2. Affine connections on manifolds with almost complex, quaternion or Hermitian structure.

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The existence of an affine connection with respect to which given vector or tensor fields are covariant constant is one of the most important problems in the theory of connections. The problem of finding a metric connection in a given Riemannian manifold with the metric $g_{ab}$\(^1\)\(^2\) has been completely solved, i.e. the Christoffel symbol $\{^h_i^j\}$ belonging to $g_{ab}$ gives the unique metric connection without torsion and any metric connection $\Gamma^h_{ij}$ with the torsion $S^h_{ij}$ is given by

$$\Gamma^h_{ij} = \{^h_i^j\} + g^{kh}(S_{jkh} - S_{kjh} + S_{hjk})^{ij},$$

where $S_{jkh} = S_{jk}^h g_{ah}$. Some considerations have been also given on affine connections with respect to which given vector fields or general linear fields\(^4\) are covariant constant.

On the other hand, an almost complex structure in an even-dimensional manifold is defined by the existence of a tensor field $\phi^h_i$ satisfying $\phi^a_i \phi^h_a = -\delta^h_i$ and consequently affine connections leaving invariant the almost complex structure, called $\phi$-connections in this paper, are very useful for the study of the differential geometry on the almost complex manifold.\(^5\) Much have been studied concerning the existence of $\phi$-connections. Among others A. Frölicher has given a method of constructing a $\phi$-connection from any affine connection given a priori. Remarkable is that applying his method to a symmetric affine connection a $\phi$-connection with the torsion tensor proportional to the Nijenhuis tensor\(^6\) is obtained.

In this paper we shall treat in a different way from Frölicher's and systematically the problem on manifolds with almost complex, quaternion\(^7\) and almost Hermitian structure. In all cases, the problem is, on the common principle, reduced to solving linear equations and by means of their general solutions every affine connection is characterized.

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1) As to the notations, we follow Schouten \([23]\) in principle. The numbers between brackets refer to the Bibliography at the end of the paper.

2) In this paper we shall restrict ourselves to manifolds which are of differentiable class $C^\infty$ and satisfy the second axiom of countability. In such a manifold there always exists a Riemannian metric \((\text{Steenrod} \ [26])\).

3) For instance, see Schouten \([23]\).

4) Cf. Ehresmann \([8]\), Ishihara and Obata \([13]\), Willmore \([27]\).

5) Cf. Bochner \([2]\), Chern \([4, 5]\), Frölicher \([9]\), Fukami and Ishihara \([10]\), Guggenheimer \([11]\), Libermann \([15, 16]\), Lichnerowicz \([17, 18]\), Obata \([21]\), Patterson \([22]\), Schouten and van Dantzig \([24]\), Schouten and Yano \([25]\), Yano \([29, 30, 31]\).

6) Cf. Eckmann and Frölicher \([6]\), Nijenhuis \([19]\).

7) Cf. Ehresmann \([7]\), Libermann \([15, 16]\).
connection leaving invariant the given structures is written out explicitly. It will be shown that the torsion tensor of such a connection is completely characterized by its relations with the tensors intrinsically defined by the structures, for instance the Nijenhuis tensor in an almost complex manifold and the tensor $\phi_{\mu}{}^{\alpha}{}_{\beta}$ in an almost Hermitian manifold. Furthermore we shall define the special connections with the torsion tensors constructed by only tensors intrinsically defined by the structures. In the complex analytic case, our process of construction is proved to be natural.

It is to be remarked that any affine connection constructed in this paper is always defined globally over the manifold, because we use only quantities whose existence in the large is proved.

In Chap. I we shall study the quaternion structure in a complex analytic manifold and give an integrability condition of the structure. This is expressed by the vanishing of three tensors closely related to the torsion tensor and the curvature tensor.

In Chap. II we shall explain our method in an almost complex manifold, which is the model in the other cases.

In Chap. III the process analogous to Chap. II will be given in the case of quaternion structure. We shall give three special connections whose torsion tensors are written by three Nijenhuis tensors derived from the quaternion structure and also show that there exists a one-to-one correspondence between the set of all connections in a quaternion manifold and a set of tensors of the type (1,2).

In Chap. IV applying our method to metric connections we shall study affine connections in an almost Hermitian manifold. A remarkable connection, possibly new, is obtained. Its torsion tensor is in intimate relation with the integrability of the almost Hermitian structure. As an appendix a pseudo-Kählerian space with quaternion structure will be referred to.

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**CHAPTER I**  **Almost complex structure and quaternion structure**

1. Almost complex structure

We consider a $2n$-dimensional differentiable manifold of class $C^\infty$. By an *almost complex structure* we shall mean a tensor field $\phi_{\mu}{}^{\alpha}{}_{\beta}$ of class $C^\infty$ satisfying

$$\phi_{\mu}{}^{\alpha}{}_{\beta} = -\partial_{\mu}{}^{\alpha}.$$

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8) $\alpha, b, c, \cdots, i, j, k, \cdots = 1, 2, \cdots, n$, $\bar{1}, \bar{2}, \cdots, \bar{n}$;
$\alpha, \beta, \gamma, \cdots, \kappa, \lambda, \mu, \cdots = 1, 2, \cdots, n$;
$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots, \bar{\kappa}, \bar{\lambda}, \bar{\mu}, \cdots = 1, 2, \cdots, \bar{n}$. 

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The following theorem is well-known. 45

**Theorem 1.1** In order that an almost complex structure $\phi_i^h$ of class $C^\infty$ in a manifold of class $C^\infty$ give a complex analytic structure, i.e., that there exist a complex coordinate system $(z^k, \bar{z}^\ell)_{10}$ in which $\phi_i^h$ has numerical components

$$ (I_i^h) = \begin{pmatrix} i\partial_{x^k} & 0 \\ 0 & -i\partial_{\bar{x}^\ell} \end{pmatrix}, $$

it is necessary and sufficient that the tensor $\phi_i^h$ satisfy

$$ N_{ij}^h(\phi) = \frac{1}{2}(\phi_{[i}^c \partial_{j]}^a \phi_{i}^b - \phi_{c}^j \partial_{[i}^a \phi_{j]}^b) = 0. $$

If, in an almost complex manifold of class $C^\infty$, the Nijenhuis tensor $N_{ij}^h(\phi)$ vanishes identically, the manifold is called a pseudo-complex manifold and $\phi_i^h$ is called a pseudo-complex structure.

It has been also known that in order for $I_i^h$ to be numerical components of a tensor, it is necessary and sufficient that the coordinate transformation $z^k = f^k(z, \bar{z})$, $\bar{z}^{\ell} = \bar{f}^{\ell}(z, \bar{z})$ satisfy

$$ \frac{\partial f^k}{\partial z^{\ell}} = -\frac{\partial \bar{f}^{\ell}}{\partial z^k} = 0, $$

or equivalently that $z^k$ be complex analytic functions of $\bar{z}^{\ell}$:

$$ z^k = f^k(z) $$
and consequently $\bar{z}^{\ell} = \bar{f}^{\ell}(z)_{11}$.

Now if $T_{ik}^{\alpha h} = (T_{ik}^{\alpha x}, T_{ik}^{\alpha \bar{x}}, T_{ik}^{\alpha x}, \ldots, T_{ik}^{\alpha \bar{x}})$ are components of a tensor, then $(T_{ik}^{\alpha x}, 0, \ldots, 0), (0, T_{ik}^{\alpha \bar{x}}, 0, \ldots, 0)$ etc. are also components of tensors of the same kind as the original one. Moreover $T_{ik}^{\alpha h}_{12}$ are also components of a tensor. We call any quantity $T_{ik}^{\alpha h}$ self-adjoint, if

$$ T_{ik}^{\alpha h} = \overline{T_{ik}^{\alpha h}} $$

The self-adjoint quantity represents a real quantity in the real coordinate system and vice versa.

2. Quaternion structure

Let us consider a $2n$-dimensional differentiable manifold of class $C^\infty$ admitting two almost complex structures $\phi_i^h$ and $\psi_i^h$ of class $C^\infty$ satisfying

$$ \phi_i^h \psi_a^b + \psi_i^h \phi_a^b = 0. $$

Such a manifold is called a quaternion manifold and the pair $(\phi_i^h, \psi_i^h)$ satisfying (2.1) is called a quaternion structure.

The tensors $\psi_i^h$ and $\psi_i^h$ define linear transformations $I_p$ and $J_p$ respectively in

9) Cf. Calabi and Spencer [3], Eckmann and Fröllicher [6], Frölicher [9], Guggenheimer [11], Hodge [12], Libermann [15], Yano [29].
10) The bar on the central letter denotes the complex conjugate.
11) $\bar{f}^k(\bar{z})$ denotes the complex conjugate of the complex analytic function $f^k(z)$.
12) $\bar{k} = \bar{x}$ if $h = x$ and $\bar{k} = \bar{x}$ if $h = \bar{x}$.
13) Cf. Ehresmann [7], Libermann [15, 16].
the tangent space $T_p$ at each point $p$ of the manifold. Let $T_p^c$ be the complexification of $T_p$. The linear transformations $I_p$ and $J_p$ are naturally extended to the transformations in $T_p^c$. Since $I_p^2 = -E_p$, $E_p$ being the identity transformation, $I_p$ has $n$ eigenvalues $\pm i$ and $n$ eigenvalues $-i$. We denote by $C_p$ and $\overline{C}_p$ the eigenspaces of $+i$ and $-i$ respectively. Then $C_p + \overline{C}_p = T_p^c$ (direct). If $\{e_1, \ldots, e_n\}$ is a base of $C_p$, then $\{\overline{e}_1, \ldots, \overline{e}_n\}$ is a base of $\overline{C}_p$. We have, by the definition of the base,

\[(2.2) \quad I_pe_\alpha = +ie_\alpha, \quad I_p\overline{e}_\alpha = -i\overline{e}_\alpha\]

i.e. $I_p$ is represented by the matrix

\[(2.3) \quad \begin{pmatrix} \partial_{\alpha} & 0 \\ 0 & -i\partial_{\alpha} \end{pmatrix}.
\]

