \( \pi \). We put
\[ \omega'(n) = \omega'(n, x) = \sum_{\substack{p \in \pi \atop p \equiv \alpha}} 1, \]
that is, \( \omega'(n) \) is the number of distinct prime factors of \( n \) which belong to the set \( \pi \).

We first prove the following lemma.

Lemma 3.1. The number of positive integers \( n \leq x \), which satisfy at least one of the inequalities
\[ \omega(f_i(n)) - \omega'(f_i(n)) > (\log \log \log n)^2 \quad (1 \leq i \leq k), \]
is \( o(x) \).

Proof. We shall prove that
\[ (3.2) \]
\[ \sum_{n=0}^{x^a} (\omega(f_i(n)) - \omega'(f_i(n))) = O(x \log \log \log x). \]
Let \( n \) be a fixed positive integer not exceeding \( x \). Then the number of primes \( p \) such that \( p | f_1(n), p > x \) is at most \( l_i \), the degree of \( f_i(\xi) \), for sufficiently large \( x \).
In order to see this, suppose there are \( m \) such primes, say \( p_1, \ldots, p_m \), then we have, for sufficiently large \( x \),
\[ x^{a_5} < p_1, \ldots, p_m \leq f_1(n) \leq f_1(x). \]
It follows from this that \( m \) cannot be greater than \( l_i \), for \( x \) large enough.
Now, from this fact, noting \( \nu_1(p) \leq l_i \), we derive, for \( x \) large enough,
(3.3) \[ \sum_{n \leq x} \left( \omega(f_1(n)) - \omega'(f_1(n)) \right) = \sum_{n \leq x} \sum_{p \not\in \pi} 1 = \sum_{n \leq x} \sum_{p \not\in \pi} \sum_{i=1}^{\nu_i(p)} 1 + \sum_{n \leq x} \sum_{p \not\in \pi} 1 \leq \sum_{p \leq x, \nu_i(p)} \nu_i(p) \left( \frac{x}{p} + 1 \right) + 1 = \frac{x}{\nu_i(p)} + \frac{1}{p} + \frac{1}{l_1 x} . \]

Here we quote the well-known formula (3.4)

\[ \sum_{p} \frac{1}{p} = \log \log y + O(1) . \]

Using this formula and noting that the primes belonging to the set \( \pi \) satisfy (3.1), we deduce

\[ \sum_{p \leq x, \nu_i(p)} \frac{1}{p} = \sum_{p \leq x} \frac{1}{p} + \sum_{x^{1/8} \leq p \leq x} \frac{1}{p} = O(\log \log \log x) . \]

Now this and (3.3) give (3.2).

The corresponding results for \( f_i(\xi) \), \( 2 \leq i \leq k \), will be obtained similarly, and the lemma will easily be derived from these results.

Next, for \( x \) so large that any prime belonging to the set \( \pi \) is greater than \( l \), we put

\[ \sum_{p \leq x} \frac{1}{p} = \sum_{p \leq \log \log x} \frac{1}{p} + \sum_{x^{1/8} \leq p \leq x} \frac{1}{p} = O(\log \log \log x) . \]

We shall denote by \( M(t) \), where \( t \) is a positive integer, the set of positive numbers \( n \) subject to the following conditions:

1. \( n \) is composed only of primes belonging to the set \( \pi \);
2. \( n \) is squarefree;
3. \( n \) has \( t \) prime factors.

Lemma 3.3. Let \( t_1, \ldots, t_k \) be positive integers such that \( t_i < 2r_i \log \log x \) for\( 1 \leq i \leq k \), then we have

\[ \sum_{p \leq x} \frac{\nu_i(p)}{p} = O(\log \log \log x) . \]

From this and (2.1), we have, for \( 1 \leq i \leq k \),

\[ \sum_{p \leq x} \frac{\nu_i(p)}{p} = \frac{r_i \log \log x + O(\log \log \log x)}{x^{1/8} \log x} . \]

On the other hand, since \( \nu_i(p) = O(1) \) and \( \nu(p) = O(1) \), we have, for \( 1 \leq i \leq k \),

\[ \nu_i - \sum_{p \leq x} \frac{\nu_i(p)}{p} = \sum_{p \leq x} \frac{\nu(p) - 1}{p} \frac{\nu_i(p)}{p} = O\left( \sum_{p \leq x} \frac{1}{p^2} \right) = O(1) . \]

Now from this and (3.6), the lemma follows at once.
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signified by the dash attached to $\sum$. The meanings of $\nu_1(m_1), \ldots, \nu_k(m_k); m_1', \ldots, m_k'$ are as follows:

$$\nu_i(m) = \prod_{p \mid m_i} \nu_i(p)^{\nu_i(p)} \quad \text{for } 1 \leq i \leq k,$$

$$m_i' = \prod_{p \mid m_i} \frac{p - \nu_i(p)}{p - 1} \quad \text{for } 1 \leq i \leq k.$$

The $O$-term on the right-hand side is uniform with respect to the numbers $t_1, \ldots, t_k$ such that $t_i < 2r_i \log \log x$ for $1 \leq i \leq k$.