Now (2.1) gives

\[(2.4) \quad I_pJ_p = -J_pI_p,
\]

from which it follows that $J_p$ is represented by the matrix of the form

\[(2.5) \quad \begin{pmatrix} \psi_{\alpha}^{\alpha} \\ \psi_{\alpha}^{\overline{\alpha}} \\ \psi_{\overline{\alpha}}^{\alpha} \\ \psi_{\overline{\alpha}}^{\overline{\alpha}} \end{pmatrix}, \quad \psi_{\alpha}^{\alpha} = \psi_{\overline{\alpha}}^{\overline{\alpha}}.
\]

This shows that $J_pC_p \subset \overline{C}_p$ and $J_p\overline{C}_p \subset C_p$. Moreover it is known\(^{14}\) that $n$ is even, $n = 2m$, and we can find a base $\{f_1, \ldots, f_n, \overline{f}_1, \ldots, \overline{f}_n\}$ such that

\[(2.6) \quad \begin{align*}
I_pf_\alpha &= +if_\alpha, \quad I_p\overline{f}_\alpha = -i\overline{f}_\alpha, \\
J_pf_r &= \overline{f}_{r+m}, \quad J_p\overline{f}_{r+m} = -\overline{f}_r \quad (r = 1, 2, \ldots, m), \\
J_pf_r &= f_{r+m}, \quad J_p\overline{f}_{r+m} = -f_r,
\end{align*}
\]

i.e. $J_p$ is represented, with respect to this base, by the matrix

\[(2.7) \quad \begin{pmatrix} 0 & 0 & 0 & -\partial_{\alpha} \\ 0 & 0 & \partial_{\alpha} & 0 \\ 0 & -\partial_{\alpha} & 0 & 0 \\ \partial_{\alpha} & 0 & 0 & 0 \end{pmatrix}.
\]

Conversely if in an almost complex manifold there exists a differentiable field of linear transformations $J_p$ such that

\[(2.8) \quad J_pC_p \subset \overline{C}_p, \quad J_p\overline{C}_p \subset C_p \quad \text{and} \quad J_p^{*} = -E_p,
\]

in the tangent space $T_p$ at $p$, then $n$ is even, $n = 2m$, and we have

\[(2.9) \quad I_pJ_p = -J_pI_p.
\]

Thus the field of $J_p$ defines a tensor field $\psi_i^h$ satisfying (2.1) and $\phi_i^a\psi_a^h = -\partial_i^h$. That is, in an almost complex manifold the fields of $I_p$ and $J_p$ satisfying (2.8) define a quaternion structure $(\phi_i^a, \psi_i^h)$, which is evidently unique.

Now if we put $\kappa_i^h = \phi_i^a\psi_a^h$, we see that $\kappa_i^h$ is also an almost complex structure $\kappa_i^h\kappa_a^h = -\partial_i^h$ and we have

\(^{14}\) Cf. Obata [20, 21].
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To the tensor fields $\phi^i_h$, $\psi^i_h$, $\kappa^i_h$ in a quaternion manifold we can assign the usual base $1, i, j, k$ of the algebra of quaternions over the real number field:

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The term "quaternion" is used on this account.

3. Integrability of $\psi^i_h$

We consider a $4m$ $(n=2m)$ dimensional quaternion manifold defined by $(\phi^i_h, \psi^i_h)$. We assume in §§ 3-5 that $\phi^i_h$ and $\psi^i_h$ as well as the manifold are of class $C^\omega$.

We know that the almost complex structure $\phi^i_h$ is integrable, i.e. it gives a complex analytic structure if and only if it satisfies

$$(3.1) \quad N_{ij}^h(\phi) = \frac{1}{2} (\psi_{ij}^a \partial_{\alpha} \psi^a - \psi_{ij}^a \partial_{\alpha} \phi^a) = 0.$$

Next we assume that the almost complex structure $\phi^i_h$ gives a complex analytic structure and choose a complex analytic coordinate system $(z^\kappa, \bar{z}^\kappa)$. Then $\phi^i_h = I^i_h$ and from (2.1) we see that $\psi^i_h$ has components of the form

$$\psi^i_h = \begin{pmatrix} 0 & \psi^i_{\kappa} \bar{z}^\kappa \\ \psi^i_{\bar{\kappa}} z^\kappa & 0 \end{pmatrix}, \quad \psi^i_{\kappa} = \overline{\psi^i_{\bar{\kappa}}},$$

in all complex analytic coordinate systems. Here $\psi^i_{\kappa}(z, \bar{z})$ are analytic functions of $z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n$.

Differentiating (3.3) with respect to $z^\kappa$, we obtain

$$(3.4) \quad (\partial_{\kappa} \psi^i_{\bar{\kappa}}) \psi^i_{\bar{z}^\kappa} + \psi^i_{\bar{\kappa}} \partial_{\kappa} \phi^i_{\bar{z}^\kappa} = 0.$$

If, taking account of (3.3), (3.4) and the tensor character of $N_{ij}^h(\phi)$, we write out $N_{ij}^h(\psi)$ in a complex analytic coordinate system, we have

$$(3.5a) \quad 2N_{\kappa \bar{\mu}}^h(\psi) = -\phi^i_{\kappa \bar{\mu}} \partial_{\kappa \bar{\mu}} \phi^i_{\bar{z}^\kappa} = - (\partial_{\bar{\kappa}} \phi^i_{\bar{\mu} \bar{\kappa}}) \phi^i_{\bar{z}^\kappa},$$

$$(3.5b) \quad 2N_{\kappa \bar{\mu}}^h(\psi) = -\frac{1}{2} (\phi^i_{\kappa \bar{\mu}} \partial_{\kappa \bar{\mu}} \psi^i_{\bar{z}^\kappa} - \phi^i_{\bar{\mu} \bar{\kappa}} \partial_{\kappa \bar{\mu}} \psi^i_{\bar{z}^\kappa}) = \phi^i_{\bar{\mu}} \partial_{\kappa \bar{\mu}} \psi^i_{\bar{z}^\kappa},$$

$$(3.5c) \quad 2N^h_{\bar{\kappa} \bar{\mu}}^h(\psi) = \phi^i_{\bar{\mu}} \partial_{\bar{\kappa} \bar{\mu}} \phi^i_{\bar{z}^\kappa} = \psi^i_{\bar{\kappa}} \partial_{\bar{\kappa} \bar{\mu}} \phi^i_{\bar{z}^\kappa} = \psi^i_{\bar{\mu}} \partial_{\bar{\kappa} \bar{\mu}} \phi^i_{\bar{z}^\kappa},$$

the others being given by skew-symmetry with respect to the lower indices and self-adjointness. Since $\psi^i_{\kappa}$ has the inverse, the condition $N_{ij}^h(\psi) = 0$ is equivalent to

$$(3.6) \quad \partial_{\kappa} \phi^i_{\bar{\kappa}} = 0; \quad \text{conj.}$$

Thus we have

**Theorem 3.1** If a 4m-dimensional real analytic manifold has a quaternion structure $(\phi^i_h, \psi^i_h)$ of class $C^\omega$ and if $\phi^i_h$ is integrable, then a necessary and suf-
4. Integrability of a quaternion structure

We consider a 4m-dimensional quaternion manifold defined by \((\phi^h, \psi^h)\). If there exists a coordinate system in which the fields \(\phi^h\) and \(\psi^h\) have simultaneously numerical components, the quaternion structure \((\phi^h, \psi^h)\) is said to be integrable. If \(\phi^h\) has numerical components in some coordinate system, then there exists a system in which \(\phi^h = I^h\). Since this applies for \(\psi^h\), for a quaternion structure \((\phi^h, \psi^h)\) to be integrable, it is necessary that the Nijenhuis tensors \(N_{ij}^h(\phi)\) and \(N_{ij}^h(\psi)\) vanish identically.

Now we assume that \(\phi^h\) is integrable and choose a complex analytic coordinate system \((z^a, \bar{z}^a)\). Then \(\phi^h = I^h\) and \(\psi^h = (\psi^h_a, \psi^h_{\bar{a}})\). We construct

\[
M_{\mu \nu}^{\kappa} = \partial_{\nu}(\partial_{\mu} \phi^h_a) \phi^h_{\bar{a}} = (\partial_{\nu}(\partial_{\mu} \phi^h_a) \phi^h_{\bar{a}} + (\partial_{\nu} \phi^h_{\bar{a}})(\partial_{\mu} \phi^h_a)); \quad \text{conj.}
\]

The quantity \(M_{\mu \nu}^{\kappa} = (M_{\mu \kappa}^{\nu}, M_{\nu \kappa}^{\mu})\) is self-adjoint and has the tensor character.

To prove this we use the covariant differentiation \(\nabla_{\alpha} \) with respect to an arbitrary affine connection\(^{16}\) \(\Gamma_{\mu \nu}^\alpha = (\Gamma_{\mu \nu}^\alpha, \Gamma_{\nu \mu}^\alpha)\), which can always be introduced in a complex analytic manifold. We construct analogously to (4.1)

\[
\tilde{M}_{\mu \kappa}^{\nu} = \nabla_{\nu}(\nabla_{\mu} \phi^h_a) \phi^h_{\bar{a}} = (\nabla_{\nu}(\nabla_{\mu} \phi^h_a) \phi^h_{\bar{a}} + (\nabla_{\nu} \phi^h_{\bar{a}})(\nabla_{\mu} \phi^h_a)); \quad \text{conj.}
\]

Then the quantity \(\tilde{M}_{\mu \kappa}^{\nu} = (\tilde{M}_{\mu \kappa}^{\nu}, \tilde{M}_{\nu \kappa}^{\mu})\) is self-adjoint and has evidently the tensor character. We shall show that

\[
M_{\mu \kappa}^{\nu} = \tilde{M}_{\mu \kappa}^{\nu} - R_{\mu \kappa}^{\nu},
\]

where \(R_{\mu \kappa}^{\nu}\) are components of the curvature tensor, from which the proof follows.

We have

\[
\nabla_{\nu} \phi^h_a = \partial_{\nu} \phi^h_a - \Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h \phi^h_{\alpha} \\
\nabla_{\nu} \phi^h_{\bar{a}} = \partial_{\nu} \phi^h_{\bar{a}} - \Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h \phi^h_{\bar{a}}
\]

\[
\nabla_{\nu}(\nabla_{\mu} \phi^h_a) \phi^h_{\bar{a}} = \partial_{\nu}(\partial_{\mu} \phi^h_a) \phi^h_{\bar{a}} + (\partial_{\nu} \phi^h_{\bar{a}})(\partial_{\mu} \phi^h_a)\Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h - \Gamma_{\mu \nu}^{\alpha} (\partial_{\nu} \phi^h_a)(\partial_{\mu} \phi^h_{\bar{a}}) + (\partial_{\nu} \phi^h_{\bar{a}})(\partial_{\mu} \phi^h_a)\Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h + (\partial_{\nu} \phi_{\mu}^h)(\partial_{\mu} \phi^h_{\bar{a}})\Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h - \Gamma_{\mu \nu}^{\alpha} (\partial_{\nu} \phi_{\mu}^h)(\partial_{\mu} \phi^h_{\bar{a}})
\]

and hence

\[
\tilde{M}_{\mu \kappa}^{\nu} = (\partial_{\nu}(\partial_{\mu} \phi^h_a) \phi^h_{\bar{a}} - (\partial_{\nu} \Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h \phi^h_{\alpha}) + (\partial_{\nu} \phi^h_a)(\partial_{\mu} \phi^h_{\bar{a}})\Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h + (\partial_{\nu} \phi_{\mu}^h)(\partial_{\mu} \phi^h_{\bar{a}})\Gamma_{\mu \nu}^{\alpha} \phi_{\mu}^h - \Gamma_{\mu \nu}^{\alpha} \phi^h_a \phi^h_{\bar{a}})
\]

\[
-R_{\mu \kappa}^{\nu} = \partial_{\nu} \Gamma_{\mu \nu}^{\alpha} - \partial_{\mu} \Gamma_{\mu \nu}^{\alpha} + \Gamma_{\mu \nu}^{\alpha} \Gamma_{\nu \mu}^{\beta} - \Gamma_{\mu \mu}^{\alpha} \Gamma_{\nu \nu}^{\beta} = \nabla_{\nu} \nabla_{\mu} \phi^h_a \phi^h_{\bar{a}}
\]

In fact, from the definition, we get

\[
R_{\mu \kappa}^{\nu} = \partial_{\nu} \Gamma_{\mu \nu}^{\alpha} - \partial_{\mu} \Gamma_{\mu \nu}^{\alpha} + \Gamma_{\mu \nu}^{\alpha} \Gamma_{\nu \mu}^{\beta} - \Gamma_{\mu \mu}^{\alpha} \Gamma_{\nu \nu}^{\beta} = \nabla_{\nu} \nabla_{\mu} \phi^h_a \phi^h_{\bar{a}}.
\]

\(^{16}\) Cf. Chern [4, 5], Libermann [15], Schouten and van Dantzig [24]. See also § 8. With respect to this connection we have \(\nabla_{\nu} \phi_{\bar{a}}^h = 0\).
Thus we have

\[(4.3) \quad M_{i\alpha}^h = \tilde{M}_{i\alpha}^h - R_{i\alpha}^h; \text{ conj.}\]

If the quaternion structure \((\phi^h, \psi^h)\) is integrable, we choose a complex analytic coordinate system in which \(\psi^h\) are constant. In this coordinate system, \(\psi^h\), \(\psi^h\) being constant, all their partial derivatives vanish and consequently

\[M_{i\alpha}^h = 0.\]

Since \(M_{i\alpha}^h\) are components of a tensor, this result is independent of the choice of the coordinate system used. Thus we have

**Theorem 4.1** In order that a quaternion structure \((\phi^h, \psi^h)\) of class \(C^\omega\) in a manifold of class \(C^\omega\) be integrable, it is necessary that the tensors \(N_{i\alpha}^h(\phi), N_{i\alpha}^h(\psi)\) and \(M_{i\alpha}^h\) vanish identically.

Next we assume that a quaternion structure is integrable and choose a coordinate system in which \(\phi^h\) and \(\psi^h\) have numerical components. Then by linear transformations of coordinates we can find a complex analytic coordinate system \((w^\alpha, \bar{w}^\alpha)\) in which

\[\phi^h = I^h, \quad \psi^h = J^h,\]

In order that \(J^h\) be components of a tensor, it is necessary and sufficient that the coordinate transformation \(w^\alpha = f^\alpha(w), \bar{w}^\alpha = \bar{f}^\alpha(\bar{w})\) satisfy

\[J^h = \frac{\partial w^\alpha}{\partial w^\alpha} \frac{\partial w^\alpha}{\partial w^\alpha} J^k; \text{ conj.}\]

or what amount to the same

\[\frac{\partial w^r}{\partial w^s} = \frac{\partial \bar{w}^{r+m}}{\partial \bar{w}^{r+m}}, \quad \frac{\partial w^r}{\partial \bar{w}^{s+m}} = -\frac{\partial \bar{w}^{r+m}}{\partial \bar{w}^s}; \text{ conj. (}r, s = 1, 2, \ldots, m).\]

Since the left hand sides are the complex analytic functions of \(w^1, \ldots, w^n\) and the right hand sides are those of \(\bar{w}^1, \ldots, \bar{w}^n\), both sides must be constant. This shows that \(w^\alpha\) are linear functions of \(w^\alpha\) in a connected domain and therefore the manifold is covered by a system of affine coordinates, i.e. the manifold is locally affine. Conversely, if a \(4m\)-dimensional analytic manifold which is locally affine has a quaternion structure \((\phi^h, \psi^h)\), then in a system of affine coordinates \(\phi^h\) and \(\psi^h\) have both numerical components and consequently the structure is integrable. Thus we have

**Theorem 4.2** In order that a quaternion structure of class \(C^\omega\) in a \(4m\)-dimensional real analytic manifold be integrable, it is necessary and sufficient that the manifold be locally affine.

### 5. A sufficient condition for integrability of a quaternion structure

We have seen in the preceding section that for the integrability of a quaternion...
structure \((\phi_i^h, \psi_i^h)\) the vanishing of the tensors \(N_i^h(\phi), N_j^h(\psi)\) and \(M_{ij}^h\) is necessary. In this section we shall prove that these conditions are also sufficient under the assumption that the quaternion structure is real analytic as well as the manifold.

If \(N_i^h(\phi)=0\) we can find a complex analytic coordinate system \((z^s, \bar{z}^s)\), in which \(\phi_i^h\) and \(\psi_i^h\) have components

\[
\phi_i^h = I_i^h \quad \text{and} \quad (\psi_i^h) = \begin{pmatrix} 0 & \phi_i^h \bar{\psi}_i^h \\ \bar{\phi}_i^h \psi_i^h & 0 \end{pmatrix}.
\]

In such a coordinate system, \(N_i^h(\psi)=0\) are equivalent to

\[
(\ref{eq:5.1}) \quad \partial_{\bar{z}^s} \partial_z \psi_i^h = 0; \quad \text{conj.}
\]

Now we consider a system of differential equations

\[
(\ref{eq:5.2} \ a) \quad \partial_z \partial_z \psi_i^h = -(\partial_z \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_z \partial_z \psi_i^h,
\]

\[
(\ref{eq:5.2} \ b) \quad \partial_{\bar{z}^s} \partial_z \psi_i^h = 0,
\]

in which \(\psi_i^h, \partial_z \psi_i^h\) are analytic functions of \(z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n\). Differentiating \((\ref{eq:5.2} \ a)\) with respect to \(z^r\) we obtain

\[
\partial_z \partial_z \psi_i^h = -(\partial_z \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_z \partial_z \psi_i^h
\]

\[
= -(\partial_z \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_z \partial_z \psi_i^h,
\]

because we have \((\partial_z \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_z \partial_z \psi_i^h = 0\).

Next differentiating \((\ref{eq:5.2} \ a)\) with respect to \(\bar{z}^r\) and using \((\ref{eq:5.2} \ b)\) we get

\[
\partial_{\bar{z}^s} \partial_z \psi_i^h = -(\partial_{\bar{z}^s} \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_{\bar{z}^s} \partial_z \psi_i^h
\]

\[
= -(\partial_{\bar{z}^s} \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_{\bar{z}^s} \partial_z \psi_i^h
\]

Since \(\partial_{\bar{z}^s} \partial_z \psi_i^h = 0\) and \(\partial_{\bar{z}^s} \partial_z \psi_i^h = 0\), if the tensor \(M_{ij}^h = (M_{ij}^h, M_{ij}^\bar{h})\) vanishes identically, the conditions of integrability of \((\ref{eq:5.2})\) are satisfied identically and a set of solutions of \((\ref{eq:5.2})\) is found as that of complex analytic functions \(p_i^h\) of \(z^1, \cdots, z^n\) by virtue of \((\ref{eq:5.2} \ b)\). By analyticity of \(p_i^h(\bar{z})\) their complex conjugates \(p_i^h(\bar{z})\), denoted by \(p_i^\bar{h}\), satisfy

\[
(\ref{eq:5.3} \ a) \quad \partial_{\bar{z}^s} \partial_z p_i^h = -(\partial_{\bar{z}^s} \partial_z \psi_i^h) \psi_i^h \bar{\psi}_i^h \partial_{\bar{z}^s} \partial_z p_i^h
\]

\[
(\ref{eq:5.3} \ b) \quad \partial_z p_i^h = 0.
\]

From \((\ref{eq:5.2} \ a)\) we have

\[
\partial_z p_i^h - \psi_i^h \bar{\psi}_i^h \partial_z p_i^h = 0,
\]

from which

\[
\psi_i^h \partial_z p_i^h + \bar{\psi}_i^h p_i^h = 0,
\]

or

\[
(\ref{eq:5.4}) \quad \partial_z (p_i^h \bar{p}_i^h) = 0,
\]

which means that the functions \(\psi_i^h p_i^h\) are complex analytic functions of \(\bar{z}^1, \cdots, \bar{z}^n\). Since the functions \(\psi_i^h p_i^h\) are complex conjugates of \(\bar{\psi}_i^h p_i^h\), they are complex analytic functions of \(z^1, \cdots, z^n\):

\[
(\ref{eq:5.5}) \quad \partial_{\bar{z}^s} (\psi_i^h p_i^h) = 0.
\]