**Proof.** Noting (3.5) and the fact that $\nu_i(p) \leq l$ for $1 \leq i \leq k$, we easily see by multinomial theorem that

$$\sum_{m_i \in \mathbb{N}(t_i)} \frac{\nu_1(m_1) \cdots \nu_k(m_k)}{m_1' \cdots m_k'} \leq \frac{\nu_1^i \cdots \nu_k^k}{t_1! \cdots t_k!} \sum_{w' \in \mathcal{W}'} \frac{1}{w'},$$

where $\sum_{w'}$ signifies that the summation-variable $w$ runs through the positive integers subject to the following conditions:

- $w$ is composed only of primes belonging to the set $\pi$;
- $w$ is not squarefree;
- $w$ has $t_1 + \cdots + t_k$ prime factors, multiple factors being counted multiply.

$w'$ has the meaning: when $w = p^aq^b \cdots$, then $w' = \left( \frac{p - \nu_i(p)}{p - 1} \right)^{\nu_i(q - \nu_i(q))} \cdots$.

We shall consider the values of $x$ so large that any prime $p$ belonging to the set $\pi$ is greater than $2l$. Then

$$w' > \left( \frac{p - \nu_i(p)}{p - 1} \right)^{\nu_i(q - \nu_i(q))} \cdots = \frac{w}{2^{t_1 + \cdots + t_k}},$$

and hence

$$(3.8) \quad (t_1 + \cdots + t_k)!^2 (t_1 + \cdots + t_k)!^2 \sum_{w'} \frac{1}{w'} \leq (t_1 + \cdots + t_k)! (2l)^{t_1 + \cdots + t_k} \sum_{w'} \frac{1}{w'}.$$

But from the condition $t_i < 2r_i \log \log x$ for $1 \leq i \leq k$, we have $t_1 + \cdots + t_k < 2r \log \log x$, and hence

$$(3.9) \quad (t_1 + \cdots + t_k)! (2l)^{t_1 + \cdots + t_k} \leq (2l)^{t_1 + \cdots + t_k} (t_1 + \cdots + t_k)! < (4l \log x)^{2r \log x}.$$

On the other hand, noting the meaning of the set $\pi$, the conditions imposed upon $w$, and the inequality $t_1 + \cdots + t_k < 2r \log \log x$, we get

$$w < \left( x^{1/8r \log x} \right)^{2r \log \log x} < x,$$

so that we can put $w = p^d$ where $p \in \pi$ and $d < x$. On noting again the meaning of the set $\pi$, we deduce

$$(3.10) \quad \sum_{w'} \frac{1}{w} \leq \sum_{p \in \pi} \frac{1}{p^a} \sum_{d \leq x} \frac{1}{d} \leq \sum_{p \geq e^{(\log \log x)^2} \log x} \frac{1}{p^a} \sum_{d \leq x} \frac{1}{d} = O(e^{-(\log \log x)^2} \log x).$$

It follows from (3.8), (3.9) and (3.10) that

$$(3.11) \quad (t_1 + \cdots + t_k)! (t_1 + \cdots + t_k)!^2 \sum_{w'} \frac{1}{w'} = O((4l \log x)^{2r \log \log x} \cdot e^{-(\log \log x)^2} \log x).$$
Finally (3.7) and (3.11) establish the lemma including the uniformity of the $O$-term.

Henceforward we consider those values of $x$ for which $e^{\log \log x)^2 > \gamma$ defined as
$max(\gamma_1, \gamma_2)$ in section 2, so that any prime belonging to the set $\pi$ is greater than $\gamma$. Now we have

**Lemma 3.4.** The number of positive integers $n \leq x$, for which at least one of $f_i(n)$, $1 \leq i \leq k$, is divisible by the square of a prime belonging to the set $\pi$, is $O(x/\log x)$.

**Proof.** On account of the stipulation made above, each of the congruences
$f_i(z) \equiv 0 \pmod{p^2}$ has $\nu_i(p)$ incongruent solutions modulo $p^2$ for any prime $p \in \pi$.
It follows easily from this that the number of positive integers $n \leq x$, for which there is at least one prime $p \in \pi$ such that at least one of the congruences $f_i(n) \equiv 0 \pmod{p^2}$, $1 \leq i \leq k$, is satisfied, is not greater than

$$\sum_{p \leq x} \sum_{i=1}^k \nu_i(p) \left( \left\lfloor \frac{x}{p^2} \right\rfloor + 1 \right) \leq x \sum_{p \leq x} \frac{\nu(p)}{p^2} + \sum_{p \leq x} \nu(p).$$

Since $\nu(p) = O(1)$ and the primes belonging the set $\pi$ satisfy (3.1), the expression on the right-hand side is of order

$$x \sum_{p \leq x} \frac{1}{p^2} + x^{1/2} \log \log x = O\left( \frac{x}{\log x} \right).$$

Thus the lemma is proved.

Let $t_1, \ldots, t_k$ be positive integers, and let $G(x; t_1, \ldots, t_k)$ denote the number of positive integers $n \leq x$ such that $\omega(f_i(n)) = t_i$ for $1 \leq i \leq k$.