Now if the determinant \(p_i^h\) of the matrix \((p_i^h)\) does not vanish, we can find complex analytic functions \(F_i^h(\bar{z})\) of \(\bar{z}^1, \cdots, \bar{z}^n\) such that
Affine connections on manifolds with almost complex, quaternion or Hermitian structure.

\[ (5.6) \quad \phi_{x}^{\alpha} p_{x}^{\beta} = - p_{x}^{\alpha} F_x^{\alpha} , \]
from which we obtain
\[ (5.7) \quad p_{x}^{\alpha} = \phi_{x}^{\beta} p_{\bar{z}} \tilde{F}_x^{\alpha} . \]
If, taking account of (5.2 b) and (5.5), we differentiate (5.7) with respect to \( \bar{z} \), we get
\[ 0 = \psi_{x}^{\beta} p_{\bar{z}} \tilde{\partial}_{\beta} F_x^{\alpha} . \]
Since the determinants of \( (p_{x}^{\alpha}) \) and \( (\phi_{x}^{\alpha}) \) do not vanish, it follows
\[ (5.8) \quad \partial_{\beta} F_x^{\alpha} = 0 , \]
which means that the complex analytic functions \( F_x^{\alpha}(\bar{z}) \) of \( \bar{z}_1, \ldots, \bar{z}_n \) are constant. If we choose functions \( q_{x}^{\alpha} \) such that
\[ (5.9) \quad q_{x}^{\alpha} p_{x}^{\beta} = \partial_{\beta} F_x^{\alpha} , \]
then \( q_{x}^{\alpha} \) are complex analytic functions of \( z_1, \ldots, z_n \). From (5.6) and (5.8) it follows that
\[ (5.10) \quad q_{x}^{\alpha} \psi_{x}^{\beta} p_{x}^{\alpha} = \text{const.} \]
where \( q_{x}^{\alpha} \) are complex conjugates \( \bar{q}_{x}^{\alpha}(\bar{z}) \) of the functions \( q_{x}^{\alpha}(z) \). Hence if we choose a set of solutions \( p_{x}^{\alpha} \) in such a way that the initial values satisfy
\[ (5.11\ a) \quad \psi_{x}^{\alpha} p_{x}^{\beta} - p_{x}^{\alpha} J_x^{\alpha} = 0 , \quad |p_{x}^{\alpha}| = 0^{(0)} , \]
or
\[ (5.11\ b) \quad q_{x}^{\alpha} \psi_{x}^{\beta} p_{x}^{\alpha} = J_x^{\alpha} , \]
then the solutions \( p_{x}^{\alpha} \) satisfy (5.11) identically.

Next using these solutions \( p_{x}^{\alpha} \) of (5.2), we consider a system of differential equations
\[ (5.12) \quad \partial_{\beta} w^{\alpha} = p_{x}^{\alpha} . \]
Differentiating (5.12) with respect to \( z^e \) and using (5.2) we obtain
\[ \partial_{\alpha} \partial_{\beta} w^{\alpha} = \partial_{\alpha} p_{x}^{\alpha} = - (\partial_{\alpha} \phi_{x}^{\beta}) \psi_{x}^{\alpha} p_{x}^{\alpha} . \]
If \( N_{i}^{\alpha}(\phi) = 0 \), we have \( \partial_{\alpha} \psi_{x}^{\beta} = 0 \) and consequently the conditions of integrability of (5.12) are identically satisfied.

Let \( \{ w^1, \ldots, w^n \} \) be a set of solutions of (5.12) and \( \{ \bar{w}^1, \ldots, \bar{w}^n \} \) be their complex conjugates. Then \( \{ w^a, \bar{w}^a \} \) may be regarded as a complex coordinate system, in which, by (5.11 b),
\[ \phi_{i}^{a} = I_{i}^{a} , \quad \psi_{i}^{a} = J_{i}^{a} . \]
The above consideration shows that the vanishing of the tensors \( N_{i}^{\alpha}(\phi) \), \( N_{i}^{\alpha}(\psi) \) and \( M_{ij}^{k}(\phi) \) is a sufficient condition for the integrability of a quaternion structure \( (\phi_{i}^{a}, \psi_{i}^{a}) \).

Together with Theorem 4.1 we have

**Theorem 5.1** In order that a quaternion structure \( (\phi_{i}^{1}, \psi_{i}^{1}) \) of class \( C^m \) in a manifold of class \( C^m \) be integrable, it is necessary and sufficient that the tensors

19) This is always possible, see § 2 and Obata [20, 21].
We consider an almost complex manifold defined by the tensor field \( \phi^h_i \). We define six linear operators \( \Phi_1, \Phi_2, \Phi_2^s, \Phi_3, \Phi_4 \), operating on the space of tensors; for later purpose we operate them on the tensors of type (1, 2).

**Definition.** Let \( P_{ji}^h \) be a tensor. We define

\[
\begin{align*}
\Phi_1 P_{ji}^h &= \frac{1}{2} (P_{ji}^h - \phi^h_i P_{j\alpha}^s \phi^h_{\alpha}) \\
\Phi_2 P_{ji}^h &= \frac{1}{2} (P_{ji}^h + \phi^h_i P_{j\alpha}^s \phi^h_{\alpha}) \\
\Phi_3 P_{ji}^h &= \frac{1}{2} (P_{ji}^h - \phi^h_j P_{i\alpha}^s \phi^h_{\alpha}) \\
\Phi_4 P_{ji}^h &= \frac{1}{2} (P_{ji}^h + \phi^h_j P_{i\alpha}^s \phi^h_{\alpha}) \\
\Phi_5 P_{ji}^h &= \frac{1}{2} (P_{ji}^h - \phi^h_i \phi^h_j P_{i\alpha}^s) \\
\Phi_6 P_{ji}^h &= \frac{1}{2} (P_{ji}^h + \phi^h_i \phi^h_j P_{i\alpha}^s)
\end{align*}
\]

From the definition they are linear and the following three propositions are evident.

(6.1) \( \Phi_1 + \Phi_2 = \text{identity}, \quad \Phi_1^\# + \Phi_2^\# = \text{identity}, \quad \Phi_3 + \Phi_4 = \text{identity}. \)

(6.2) \( \Phi_1 \Phi_2 = \Phi_1, \quad \Phi_2 \Phi_2 = \Phi_2, \quad \Phi_2^s \Phi_1^s = \Phi_1^s, \quad \Phi_2^s \Phi_2^s = \Phi_2^s, \quad \Phi_3 \Phi_3 = \Phi_3, \quad \Phi_3 \Phi_4 = \Phi_3. \)

(6.3) \( \Phi_1 \Phi_2 = 0, \quad \Phi_2 \Phi_1 = 0, \quad \Phi_2^\# \Phi_1^\# = 0, \quad \Phi_3 \Phi_3 = 0, \quad \Phi_4 \Phi_4 = 0. \)

(6.4) \( \Phi_1 P_{ji}^h = 0 \) (\( \Phi_2 P_{ji}^h = 0 \)) implies \( \Phi_2 P_{ji}^h = P_{ji}^h \) (\( \Phi_1 P_{ji}^h = P_{ji}^h \)).

**Proof.** By (6.1) we have \( \Phi_2 P_{ji}^h = \Phi_1 P_{ji}^h + \Phi_2 P_{ji}^h = (\Phi_1 + \Phi_2) P_{ji}^h = P_{ji}^h. \)

If we state (6.4) more precisely, we have

(6.5) Given a tensor \( P_{ji}^h \), in order that there exist a tensor \( Q_{ji}^h \) such that \( \Phi_1 Q_{ji}^h = P_{ji}^h \) it is necessary and sufficient that \( \Phi_4 P_{ji}^h = 0. \)

**Proof.** \( \Phi_1 Q_{ji}^h = P_{ji}^h, \) then by (6.3) we have \( \Phi_2 P_{ji}^h = \Phi_2 \Phi_1 Q_{ji}^h = 0. \) Conversely if \( \Phi_2 P_{ji}^h = 0, \) then by (6.4) we have \( \Phi_2 P_{ji}^h = P_{ji}^h, \) which shows that we may choose \( P_{ji}^h \) itself as a solution of the equation \( \Phi_1 Q_{ji}^h = P_{ji}^h. \)

From (6.5) it follows that the image of the tensor space of type (1, 2) by \( \Phi_2(\Phi_1) \) is nothing but the subspace defined by \( \Phi_1 P_{ji}^h = 0 \) (\( \Phi_2 P_{ji}^h = 0 \)).

(6.6) For the pairs \( (\Phi_1^s, \Phi_2^s) \) and \( (\Phi_3, \Phi_4) \) the propositions analogous to (6.5) hold good.

---

20) These operations are the same as \( O_{\alpha\beta}^{\#}, *O_{\alpha\beta}^{\#} \) defined by Schouten and Yano [25].
After some calculations we have

$$\Phi_{ij}^{kl} = \Phi_{ij}^{kl} = \Phi_{ij}^{kl} = \Phi_{ij}^{kl} = \Phi_{ij}^{kl} = \Phi_{ij}^{kl}.$$  

The transform of a tensor $P_{ji}^{kh}$ by them is

$$\frac{1}{4} \left( P_{ji}^{kh} - \phi_i^k P_{ji}^{ja} \phi_a^h - \phi_j^k P_{ji}^{ca} \phi_a^h - \phi_j^a P_{ij}^{ca} \phi_k^h \right).$$

(6.7)

The transform of a tensor $P_{ji}^{ik}$ by them is

$$\frac{1}{4} \left( P_{ji}^{ik} - \phi_i^k P_{ji}^{ja} \phi_a^h + \phi_j^k P_{ji}^{ca} \phi_a^h + \phi_j^a P_{ij}^{ca} \phi_k^h \right).$$

(6.8)

The transform of a tensor $P_{ji}^{ik}$ by them is

$$\frac{1}{4} \left( P_{ji}^{ik} + \phi_i^k P_{ji}^{ja} \phi_a^h - \phi_j^a P_{ij}^{ca} \phi_k^h \right).$$

(6.9)

The transform of a tensor $P_{ji}^{ik}$ by them is

$$\frac{1}{4} \left( P_{ji}^{ik} + \phi_i^k P_{ji}^{ja} \phi_a^h - \phi_j^a P_{ij}^{ca} \phi_k^h \right).$$

(6.11)

If $P_{ji}^{kh} = 0$, the condition $\Phi_{ij}^{kh} = 0$ is equivalent with the condition $\Phi_{ij}^{kh} = 0$, $\Phi_{ij}^{kh} = 0$.

**Proof.** $\Phi_{ij}^{kh} = 0$ follows immediately from $\Phi_{ij}^{kh} = 0$, because of $\Phi_{ij}^{kh} = 0$.

From (6.8) and (6.9) we have

$$\Phi_{ij}^{kh} = \Phi_{ij}^{kh} = \Phi_{ij}^{kh} = \Phi_{ij}^{kh} = \Phi_{ij}^{kh}.$$

But by the assumed anti-symmetry of $P_{ji}^{kh}$, $\Phi_{ij}^{kh} = 0$ implies $\Phi_{ij}^{kh} = 0$ and also $\Phi_{ij}^{kh} = 0$. Thus we have

$$(\Phi_{ij}^{kh} + \Phi_{ij}^{kh}) P_{ij}^{kh} = \Phi_{ij}^{kh} = 0.$$

From $\Phi_{ij}^{kh} = 0$ it follows $\Phi_{ij}^{kh} = 0$. Since $\Phi_{ij}^{kh} = 0$, we have $\Phi_{ij}^{kh} = 0$.

Conversely if $\Phi_{ij}^{kh} = 0$, $\Phi_{ij}^{kh} = 0$, the second equation implies $\Phi_{ij}^{kh} = 0$, from which we have $\Phi_{ij}^{kh} = 0$.

Following Schouten and Yano [25], if $\Phi_{ij}^{kh} = 0$ ($\Phi_{ij}^{kh} = 0$) we say that $P_{ij}^{kh}$ is hybrid (pure) in $ik$ and if $\Phi_{ij}^{kh} = 0$, $\Phi_{ij}^{kh} = 0$, we say that $P_{ij}^{kh}$ is hybrid (pure) in $ji$.

7. Affine connections in an almost complex manifold

Let us consider an almost complex manifold defined by the tensor $\phi_i^k$ and introduce an affine connection in which $\phi_i^k$ is covariant constant. We call such an affine connection a $\phi$-connection for convenience. We shall give a general method to get all $\phi$-connections.

As is well-known, there always exists an affine connection $\Gamma_{ij}^{kh}$ in the manifold. If we introduce a second affine connection $\Gamma_{ij}^{kh}$, then it is written as
where $Q_{ji}^{h}$ is a tensor. Conversely if $Q_{ji}^{h}$ is a tensor, $\Gamma_{ji}^{h} = \hat{\Gamma}_{ji}^{h} + Q_{ji}^{h}$ is an affine connection. By $\overrightarrow{\nabla}_{ji}$ and $\nabla_{ji}$ we denote the operators of covariant differentiations with respect to $\hat{\Gamma}_{ji}^{h}$ and $\Gamma_{ji}^{h}$ respectively. We have then

$$\overrightarrow{\nabla}_{ji} \phi_{a}^{h} = \overrightarrow{\nabla}_{ji} \phi_{a}^{h} + \phi_{a}^{h} Q_{ji}^{h} - Q_{ji}^{h} \phi_{a}^{h}.$$ 

In order to have $\overrightarrow{\nabla}_{ji} \phi_{a}^{h} = 0$, it is then necessary and sufficient that we have

$$\overrightarrow{\nabla}_{ji} \phi_{a}^{h} + \phi_{a}^{h} Q_{ji}^{h} - Q_{ji}^{h} \phi_{a}^{h} = 0,$$

which is equivalent to

\begin{equation}
Q_{ji}^{h} + \phi_{a}^{h} Q_{ji}^{h} \phi_{a}^{h} = -(\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h}.
\end{equation}

If we put $P_{ji}^{h} = -\frac{1}{2} (\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h}$, then (7.1) is written in the form

\begin{equation}
\Phi_{ji} Q_{ji}^{h} = P_{ji}^{h}.
\end{equation}

By (6.5) there exists a tensor $Q_{ji}^{h}$ satisfying (7.2) if and only if $\Phi_{ji} P_{ji}^{h} = 0$. It follows, however, from $\phi_{a}^{h} \phi_{a}^{h} = -\delta_{a}^{h}$ that

$$\overrightarrow{\nabla}_{ji} \phi_{a}^{h} + \phi_{a}^{h} \overrightarrow{\nabla}_{ji} \phi_{a}^{h} = 0,$$

from which

$$\overrightarrow{\nabla}_{ji} \phi_{a}^{h} - \phi_{a}^{h} (\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h} = 0,$$

what amounts to the same

$$\Phi_{ji} P_{ji}^{h} = 0.$$

Thus in order to have $\overrightarrow{\nabla}_{ji} \phi_{a}^{h} = 0$ it is necessary and sufficient that $Q_{ji}^{h}$ be of the form

$$Q_{ji}^{h} = P_{ji}^{h} + \phi_{a}^{h} A_{ji}^{h},$$

where $A_{ji}^{h}$ is an arbitrary tensor. Thus we have

**Theorem 7.1** Let $\hat{\Gamma}_{ji}^{h}$ be an arbitrary but fixed affine connection in an almost complex manifold. Then in order that an affine connection $\Gamma_{ji}^{h}$ in the manifold be a $\phi$-connection it is necessary and sufficient that $\Gamma_{ji}^{h}$ be written in the form

$$\Gamma_{ji}^{h} = \hat{\Gamma}_{ji}^{h} - \frac{1}{2} (\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h} + \frac{1}{2} (A_{ji}^{h} - \phi_{a}^{h} A_{ji}^{h} \phi_{a}^{h}),$$

$A_{ji}^{h}$ being an arbitrary tensor.

**Corollary 1.** In an almost complex manifold there always exists a $\phi$-connection.

**Corollary 2.** Let $\Gamma_{ji}^{h}$ be a $\phi$-connection in an almost complex manifold. Then $\hat{\Gamma}_{ji}^{h} = \Gamma_{ji}^{h} + A_{ji}^{h}$, $A_{ji}^{h}$ being a tensor field, is a $\phi$-connection if and only if $\Phi_{ji} A_{ji}^{h} = 0$ or equivalently there exists a tensor field $B_{ji}^{h}$ such that $A_{ji}^{h} = \Phi_{ji} B_{ji}^{h}$.

Now let $\hat{\Gamma}_{ji}^{h}$ be an affine connection defined by

$$\hat{\Gamma}_{ji}^{h} = \hat{\Gamma}_{ji}^{h} + A_{ji}^{h}.$$

We have then

$$(\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h} = (\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h} + \phi_{a}^{h} A_{ji}^{h} \phi_{a}^{h} + A_{ji}^{h},$$

from which we have

$$\hat{\Gamma}_{ji}^{h} - \frac{1}{2} (\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h} = \hat{\Gamma}_{ji}^{h} - \frac{1}{2} (\overrightarrow{\nabla}_{ji} \phi_{a}^{h}) \phi_{a}^{h} + \frac{1}{2} (A_{ji}^{h} - \phi_{a}^{h} A_{ji}^{h} \phi_{a}^{h}).$$

21) Cf. Frölicher [9].
If we write
\[ \Gamma^b_{ji} = \Phi \tilde{\Gamma}^b_{ji} - \tilde{\Gamma}^b_{ji} - \frac{1}{2} (\nabla_j \phi_i^a) \phi_a^b, \]
then \( \Phi \tilde{\Gamma}^b_{ji} \) is a \( \phi \)-connection and we get
\[ \Phi \tilde{\Gamma}^b_{ji} = \Phi (\tilde{\Gamma}^b_{ji} + A_{ji}^b) = \Phi \tilde{\Gamma}^b_{ji} + \Phi A_{ji}^b. \]

Since we have \( \Phi \Phi = \Phi \) i.e. if \( I_{ji}^b \) is a \( \phi \)-connection, \( \Phi I_{ji}^b \) is identical with \( I_{ji}^b \) itself, every \( \phi \)-connection is written in the form \( \Phi \tilde{\Gamma}^b_{ji} \) for some affine connection \( \tilde{\Gamma}^b_{ji} \).

**Theorem 7.2** In an almost complex manifold, an affine connection \( I_{ji}^b \) is a \( \phi \)-connection if and only if there exists an affine connection \( \tilde{\Gamma}^b_{ji} \) such that \( \tilde{\Gamma}^b_{ji} = \Phi I_{ji}^b \).

Now we consider a \( \phi \)-connection \( I_{ji}^b \) in the manifold and denote by \( S_{ji}^b \) the torsion tensor of \( I_{ji}^b : S_{ji}^b = I_{ji}^b \). Making use of \( \nabla_j \phi_i^a = 0 \) and \( \phi_i^a \phi_a^b = -\delta_i^b \) we have
\[
2 N_{ji}^b(\phi) = \phi_j^c \nabla_i \phi_c^b - \phi_j^c \phi_i^b \Gamma^c_{ij} + \phi_j^c \Gamma^c_{ij} \phi_i^b - \phi_j^c \phi_i^b \phi_i^c \phi_a^b - \phi_j^c \phi_i^b S_{ji}^b = 2 \phi_j^c \phi_i^b S_{ji}^b. \]

**Theorem 7.3** Let \( I_{ji}^b \) be any \( \phi \)-connection and \( S_{ji}^b \) be its torsion tensor. Then we have
\[
N_{ji}^b(\phi) = 2 \phi_j^c \phi_i^b S_{ji}^b. \]

Taking account of (6.10) and (6.2) we have

**Corollary 1.** The Nijenhuis tensor \( N_{ji}^b(\phi) \) has the property
\[
\phi_1 N_{ji}^b(\phi) = \phi_2 N_{ji}^b(\phi) = \phi_3 N_{ji}^b(\phi) = 0,
\]
or equivalently
\[
\phi_2 N_{ji}^b(\phi) = \phi_3 N_{ji}^b(\phi) = \phi_3 N_{ji}^b(\phi) = N_{ji}^b(\phi).
\]

**Corollary 2.** If the relation \( N_{ji}^b(\phi) = k S_{ji}^b \) holds, where \( k \) is constant, then \( k \) must be 2 : \( N_{ji}^b(\phi) = 2 S_{ji}^b \).