Now we proceed to the following lemma, the proof of which is based on the result obtained by Brun's method.

**Lemma 3.5.** Let $t_1, \ldots, t_k$ be positive integers such that $t_i < 2r_i \log \log x$ for $1 \leq i \leq k$, then

$$G(x; t_1, \ldots, t_k) = (1 + o(1)) x^{\sum_{i=1}^k t_i} \prod_{p \in \pi} \left( 1 - \frac{\nu(p)}{p} \right) + O\left( \frac{x}{\log x} \right).$$

Both the terms $o(1)$ and $O(x/\log x)$ are uniform with respect to $t_1, \ldots, t_k$.

**Proof.** Let $G'(x; t_1, \ldots, t_k)$ be the number of positive integers $n \leq x$ such that $\omega(f_i(n)) = t_i$ for $1 \leq i \leq k$ and furthermore such that $f_i(n)$, $1 \leq i \leq k$, are not divisible by the square of any prime belonging to the set $\pi$, in other words, $f_i(n)$, $1 \leq i \leq k$, are squarefree in respect to the prime factors belonging to the set $\pi$. It follows from Lemma 3.4 that

$$(3.12) \quad G(x; t_1, \ldots, t_k) = G'(x; t_1, \ldots, t_k) + O\left( \frac{x}{\log x} \right).$$

Hence it suffices for us to deal with $G'(x; t_1, \ldots, t_k)$.

Corresponding to each $n$ counted in $G'(x; t_1, \ldots, t_k)$, there is a system of positive integers $m_1, \ldots, m_k$ with the following properties:

- $m_i \in \mathbb{N}(t_i)$ for $1 \leq i \leq k$;
- $m_i \not| f_i(n)$ for $1 \leq i \leq k$;
- $f_i(n)/m_i$, $1 \leq i \leq k$, are not divisible by any prime belonging to the set $\pi$;
- $m_1, \ldots, m_k$ are relatively prime in pairs.
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The last property follows from the stipulation made just before Lemma 3.4 and the meaning of the number $\gamma_1$.

Shifting our position, we now suppose there is given a system of positive integers $m_1, \ldots, m_k$ subject to the following conditions:

$m_i \in \mathbb{M}(t_i)$ for $1 \leq i \leq k$;

$m_1, \ldots, m_k$ are relatively prime in pairs.

Further we denote by $H(x; m_1, \ldots, m_k)$ the number of integers $n \leq x$ satisfying the following conditions:

$m_i | f_i(n)$ for $1 \leq i \leq k$;

$f_i(n)/m_i, 1 \leq i \leq k$, are not divisible by any prime belonging to the set $\pi$.

Then we have

\[
(3.13) \quad G'(x; t_1, \ldots, t_h) = \sum_{m_1 \in \mathbb{M}(t_i)} H(x; m_1, \ldots, m_k),
\]

where the range of summation-variables is the same as in Lemma 3.3.

Next we must deal with $H(x; m_1, \ldots, m_k)$. For this purpose we first consider the set of positive integers $n$ subject to the following conditions:

1) $m_i | f_i(n)$ for $1 \leq i \leq k$;

2) Each of the $f_i(n)/m_i, 1 \leq i \leq k$, is prime to the product $m_1 \cdots m_k$.

Since the primes belonging to the set $\pi$ are supposed to be greater than $\gamma_1$, the condition 2) can be replaced by the following condition:

2') For each $i, 1 \leq i \leq k$, $f_i(n)/m_i$ is prime to $m_i$.

Now, since the primes belonging to the set $\pi$ are supposed to be greater than $\gamma_2$, we see that, for any prime factor of $m_i$, say $p$, the set of integers $n$ for which $f_i(n) \equiv 0 \pmod{p}$, but $f_i(n) \equiv 0 \pmod{p^2}$, constitutes $(p-1)\nu_i(p)$ residue classes modulo $p^2$. Hence the set of integers satisfying 1) and 2), or equivalently 1) and 2'), constitutes $\phi(m_1\cdots m_k)\nu_1(m_1)\cdots\nu_k(m_k)$ residue classes modulo $m_1^2\cdots m_k^2$.

Let these residue classes be expressed by the congruences

\[
n \equiv a_s \pmod{m_1^2 \cdots m_k^2} \quad (a=1, \ldots, s),
\]

where

\[
s = \phi(m_1\cdots m_k)\nu_1(m_1)\cdots\nu_k(m_k).
\]

We denote by $p_1, \ldots, p_h$ the primes which belong to the set $\pi$ and do not divide $m_1, \ldots, m_k$, then the number of integers $n \leq x$, for which

\[
n \equiv a_s \pmod{m_1^2 \cdots m_k^2},
\]

\[
f_1(n) \cdots f_h(n) \equiv 0 \pmod{p_j} \quad \text{for} \quad 1 \leq j \leq h,
\]

is nothing but the number $F(x; a, d; p_1, \ldots, p_h)$ considered in section 2, with $a=a_s$ and $d=m_1^2\cdots m_k^2$, and we have

\[
(3.14) \quad H(x; m_1, \ldots, m_k) = \sum_{s=1}^{\phi(m_1\cdots m_k)\nu_1(m_1)\cdots\nu_k(m_k)} F(x; a_s, m_1^2\cdots m_k^2; p_1, \ldots, p_h).
\]

Now we shall show that Lemma 2.10 can be applied to the summands on the right-hand side of (3.14).

Firstly, when $\varepsilon$ is given arbitrarily in the interval $0 < \varepsilon < 0.1$, and then $d=\delta(\varepsilon)$ and $x_i > x_1(\varepsilon)$ are taken as in Lemma 2.10, we consider the values of $x$ so large that $x > x_1$ and the inequalities
are satisfied. Then, since the primes \( p_1, \ldots, p_h \) belong to the set \( \pi \) and therefore satisfy (3.1), we have
\[
\max \left( \gamma_1, \frac{2l}{\varepsilon} \right) < x^{\log \log x}, \quad x^{1/3y \log \log x} \leq x^\theta,
\]

Secondly, if we recollect how we define the set \( \pi \), the numbers \( t_1, \ldots, t_k \), the sets \( M(t_1), \ldots, M(t_k) \) and the numbers \( m_1, \ldots, m_k \), we easily see that the numbers \( m_1, \ldots, m_k \) have the following properties:

The number of prime factors of \( m_i \) is less than \( 2r_i \log \log x \) for \( 1 \leq i \leq k \), and hence the number of prime factors of \( m_1^{\gamma_1} \cdots m_k^{\gamma_k} \) is less than \( 4r \log \log x \);

The prime factors of \( m_i \) are less than \( x^{1/3y \log \log x} \) for \( 1 \leq i \leq k \). Consequently we have
\[
m_1^{\gamma_1} \cdots m_k^{\gamma_k} < (x^{1/3y \log \log x})^{4r \log \log x} = x^\gamma.
\]

Now we can surely infer the conclusion of Lemma 2.10. Thus, for arbitrarily given \( \varepsilon \) in the interval \( 0 < \varepsilon < 0.1 \), we have
\[
F(x; \sigma, m_1^\gamma, \ldots, m_k^\gamma; p_1, \ldots, p_h) = \begin{cases} 
<(1+\varepsilon) \frac{x}{m_1^{\gamma_1} \cdots m_k^{\gamma_k}} \prod_{j=1}^h \left( 1 - \frac{\nu(p_j)}{p_j} \right), \\
>(1-\varepsilon) \frac{x}{m_1^{\gamma_1} \cdots m_k^{\gamma_k}} \prod_{j=1}^h \left( 1 - \frac{\nu(p_j)}{p_j} \right),
\end{cases}
\]
when \( x \) is sufficiently large. Further this result can be rewritten in the form
\[
F(x; \sigma, m_1^\gamma, \ldots, m_k^\gamma; p_1, \ldots, p_h) = (1+o(1)) \frac{x}{m_1^{\gamma_1} \cdots m_k^{\gamma_k}} \prod_{j=1}^h \left( 1 - \frac{\nu(p_j)}{p_j} \right),
\]
where the term \( o(1) \) tends to zero, as \( x \) tends to infinity, uniformly with respect to \( t_1, \ldots, t_k, m_1, \ldots, m_k \). From (3.14) and (3.15) we have
\[
H(x; m_1, \ldots, m_k) = (1+o(1)) \frac{x}{m_1^{\gamma_1} \cdots m_k^{\gamma_k}} \prod_{j=1}^h \left( 1 - \frac{\nu(p_j)}{p_j} \right).
\]
Since \( p_1, \ldots, p_h \) are the primes which belong to the set \( \pi \), but do not divide \( m_1, \ldots, m_k \), we can rewrite this equality in the form
\[
H(x; m_1, \ldots, m_k) = (1+o(1)) \frac{x}{m_1^{\gamma_1} \cdots m_k^{\gamma_k}} \prod_{p \in \pi, \nu(p) \neq 0} \left( 1 - \frac{\nu(p)}{p} \right),
\]
where \( m_1', \ldots, m_k' \) have the same meaning as in Lemma 3.3. From this and (3.13) we have further
\[
G'(x; t_1, \ldots, t_k) = (1+o(1)) \frac{x}{\prod_{p \in \pi, \nu(p) \neq 0} \left( 1 - \frac{\nu(p)}{p} \right)} \sum_{m_1, \ldots, m_k} \frac{\nu_1(m_1) \cdots \nu_k(m_k)}{m_1' \cdots m_k'}.\]
Since we assume that \( t_i < 2r_i \log \log x \) for \( 1 \leq i \leq k \), we can apply Lemma 3.3 to the sum on the right-hand side. Thus it follows that
\[
G'(x; t_1, \ldots, t_k) = (1+o(1)) \left\{ \frac{y_1^{t_1} \cdots y_k^{t_k}}{t_1! \cdots t_k!} + O \left( \frac{1}{\log x} \right) \right\} \frac{x}{\prod_{p \in \pi, \nu(p) \neq 0} \left( 1 - \frac{\nu(p)}{p} \right)} + O \left( \frac{x}{\log x} \right).
\]
Finally, from this and (3.12), we obtain the lemma including the uniformity of the terms \( o(1) \) and \( O(x/\log x) \).
Lemma 3.6. Let \( t_1, \ldots, t_k \) be positive integers such that \( t_i < 2r_i \log \log x \) for \( 1 \leq i \leq k \), then
\[
G(x; t_1, \ldots, t_k) = x^{\frac{e^{-(i_1 + \cdots + i_k) / 2} - \frac{1}{2}(i_1^2 + \cdots + i_k^2)}{t_1! \cdots t_k!}} \left( 1 + o(1) \right) + O \left( \frac{x}{\log x} \right).
\]
Both the terms \( o(1) \) and \( O(x/\log x) \) are uniform with respect to \( t_1, \ldots, t_k \).