In fact, \( N_{ji}^b(\phi) = k S_{ji}^b \) implies \( \phi_2 S_{ji}^b = \phi_3 S_{ji}^b = S_{ji}^b \) by (7.5), from which and (7.3) it follows
\[
N_{ji}^b(\phi) = 2 \phi_2 S_{ji}^b = 2 \phi_3 S_{ji}^b = 2 S_{ji}^b.
\]

Next we shall show that there exists really a \( \phi \)-connection \( I_{ji}^b \) such that \( N_{ji}^b(\phi) = 2 S_{ji}^b \).

**Lemma 7.1** Let \( \tilde{\Gamma}^b_{ji} \) be an arbitrary symmetric affine connection and \( S_{ji}^b \) be the torsion tensor of the \( \phi \)-connection \( I_{ji}^b = \Phi \tilde{\Gamma}^b_{ji} \). Then we have
\[
N_{ji}^b(\phi) = 2 \phi_2 S_{ji}^b.
\]

**Proof.** From the definition we have
\[
\Gamma^b_{ji} = \Phi \tilde{\Gamma}^b_{ji} = \tilde{\Gamma}^b_{ji} - \frac{1}{2} (\nabla_j \phi_i^c) \phi_c^b,
\]

22) Cf. Yano and Mogi [28], Yano [29, 31], Schouten and Yano [25].
from which

\[ S_{ij}^{h} = -\frac{1}{2} \left( \nabla_{[j} \phi_{i]}^{a} \right) \phi_{a}^{h} = -\frac{1}{2} \phi_{[j}^{a} \nabla_{i]} \phi_{a}^{h} \cdot \]

We have then

\[ 2\phi_a S_{ij}^{h} = -\frac{1}{2} \phi_{[j}^{a} \nabla_{i]} \phi_{a}^{h} + \frac{1}{2} \phi_{j}^{a} \phi_{i}^{b} \nabla_{a} \phi_{b}^{h} \]

\[ = \frac{1}{2} \left( -\phi_{[j}^{a} \nabla_{i]} \phi_{a}^{h} + \phi_{[j}^{a} \nabla_{a} \phi_{i]}^{h} \right) \]

But it has already been known that using the covariant differentiation \( \nabla_{j} \) with respect to any symmetric affine connection \( \tilde{\Gamma}_{j}^{i} \), the Nijenhuis tensor can be written in the form

\[ N_{ij}^{h}(\phi) = \frac{1}{2} \left( -\phi_{[j}^{a} \nabla_{i]} \phi_{a}^{h} + \phi_{[j}^{a} \nabla_{a} \phi_{i]}^{h} \right) \]

Thus we have

\[ N_{ij}^{h}(\phi) = 2\phi_{a} S_{ij}^{h}. \]

**LEMMA 7.2** Given any \( \phi \)-connection \( \Gamma_{j}^{i} \) with the torsion tensor \( S_{ij}^{h} \), then there exists a \( \phi \)-connection \( \tilde{\Gamma}_{j}^{i} \) with the torsion tensor \( \tilde{S}_{ij}^{h} \) satisfying

\[ \tilde{S}_{ij}^{h} = \phi_{a} S_{ij}^{h}. \]

**PROOF.** We construct an affine connection

\[ \tilde{\Gamma}_{j}^{i} = \Gamma_{j}^{i} - 2\phi_{a} S_{ij}^{h} \]

\[ = \Gamma_{j}^{i} - \frac{1}{2} \left( S_{ij}^{h} - \phi_{a} S_{ij}^{h} \phi_{a}^{h} + \phi_{j}^{a} S_{i[a}^{a} \phi_{b]}^{h} + \phi_{j}^{a} \phi_{i[a}^{b} S_{b]}^{h} \right). \]

Then by Corollary 2 to Theorem 7.1 \( \tilde{\Gamma}_{j}^{i} \) is a \( \phi \)-connection and has the torsion tensor \( \tilde{S}_{ij}^{h} \):

\[ \tilde{S}_{ij}^{h} = S_{ij}^{h} - \frac{1}{2} S_{ij}^{h} + \frac{1}{2} \phi_{a} S_{ij}^{h} - \frac{1}{2} \phi_{j}^{a} \phi_{i}^{b} S_{b}^{h} \]

\[ = \frac{1}{2} \left( S_{ij}^{h} - \phi_{j}^{a} \phi_{i}^{b} S_{b}^{h} \right) = \phi_{a} S_{ij}^{h}. \]

Thus the connection \( \tilde{\Gamma}_{j}^{i} \) is a connection with the required property.

**THEOREM 7.4** In an almost complex manifold there always exists a \( \phi \)-connection \( \tilde{\Gamma}_{j}^{i} \) with the torsion tensor \( \tilde{S}_{ij}^{h} \) satisfying

\[ 2\tilde{S}_{ij}^{h} = N_{ij}^{h}(\phi). \]

**PROOF.** We know that there exists a symmetric affine connection \( \tilde{\Gamma}_{j}^{i} \). Then by Lemma 7.1, the affine connection \( \Gamma_{j}^{i} = \phi \tilde{\Gamma}_{j}^{i} \) has the torsion tensor \( S_{ij}^{h} \) such that

\[ N_{ij}^{h}(\phi) = 2\phi_{a} S_{ij}^{h}. \]

According to Lemma 7.2, there always exists a \( \phi \)-connection \( \tilde{\Gamma}_{j}^{i} \) with the torsion tensor \( \tilde{S}_{ij}^{h} = \phi_{a} S_{ij}^{h} \). Thus \( \tilde{\Gamma}_{j}^{i} \) is a \( \phi \)-connection with the required property:

\[ 2\tilde{S}_{ij}^{h} = N_{ij}^{h}(\phi). \]

---

23) Cf. Frölicher [9], p. 68.
An immediate consequence of Theorem 7.4 is

**Theorem 7.5** If a 2n-dimensional manifold has an almost complex structure $\phi$, then a necessary and sufficient condition that the Nijenhuis tensor $N_{ij}^h(\phi)$ vanish identically is that it be possible to introduce in the manifold a symmetric $\phi$-connection.

Now, if, in general, $\Gamma^h_{ji}$ is any affine connection on a manifold, so also is the quantity $\Gamma^h_{ji} = \Gamma^h_{ij}$. If, however, $\Gamma^h_{ji}$ is a $\phi$-connection, then $\Gamma^h_{ji}$ is not necessarily a $\phi$-connection.

**Theorem 7.6** Let $\Gamma^h_{ji}$ be a $\phi$-connection. Then in order that the quantity $\Gamma^h_{ji} = \Gamma^h_{ij}$ be a $\phi$-connection, it is necessary and sufficient that the torsion tensor $S_{ji}^h$ of $\Gamma^h_{ji}$ satisfy

$$\Phi_2S_{ji}^h = \frac{1}{2}(S_{ji}^h + \phi_i^h S_{ji}^h \phi_j^h) = 0.$$

**Proof.** By the definition of the torsion tensor we have immediately

$$\Gamma^h_{ji} = \Gamma^h_{ji} - 2S_{ji}^h.$$

Then, $\Gamma^h_{ji}$ being a $\phi$-connection, by Corollary 2 to Theorem 7.1, $\Gamma^h_{ji}$ is a $\phi$-connection if and only if $\Phi_2S_{ji}^h = 0$.

**Theorem 7.7** In an almost complex manifold, a necessary and sufficient condition that $N_{ij}^h(\phi)$ vanish identically is that it be possible to introduce in the manifold a $\phi$-connection $\Gamma^h_{ji}$ such that $\Gamma^h_{ji} = \Gamma^h_{ij}$ is also a $\phi$-connection.

**Proof.** Let $\Gamma^h_{ji}$ be a $\phi$-connection such that $\Gamma^h_{ji}$ is also a $\phi$-connection. Then, by Theorem 7.6, the torsion tensor $S_{ji}^h$ of $\Gamma^h_{ji}$ satisfies

$$\Phi_2S_{ji}^h = 0,$$

from which together with Theorem 7.3 we have

$$N_{ij}^h(\phi) = 2\Phi_2S_{ji}^h = 0.$$

Conversely we assume that $N_{ij}^h(\phi) = 0$. Let $\Gamma_{ji}^h$ be any $\phi$-connection and $\Gamma_{ji}^h$ be a $\phi$-connection such that $S_{ji}^h = \Phi_2S_{ji}^h$, where $S_{ji}^h$ and $S_{ji}^h$ are the torsion tensors of $\Gamma_{ji}^h$ and $\Gamma_{ji}^h$ respectively. We have then

$$2\Phi_2S_{ji}^h = 2\Phi_2\Phi_2S_{ji}^h = N_{ij}^h(\phi) = 0.$$

From Theorem 7.6 we see that $\Gamma_{ji}^h$ is a $\phi$-connection with the required property.

Now if $S_{ji}^h$ is a tensor field which is anti-symmetric in its lower indices, there always exists an affine connection with the torsion tensor $S_{ji}^h$. In an almost complex manifold we have the following

**Theorem 7.8** Let $S_{ji}^h$ be a tensor field in an almost complex manifold. In order that $S_{ji}^h$ be a torsion tensor of a $\phi$-connection it is necessary and sufficient that $S_{ji}^h$ be symmetric in its lower indices and satisfy

$$2\Phi_2S_{ji}^h = N_{ij}^h(\phi).$$

24) Cf. Frölicher [9], Hodge [12], Patterson [22].
PROOF. The necessity has already been proved by Theorem 7.3. Let $\Gamma^a_{\beta\gamma}$ be a $\phi$-connection with the torsion tensor $\frac{1}{2} N_{ij}^h(\phi)$, whose existence is known from Theorem 7.4. Construct a connection

$$\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} + \Phi_\beta(S_{ij}^b + \Phi_\gamma S_{ij}^b),$$

then, it is a $\phi$-connection by virtue of Theorem 7.1. Using (6.1) we have

$$\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} + \Phi_\beta \Phi_\gamma S_{ij}^b - \Phi_\gamma \Phi_\beta S_{ij}^b - \Phi_\beta S_{ij}^b \phi^a + S_{ij}^b.$$

Since $S_{ij}^b$, $\Phi_\beta \Phi_\gamma S_{ij}^b$ and $\Phi_\gamma S_{ij}^b$ are anti-symmetric in their covariant indices, denoting by $S_{ij}^b$ the torsion tensor of $\Gamma^a_{\beta\gamma}$, we find

$$S_{ij}^b = \frac{1}{2} N_{ij}^b(\phi) + \Phi_\beta \Phi_\gamma S_{ij}^b - \Phi_\gamma \Phi_\beta S_{ij}^b - \Phi_\beta S_{ij}^b \phi^a + S_{ij}^b = 2\Phi_\beta \Phi_\gamma S_{ij}^b - 2\Phi_\gamma \Phi_\beta S_{ij}^b + S_{ij}^b = S_{ij}^b.$$

Thus the connection $\Gamma^a_{\beta\gamma}$ is a $\phi$-connection with the torsion tensor $S_{ij}^b$.