Proof. On noting that the primes belonging to the set \( \pi \) satisfy (3.1), we have
\[
\sum_{\rho \in \pi} \frac{1}{\rho^2} = o(1),
\]
and therefore, on noting (3.5) and that \( \nu(p) = \nu_1(p) + \cdots + \nu_r(p) = O(1) \), we deduce
\[
\prod_{\rho \in \pi} \left( 1 - \frac{\nu(p)}{p} \right) = \exp \left( \sum_{\rho \in \pi} \log \left( 1 - \frac{\nu(p)}{p} \right) \right)
= \exp \left( - \sum_{\rho \in \pi} \frac{\nu(p)}{p} + O \left( \sum_{\rho \in \pi} \frac{1}{p^2} \right) \right)
= \exp \left( - \sum_{\rho \in \pi} \left( \frac{p-1}{p} \nu(p) + O \left( \sum_{\rho \in \pi} \frac{1}{p^2} \right) \right) \right)
= \exp \left( - (\nu_1 + \cdots + \nu_r) + o(1) \right).
\]
From this and Lemma 3.5, we obtain the lemma including the uniformity.

Lemma 3.7. If we put \( t_i = y_i + u_i \sqrt{y_i}, \) \( \alpha_i < u_i < \beta_i \) where \( \alpha_i < \beta_i \), \( 1 \leq i \leq k \), are arbitrarily given real numbers, then
\[
G(x; t_1, \ldots, t_k) = (2\pi)^{-1/2} x^{\pi/2 - 1/2 + u_1^2 + \cdots + u_k^2} \left( 1 + o(1) \right) + O \left( \frac{x}{\log x} \right).
\]
Both the terms \( o(1) \) and \( O(x/\log x) \) are uniform with respect to \( t_1, \ldots, t_k \), that is, \( u_1, \ldots, u_k \).

Proof. When \( t \) is a positive integer, we have
\[
t! = \sqrt{2\pi} t^{t+1/2} e^{-t} \left( 1 + O \left( \frac{1}{t} \right) \right)
\]
(Stirling's formula).

Here we put \( t = y + u \sqrt{y} \) and consider large \( y \), leaving \( u \) contained in a finite interval, then we obtain by easy calculations
\[
t! = \sqrt{2\pi} y^{y+1} e^{-y+u/2} \left( 1 + O \left( \frac{1}{\sqrt{y}} \right) \right),
\]
so that
\[
\frac{e^{-uy^t}}{t!} = \frac{e^{-u^2/2}}{\sqrt{2\pi} y} \left( 1 + O \left( \frac{1}{\sqrt{y}} \right) \right).
\]
Putting \( t = t_i, y = y_i, u = u_i, 1 \leq i \leq k \), and combining the formulas thus obtained, we have
\[
(3.16) \quad \frac{e^{-uy_1^t + \cdots + u_k^t}}{t_1! \cdots t_k!} = (2\pi)^{-1/2} (y_1 \cdots y_k)^{-1/2} e^{-1/2(u_1^2 + \cdots + u_k^2)} \left( 1 + o(1) \right).
\]
Since \( u_i, 1 \leq i \leq k \), are supposed to be bounded, it follows from Lemma 3.2 that \( t_i < 2r_i \log \log x \) for \( 1 \leq i \leq k \), when \( x \) is large enough. Therefore Lemma 3.6 is applicable to the present case, and from Lemma 3.6 and (3.16), we establish the lemma including the uniformity.

Lemma 3.8. Let \( \alpha_i < \beta_i, 1 \leq i \leq k \), be arbitrarily given real numbers, and let \( A^*(x) = A^*(x; \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k) \) denote the number of positive integers \( n \leq x \) for which
\[
(3.17) \quad y_i + \alpha_i \sqrt{y_i} < \omega'(f_i(n)) < y_i + \beta_i \sqrt{y_i} \quad (1 \leq i \leq k)
\]
simultaneously, then we have
Proof. From the meaning of $G(x; t_1, ..., t_k)$, we have at once
\[(3.18) \quad A^*(x) = \sum_{i_1 + i_2 + \cdots + i_k \leq k} G(x; t_1, ..., t_k),\]
where the summation-variables are $k$ numbers $t_1, ..., t_k$, and the number of the summands is $O((\log \log x)^{k/2})$ from Lemma 3.2.

Now let the integral values of $t_i$ taken in the above sum be named by $t_{ij}$, $j=1, ..., s_i$ in the order of their magnitudes. Further we put $t_{ij} = y_i + u_{ij}/y_i$, then
\[u_{ij} = \frac{1}{\sqrt{y_i}}.\]
With these notations, from (3.18) and Lemma 3.7, we obtain
\[A^*(x) = (1 + o(1)) \sum_{i=1}^{k} \sum_{j=1}^{s_i} e^{-u_{ij}^2/2} (y_i/\sqrt{y_i}) + O\left(\frac{(\log \log x)^{k/2}}{\log x}\right).\]
Making $x \to \infty$ in this equality, we obtain the lemma.

Lemma 3.9. Let $\alpha_i < \beta_i$, $1 \leq i \leq k$, be arbitrarily given real numbers, and let $A(x) = A(x; \alpha_1, \beta_1, ..., \alpha_k, \beta_k)$ denote the number of integers $n$, $3 \leq n \leq x$, for which $r_i \log \log n + \alpha_i < \beta_i \sqrt{r_i} \log \log n < r_i \log \log n + \alpha_i$ simultaneously, then we have
\[\lim_{x \to \infty} \frac{A^*(x)}{x} = (2\pi)^{-k/2} \prod_{i=1}^{k} \int_{-\infty}^{\beta_i/2} e^{-u_i^2/2} \, du_i.\]
Proof. Notice that, for $\sqrt{x} < n \leq x$, we have $\log \log x - \log 2 < \log \log n < \log \log x$. From Lemma 3.2 and this fact, we can replace $y_i$ with $\frac{1}{\sqrt{y_i}}$. On the other hand, from Lemma 3.1, we can replace $\omega'$ with $\omega$. Thus we can easily derive Lemma 3.9 from Lemma 3.8. The details will be omitted.

Lemma 3.9 is a special case of Theorem A when the set $E$ is an interval $\alpha_i < u_i < \beta_i$, $1 \leq i \leq k$. Now Theorem A with a Jordan-measurable set $E$ can easily be deduced from this special case.

The Proof of Theorem A. First we consider the case when the set $E$ is bounded. We consider two systems of intervals finite in number, say $I_{\mu}$, $\mu=1, 2, ..., I_{\mu}'$, $\mu=1, 2, ...$, such that
\[\sum_{\mu} I_{\mu} \subset E \subset \sum_{\mu} I_{\mu}'.\]
and any two of $I_{\mu}$ do not overlap. Then we have obviously
\[\sum_{\mu} A(x; I_{\mu}) \leq A(x; E) \leq \sum_{\mu} A(x; I_{\mu}').\]
Now we apply Lemma 3.9 to the intervals $I_{\mu}$, $I_{\mu}'$, then we obtain
\[(3.19) \quad (2\pi)^{-k/2} \sum_{\mu} \int_{I_{\mu}} e^{-\frac{1}{2} \sum_{i=1}^{k} u_i^2} \, du_1 \cdots \, du_k \leq \liminf_{x \to \infty} \frac{A(x; E)}{x} \leq \limsup_{x \to \infty} \frac{A(x; E)}{x} \leq (2\pi)^{-k/2} \sum_{\mu} \int_{I_{\mu}'} e^{-\frac{1}{2} \sum_{i=1}^{k} u_i^2} \, du_1 \cdots \, du_k.\]
But, for an arbitrarily given positive $\varepsilon$, we can take the intervals $I_{\mu}$, $I_{\mu}'$ such that
\[(3.20) \quad (2\pi)^{-k/2} \int_E -\varepsilon < (2\pi)^{-k/2} \sum_{\mu} \int_{I_{\mu}} \leq (2\pi)^{-k/2} \sum_{\mu} \int_{I_{\mu}'} < (2\pi)^{-k/2} \int_E + \varepsilon,\]
omitting the common integrands. From (3.19), (3.20) it follows that 
\[(2\pi)^{-k/2} \int_E -\varepsilon < \lim \inf_{x \to \infty} \frac{A(x; E)}{x} \leq \lim \sup_{x \to \infty} \frac{A(x; E)}{x} < (2\pi)^{-k/2} \int_E +\varepsilon .\]
On making $\varepsilon \to 0$, we obtain
\[\lim_{x \to \infty} \frac{A(x; E)}{x} = (2\pi)^{-k/2} \int_E .\]