REMARK 1. On account of (6.11) we see that if $N_{ij}^b(\phi) = 0$, a $\phi$-connection $\Gamma^a_{\beta\gamma}$ such that $\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma}$ is also a $\phi$-connection is characterized by $\Phi_\beta S_{ij}^b = 0$.

REMARK 2. In Lemma 7.2 starting from any $\phi$-connection $\Gamma^a_{\beta\gamma}$ with the torsion tensor $S_{ij}^b$ we constructed the $\phi$-connection

$$\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} - 2\Phi_\gamma \Phi_\beta S_{ij}^b,$$

then the torsion tensor $S_{ij}^b$ of $\Gamma^a_{\beta\gamma}$ satisfies

$$S_{ij}^b = \Phi_\beta S_{ij}^b.$$

In Theorem 7.7 it was also proved that if $N_{ij}^b(\phi) = 0$, then $\Gamma^a_{\beta\gamma}$ has the property that $\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma}$ is also a $\phi$-connection.

In case $N_{ij}^b(\phi) = 0$, there is an alternative method to obtain this connection. Namely, we define

$$\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} - 2\Phi_\gamma \Phi_\beta S_{ij}^b = \Gamma^a_{\beta\gamma} - (S_{ij}^b + \Phi_\gamma S_{ij}^b \phi^a).$$

Then from (6.10), Theorem 7.3 and the assumption $N_{ij}^b(\phi) = 0$ we get

$$2\Phi_\gamma \Phi_\beta S_{ij}^b = 2\Phi_\gamma \Phi_\beta S_{ij}^b = N_{ij}^b(\phi) = 0.$$

Hence $\Gamma^a_{\beta\gamma}$ is a $\phi$-connection by virtue of Corollary 2 to Theorem 7.2 and has the torsion tensor $S_{ij}^b$:

$$S_{ij}^b = \Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} - \frac{1}{2} \Phi_\gamma S_{ij}^b \phi^a + \frac{1}{2} \Phi_\beta S_{ij}^b \phi^a = - \Phi_\gamma S_{ij}^b \phi^a + \Phi_\beta S_{ij}^b - 2\Phi_\gamma \Phi_\beta S_{ij}^b = \Phi_\beta S_{ij}^b - N_{ij}^b(\phi) = \Phi_\beta S_{ij}^b.$$

REMARK 3. The proof of Theorem 7.8 shows that for any $\phi$-connection $\Gamma^a_{\beta\gamma}$, the connection $\Gamma^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} - \Phi_\gamma (S_{ij}^b + \Phi_\delta S_{ij}^b)$ is a $\phi$-connection whose torsion tensor is $\frac{1}{2} N_{ij}^b(\phi)$, where $S_{ij}^b$ denotes the torsion tensor of $\Gamma^a_{\beta\gamma}$. 
8. Affine connections in a complex analytic manifold

We assume in this section that the almost complex structure $\phi_i$ of $C^n$ in a real analytic manifold is integrable and choose a complex analytic coordinate system $(z^k, \bar{z}^\bar{k})$. Then $\phi_i$ is of the form $\phi_i = I^i$.

Let $\hat{T}^i_{jk}(z, \bar{z})$ be any affine connection in the manifold; then from the special form of the coordinate transformation $z^k = f^k(z)$, $\bar{z}^\bar{k} = f^{\bar{k}}(\bar{z})$, we can see easily that

$$
\partial_j \partial_k \partial_\bar{l} \partial_\bar{m} \hat{T}^i_{jkl} = \partial_j \partial_k \partial_\bar{l} \partial_\bar{m} \hat{T}^i_{jkl},
$$

$$
\partial_j \partial_\bar{l} \partial_\bar{m} \hat{T}^i_{jkl} = \partial_j \partial_\bar{l} \partial_\bar{m} \hat{T}^i_{jkl},
$$

are all components of tensors of type $(1,2)$. Also if we assume that the affine connection is self-adjoint, then the self-adjointness of a tensor is preserved by the covariant differentiation with respect to this affine connection.

Now, writing out $\nabla_j \phi_i$ fully, from the special form of $\phi_i = I^i$, we have

$$
\nabla_j \phi_i^k = \nabla_j \phi_i^k = 0,
\nabla_j \phi_i^\bar{k} = -2i \hat{T}^i_{jkl} \nabla_j \phi_i^k = 2i \hat{T}^i_{jkl}.
$$

From these equations we see that $\nabla_j \phi_i^k = 0$ if and only if $\hat{T}^i_{jkl} = 0$.

Since $P_{ij} = (\hat{T}^i_{jkl}, \hat{T}^\bar{i}_{j\bar{k}})$ are components of a tensor, we can define an affine connection

$$
\Gamma_{ij} = \hat{T}^i_{jkl} - P_{ij}^k,
$$

which is a $\phi$-connection: $\nabla_j \phi_i = 0$. Furthermore $Q_{ij} = (\hat{T}^i_{jkl}, \hat{T}^\bar{i}_{j\bar{k}})$ being also components of a tensor

$$
\Gamma_{ij} = \hat{T}^i_{jkl} - Q_{ij}^k,
$$

is also a $\phi$-connection.

Now we apply our method explained in the previous section to this case. Let $\hat{T}^i_{jkl}$ be an arbitrary affine connection in the manifold. From the special form (8.1) of $\nabla_j \phi_i$ we see

$$
\phi_i^k \nabla_j \phi_i^k = -2i \hat{T}^i_{jkl},
\phi_i^\bar{k} \nabla_j \phi_i^{\bar{k}} = -2i \hat{T}^i_{j\bar{k}}.
$$

Hence the $\phi$-connection

$$
\Gamma_{ij} = \phi \hat{T}^i_{jkl} = \hat{T}^i_{jkl} - \frac{1}{2} \left( \nabla_j \phi_i^k \right) \phi_i^k
$$

has the components

$$
\Gamma_{j\bar{k}} = \hat{T}^\bar{i}_{j\bar{k}},
\Gamma_{j\bar{k}} = \hat{T}^{\bar{i}}_{j\bar{k}},
\Gamma_{j\bar{k}} = \hat{T}^\bar{i}_{j\bar{k}},
$$

which show that $\Gamma_{ij}$ is nothing but one defined by (8.2).

Moreover for any tensor $A_{ij}$ we have $\phi_i A_{ij} = (A_{ij}, A_{j\bar{k}})$.

Thus to construct an affine connection $\Gamma_{j\bar{k}} = \phi \hat{T}^\bar{i}_{j\bar{k}} + \phi_i A_{ij}$ from a given $\hat{T}^\bar{i}_{j\bar{k}}$ is to
add the tensor \((A_{\lambda k}, A_{\mu k})\) to the affine connection \((\tilde{\Gamma}_{\lambda j}^k, \tilde{\Gamma}_{\lambda k}^j)\).

Next, given a \(\phi\)-connection \(\Gamma_{\lambda j}^k=(\Gamma_{\lambda j}^k, \Gamma_{\lambda k}^j)\), then the torsion tensor \(S_{\lambda j}^h\) of \(\Gamma_{\lambda j}^k\) has components

\[
S_{\lambda k}^h = \Gamma_{\lambda j}^k, \quad S_{\mu k}^h = \frac{1}{2} \Gamma_{\mu j}^k; \quad \text{conj.}
\]

\[
S_{\lambda k}^h = -\frac{1}{2} \Gamma_{\lambda j}^k, \quad S_{\mu k}^h = 0; \quad \text{conj.}
\]

from which we see that the only non-zero components of \(\Phi_i \Phi_s S_{\lambda j}^h\) are

\[
S_{\lambda k}^h = \frac{1}{2} \Gamma_{\lambda j}^k, \quad S_{\mu k}^h = \frac{1}{2} \Gamma_{\mu j}^k.
\]

Then the \(\phi\)-connection \(\tilde{\Gamma}_{\lambda j}^k = \Gamma_{\lambda j}^k - 2\Phi_i \Phi_s S_{\lambda j}^h\) has components

\[
\tilde{\Gamma}_{\lambda j}^k = \Gamma_{\lambda j}^k; \quad \text{conj.}
\]

others being zero, and the torsion tensor \(S_{\lambda j}^h = \tilde{S}_{\lambda j}^h\) has components

\[
\tilde{S}_{\lambda k}^h = \tilde{S}_{\lambda j}^k = \Gamma_{\lambda j}^k = S_{\lambda k}^h, \quad \tilde{S}_{\lambda k}^h = \tilde{S}_{\lambda j}^k = S_{\lambda j}^h; \quad \text{conj.}
\]

others being zero, namely \(\tilde{S}_{\lambda j}^h = \Phi_i S_{\lambda j}^h\). This shows that the \(\phi\)-connection obtained in Lemma 7.2 corresponds to the connection (8.3) in a complex manifold, and if the original connection \(\tilde{\Gamma}_{\lambda j}^k\) is symmetric, then \(\tilde{S}_{\lambda j}^h = \Phi_i S_{\lambda j}^h = 0\), i.e. \(\tilde{\Gamma}_{\lambda j}^k\) is without torsion.