Secondly we consider the case when the set $E$ is not bounded. From Lemma 3.9 we see that, for an arbitrarily given positive $\varepsilon$, we can find an interval $I$ such that
\[\lim_{x \to \infty} \frac{A(x; I)}{x} = (2\pi)^{-k/2} \int_I > 1 - \varepsilon .\]
It follows easily from this that
\[(3.21) \quad \lim_{x \to \infty} \frac{A(x; E - E \cdot I)}{x} = \varepsilon ,
(3.22) \quad (2\pi)^{-k/2} \int_{E - E \cdot I} < \varepsilon .\]
But, since the set $E \cdot I$ is bounded, it is already proved that
\[(3.23) \quad \lim_{x \to \infty} \frac{A(x; E \cdot I)}{x} = (2\pi)^{-k/2} \int_{E \cdot I} .\]
From (3.21), (3.22) and (3.23) we deduce
\[\lim_{x \to \infty} \frac{A(x; E)}{x} \geq \lim_{x \to \infty} \frac{A(x; E \cdot I)}{x} = (2\pi)^{-k/2} \int_{E \cdot I} > (2\pi)^{-k/2} \int_E -\varepsilon ,
\lim_{x \to \infty} \frac{A(x; E)}{x} = \lim_{x \to \infty} \frac{A(x; E \cdot I)}{x} + \lim \sup_{x \to \infty} \frac{A(x; E - E \cdot I)}{x}
< (2\pi)^{-k/2} \int_{E \cdot I} +\varepsilon \leq (2\pi)^{-k/2} \int_E +\varepsilon .\]
Finally, making $\varepsilon \to 0$, we obtain
\[\lim_{x \to \infty} \frac{A(x; E)}{x} = (2\pi)^{-k/2} \int_E .\]
The proof of Theorem A is now completed.

4. Some Special Cases.

We shall mention some special cases of Theorem A.

**Theorem 1.** Let $\alpha, \beta$ be arbitrarily given real numbers, and let $A(x) = A(x; \alpha, \beta)$ denote the number of integers $n$, $3 \leq n \leq x$, for which
\[\log \log n + \alpha \sqrt{\log \log n} < \omega(n) < \log \log n + \beta \sqrt{\log \log n},\]
then we have
\[\lim_{x \to \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du .\]
This theorem was first proved by Erdős and Kac [3] by use of the central limit theorem. Subsequently LeVeque [8], Delange [1] and Halberstam [4] gave alternative proofs, also by the help of probability theory. It will be worth mentioning that this theorem contains classical result proved by Hardy and Ramanujan [5] and subsequently by Turán [11] to the effect that $\omega(n)$ has the so-called normal order $\log \log n$. 

...
Theorem 2. Let $\alpha, \beta$ be arbitrarily given real numbers, and let $A(x) = A(x; \alpha, \beta)$ denote the number of integers $n$, $3 \leq n \leq x$, for which
\[ \log \log n + \alpha \sqrt{\log \log n} < \omega(n^2+1) < \log \log n + \beta \sqrt{\log \log n} , \]
then we have
\[ \lim_{x \to \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-u^2/2} du . \]

Theorem 3. Let $\alpha, \beta$ be arbitrarily given real numbers, and let $A(x) = A(x; \alpha, \beta)$ denote the number of integers $n$, $3 \leq n \leq x$, for which
\[ 2 \log \log n + \alpha \sqrt{2 \log \log n} < \omega(n^2-1) < 2 \log \log n + \beta \sqrt{2 \log \log n} , \]
then we have
\[ \lim_{x \to \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-u^2/2} du . \]

Theorems 2 and 3 seem new to the author.

Theorem 4. Let $\alpha_i, \beta_i, 1 \leq i \leq k$, be arbitrarily given real numbers, and let $A(x) = A(x; \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)$ denote the number of integers $n$, $3 \leq n \leq x$, for which
\[ \log \log n + \alpha_i \sqrt{\log \log n} < \omega(n+i) < \log \log n + \beta_i \sqrt{\log \log n} \quad (1 \leq i \leq k) \]
simultaneously, then we have
\[ \lim_{x \to \infty} \frac{A(x)}{x} = (2\pi)^{-k/2} \prod_{i=1}^{k} \int_{\alpha_i}^{\beta_i} e^{-u^2/2} du . \]

Theorem 5. Let $\alpha_i, \beta_i, 1 \leq i \leq k$, be arbitrarily given real numbers, and let $A(x) = A(x; \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)$ denote the number of integers $n$, $3 \leq n \leq x$, for which
\[ \log \log n + \alpha_i \sqrt{\log \log n} < \omega(n^2+i) < \log \log n + \beta_i \sqrt{\log \log n} \quad (1 \leq i \leq k) \]
simultaneously, then we have
\[ \lim_{x \to \infty} \frac{A(x)}{x} = (2\pi)^{-k/2} \prod_{i=1}^{k} \int_{\alpha_i}^{\beta_i} e^{-u^2/2} du . \]

Theorem 6. Let $\alpha_i, \beta_i, 1 \leq i \leq k$, be arbitrarily given real numbers, and let $A(x) = A(x; \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)$ denote the number of integers $n$, $\max(3, \sqrt{k+1}) \leq n \leq x$, which satisfy simultaneously the inequalities
\[ \log \log n + \alpha_i \sqrt{\log \log n} < \omega(n^2-i) < \log \log n + \beta_i \sqrt{\log \log n} \]
for non-square $i$, $1 \leq i \leq k$, and
\[ 2 \log \log n + \alpha_i \sqrt{2 \log \log n} < \omega(n^2-i) < 2 \log \log n + \beta_i \sqrt{2 \log \log n} \]
for square $i$, $1 \leq i \leq k$. Then we have
\[ \lim_{x \to \infty} \frac{A(x)}{x} = (2\pi)^{-k/2} \prod_{i=1}^{k} \int_{\alpha_i}^{\beta_i} e^{-u^2/2} du . \]

Next we consider an arbitrary permutation of $k$ numbers $1, 2, \ldots, k$, say
\[ P = (a_1, a_2, \ldots, a_k) . \]

Then we have

Theorem 7. Let $B(x) = B(x; P)$ be the number of positive integers $n \leq x$ for which
\[ \omega(n+a_1-1) < \omega(n+a_2-1) < \cdots < \omega(n+a_k-1) , \]
then we have
\[ \lim_{x \to \infty} \frac{B(x)}{x} = \frac{1}{k!} . \]
Theorem 8. Let \( B(x; P) \) be the number of positive integers \( n \leq x \) for which
\[
\omega(n^2 + a_1) < \omega(n^2 + a_2) < \cdots < \omega(n^2 + a_k),
\]
then we have
\[
\lim_{x \to \infty} \frac{B(x)}{x} = \frac{1}{k!}.
\]

Theorems 4 and 7 were proved by LeVeque \cite{8} by the help of the so-called
Fourier-Stieltjes transform in the probability theory. Theorems 5, 6 and 8 seem
new to the author.

There are further a lot of curious theorems which may be obtained as special
cases of Theorem A, for instance,

Theorem 9. Let \( B_1(x), B_2(x) \) denote the numbers of positive integers \( n \leq x \), for
which \( \omega(n) < \omega(n^2 + 1) \) or \( \omega(n) > \omega(n^2 + 1) \) respectively, then we have
\[
\lim_{x \to \infty} \frac{B_1(x)}{x} = \lim_{x \to \infty} \frac{B_2(x)}{x} = \frac{1}{2}.
\]

Finally we shall briefly refer to the total number of prime factors of a positive
integer. Let us denote namely by \( \Omega(n) \) the number of prime factors of \( n \), multiple
factors being counted multiply. We shall put, for \( 1 \leq i \leq k \),
\[
\rho_i = \sum_{j=1}^{k} a_{ij}, \quad \sigma_i = \sqrt{\sum_{j=1}^{k} a_{ij}^2},
\]
where \( a_{ij} \) have the meaning as in (1.1). With these \( \rho_i \) and \( \sigma_i \), we shall put, for
\( n \geq 3 \) and \( 1 \leq i \leq k \),
\[
\frac{\Omega(f_i(n)) - \rho_i \log \log n}{\sigma_i \sqrt{\log \log n}} = u_i(n).
\]

Then we can obtain a theorem concerning \( \Omega(n) \), which reads just as Theorem A.
Consequently the theorems mentioned in this section remain true when we replace
\( \omega(n) \) by \( \Omega(n) \), and leave all the other words as they are. Further we may replace
\( \omega(n) \) in theorem 7, 8 and 9 also by \( d(n) \), the number of positive divisors of \( n \). The
detailed proof of these results will be published elsewhere.

Jiyu-Gakuen, Tokyo

References.

(1953), pp. 542-544.

(1937), pp. 308-314.

[3] P. Erdős and M. Kac: The Gaussian law of errors in the theory of additive number-


[5] G.H. Hardy and S. Ramanujan: The normal number of prime factors of a number \( n \),


Notes.

1) This condition (C3) may be weakened to the following (C3'): (C3') The leading coefficient of each $f_i(\xi)$, $1 \leq i \leq k$ is positive. Then there exists obviously a certain integer $n_0 \geq 3$ such that $f_i(\xi) > 0$ for all $\xi \geq n_0$, $1 \leq i \leq k$. We have then only to define $A(x; E)$ in the main theorem as the number of integers $n$, $n_0 \leq n \leq x$, for which $(u_1(n), \ldots, u_k(n))$ belong to the set $E$.

2) An unbounded set $E$ is said to be Jordan-measurable, if, for any interval $I$, the product $E \cdot I$ is Jordan-measurable.

3) Cf., for instance, Hardy and Wright [6], p. 97, Theorem 123.

4) This inequality may be valid for $\lambda=0$, stipulating that $0<\varepsilon<1$.

5) Landau utilizes a clever manner of abbreviating the formulas.

6) $p_1, \ldots, p_k$ need not be all the primes between the limits. By the way we mention that we can weaken the conditions by replacing $0<\varepsilon<0.1$ by $\varepsilon>0$ if we desire.

7) Cf. Landau [7], p. 43, the proof of Satz 75.

8) More precisely, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ are positive numbers depending only on the coefficients of $f_i(\xi)$, $1 \leq i \leq k$, and $\varepsilon$.


10) Notice that $m_1, \ldots, m_k$ are squarefree.

11) Notice that $m_1, \ldots, m_k$ are squarefree and relatively prime in pairs. $\varphi$ denotes Euler's function. $\nu_1(m_1), \ldots, \nu_k(m_k)$ have the same meaning as in Lemma 3.3.

12) $\pi$ is as usual the number 3.1415......

13) Lemma 2.3 may be proved without using algebraic number theory, when $\varphi(\xi)$ is quadratic.