**CHAPTER III** Affine connections in quaternion manifolds

We consider a quaternion manifold defined by \((\phi_i^h, \phi_i^h)\). If we put \(\kappa_i^h = \phi_i^a \phi_a^h\), then \(\kappa_i^h\) is also an almost complex structure and we have

\[
\phi_i^a \phi_a^h = \phi_i^a \phi_a^h = \kappa_i^h \kappa_a^h = -\delta_i^h,
\]

\[
\phi_i^a \phi_a^h = \kappa_i^h, \quad \phi_i^a \kappa_a^h = \phi_i^h, \quad \kappa_i^h \phi_a^h = \phi_i^h,
\]

\[
\phi_i^a \phi_a^h + \phi_i^h \phi_a^h = \phi_i^a \kappa_a^h + \kappa_i^h \phi_a^h = \kappa_i^a \phi_a^h + \phi_i^a \kappa_a^h = 0.
\]

From these we see that if \((\phi_i^h, \phi_i^h)\) is a quaternion structure, \((\phi_i^h, \kappa_i^h)\) and \((\kappa_i^h, \phi_i^h)\) give the same quaternion structure to the manifold. Therefore any statement which can be proved about \(\phi_i^h, \phi_i^h\) and \(\kappa_i^h\) could be established by exactly the same train of reasoning if everywhere \(\phi_i^h, \phi_i^h\) and \(\kappa_i^h\) were replaced by cyclic permutation of \((\phi_i^h, \phi_i^h, \kappa_i^h)\).

9. Algebraic lemmas II

By using the almost complex structures \(\phi_i^h\) and \(\kappa_i^h\) we define linear operations \(\Psi_i, \Psi_2, \Psi_3, \Psi_4\) and \(K_1, K_2, K_3, K_4\) respectively in the quite same way as in the case of \(\phi_i^h\). Then the corresponding propositions (6.1)-(6.10) hold true naturally in the cases of \(\phi_i^h\) and \(\kappa_i^h\).

By virtue of anti-commutativity of \(\phi_i^h, \phi_i^h, \kappa_i^h\) there are relations among \(\phi_i, \Psi_i, K_s\) \((1 \leq s \leq 4)\). We give only propositions which are used later.
Affine connections on manifolds with almost complex, quaternion or Hermitian structure. 61

(9.1) If $P_{ji}^h$ is a tensor we have
\[ \Phi_1 \Psi_{ji}^h = \Psi_{ji}^h = K_i \Phi_1 P_{ji}^h = \frac{1}{4} \left( P_{ji}^h - \phi_i^h \phi_j^h - \phi_j^h \phi_i^h \right) \left( K_i P_{ji}^h - \phi_i^h \phi_j^h \right) \]

(9.2) If $P_{ji}^h$ is a tensor we have
\[ O_1 \Psi_{ji}^h = \Psi_{ji}^h = \frac{1}{4} \left( P_{ji}^h - \phi_i^h \phi_j^h - \phi_j^h \phi_i^h \right) \left( K_i P_{ji}^h - \phi_i^h \phi_j^h \right) \]

(9.3) In order for a tensor $P_{ji}^h$ to satisfy $\Psi_{ji}^h = \Psi_{ji}^h = 0$ it is necessary and sufficient that there exist a tensor $Q_{ji}^h$ such that $P_{ji}^h = \Phi_1 Q_{ji}^h$.

PROOF. $\Phi_2 P_{ji}^h = \Psi_3 P_{ji}^h = 0$ implies by (6.4)
\[ \Phi_1 \Psi_{ji}^h = \Phi_1 P_{ji}^h = P_{ji}^h \]

Conversely if $P_{ji}^h = \Phi_1 \Psi_{ji}^h = 0$, we have by (6.3) and (9.1)
\[ O_3 P_{ji}^h = \Phi_2 \Psi_{ji}^h = \Psi_3 P_{ji}^h = 0 \]

(9.5) From which it follows immediately
\[ \Phi_2 \Psi_{ji}^h = 0 = \Phi_3 P_{ji}^h \]

10. Affine connections in quaternion manifolds

We consider a $4m$-dimensional manifold with a quaternion structure $(\phi_i^h, \psi_i^h)$ and give a method, on the analogy of the almost complex case, to obtain an affine connection in which $\phi_i^h$, $\psi_i^h$ and $\kappa_i^h$ are simultaneously covariant constant, called a $(\phi, \psi)$-connection, starting from any affine connection given in the manifold.

It is obvious that if $\phi_i^h$ and $\psi_i^h$ are covariant constant, so also is $\kappa_i^h = \phi_i^h \psi_i^h$.

Now given an affine connection $\breve{\Gamma}_{ji}^h$ in the manifold, we define
\[ \phi \breve{\Gamma}_{ji}^h = \hat{\Gamma}_{ji}^h - \frac{1}{2} (\nabla_j \phi_i^h) \phi_j^h \]

(10.2) $\psi \breve{\Gamma}_{ji}^h = \hat{\Gamma}_{ji}^h - \frac{1}{2} (\nabla_j \psi_i^h) \psi_j^h \]

(10.3) $K \breve{\Gamma}_{ji}^h = \hat{\Gamma}_{ji}^h - \frac{1}{2} (\nabla_j \kappa_i^h) \kappa_j^h \]

where $\nabla_j$ denotes the operation of covariant differentiation with respect to $\hat{\Gamma}_{ji}^h$. Then as were shown in §7, these operations have the following three properties:

(10.4) $\phi \hat{\Gamma}_{ji}^h$ is a $\phi$-connection for any connection $\hat{\Gamma}_{ji}^h$.

(10.5) $\phi \phi = \phi$, namely $\Gamma_{ji}^h$ is a $\phi$-connection if and only if $\phi \Gamma_{ji}^h = \Gamma_{ji}^h$. 
(10.6) If $A_{ji}^b$ is a tensor, $\Phi(\hat{\Gamma}_{ji}^h + A_{ji}^h) = \Phi\hat{\Gamma}_{ji}^h + \Phi A_{ji}^h$.

Furthermore we have, by using (10.1), (10.2) and (10.6),

$$
\Phi\Phi\hat{\Gamma}_{ji}^h = \Phi\left(\hat{\Gamma}_{ji}^h - \frac{1}{2}(\nabla_c\phi^h_a)\phi_a^h\right)
$$

$$
= \hat{\Gamma}_{ji}^h - \frac{1}{2}(\nabla_c\phi^h_a)\phi_a^h - \frac{1}{4}(\nabla_c\phi^h_a)\phi_a^h - \phi^h_c(\nabla_c\phi^h_a)\phi_a^h
$$

$$
= \hat{\Gamma}_{ji}^h - \frac{1}{2}(\nabla_c\phi^h_a)\phi_a^h - \frac{1}{4}(\nabla_c\phi^h_a)\phi_a^h + \frac{1}{4}(\nabla_c\kappa^h_a - (\nabla_c\phi^h_a)\phi_a^h)
$$

$$
= \hat{\Gamma}_{ji}^h - \frac{1}{4}(\nabla_c\phi^h_a)\phi_a^h + (\nabla_c\phi^h_a)\phi_a^h + (\nabla_c\kappa^h_a)\phi_a^h).
$$

Since the last term is symmetric in $\phi^h_i, \phi^h_j, \kappa^h_i$, we find

(10.8) \[ \Phi\Phi = \Phi K = K\Phi = K\Phi = \Phi K, \]

from which and (10.4), (10.5) we conclude that, given any affine connection $\hat{\Gamma}_{ji}^h$ in the manifold, $\Phi\Phi\hat{\Gamma}_{ji}^h$ is a $(\phi_i, \phi_j)$-connection.

If $\Gamma_{ji}^h$ is a $(\phi, \psi)$-connection, then by (10.5) $\Phi\Phi\Gamma_{ji}^h = \Phi\Gamma_{ji}^h = \Gamma_{ji}^h$. Thus we have proved:

**Theorem 10.1** In a quaternion manifold defined by $(\phi_i, \phi_j)$, an affine connection $\Gamma_{ji}^h$ is a $(\phi, \psi)$-connection if and only if there exists an affine connection $\hat{\Gamma}_{ji}^h$ such that $\Gamma_{ji}^h = \Phi\Phi\hat{\Gamma}_{ji}^h$.

Since in a manifold there always exists an affine connection, we have

**Corollary.** In a quaternion manifold there always exists a $(\phi, \psi)$-connection.

Now if $A_{ji}^h$ is a tensor field, then we get

(10.9) \[ \Phi\Phi(\Gamma_{ji}^h + A_{ji}^h) = \Phi(\Phi\Gamma_{ji}^h + \Phi A_{ji}^h) \]

$$
= \Phi\Phi\Gamma_{ji}^h + \Phi\Phi A_{ji}^h,
$$

from which and (9.4) we have

**Theorem 10.2** Let $\Gamma_{ji}^h$ be a $(\phi, \psi)$-connection in a quaternion manifold, then $\hat{\Gamma}_{ji}^h = \Gamma_{ji}^h + A_{ji}^h$, $A_{ji}^h$ being a tensor field, is a $(\phi, \psi)$-connection if and only if $\Phi\Phi A_{ji}^h = \Psi A_{ji}^h = 0$ or equivalently there exists a tensor field $B_{ji}^h$ such that $A_{ji}^h = \Phi B_{ji}^h$.

**Theorem 10.3** Let $\hat{\Gamma}_{ji}^h, \Sigma_{ji}^h$ be $(\phi, \psi)$-connections in a quaternion manifold and $S_{ji}^h$ and $\tilde{S}_{ji}^h$ be their torsion tensors. If $S_{ji}^h = \tilde{S}_{ji}^h$, then two connections coincide with each other.

**Proof.** If we write $\hat{\Gamma}_{ji}^h = \Gamma_{ji}^h + A_{ji}^h$, then $A_{ji}^h$ is a tensor field and the torsion tensor is given by

$$
\Sigma_{ji}^h = \frac{1}{2}(\nabla_{[j]} A_{ji}^h) + A_{[j]ji}^h = S_{ji}^h + A_{[j]ji}^h.
$$

If $S_{ji}^h = \tilde{S}_{ji}^h$, then we have

(10.10) \[ A_{[j]ji}^h = 0 \quad \text{or} \quad A_{ji}^h = A_{ji}^h. \]

Since $\hat{\Gamma}_{ji}^h$ and $\Gamma_{ji}^h$ are both $(\phi, \psi)$-connections, by Theorem 10.2 we have

(10.11) \[ 2\Phi A_{ji}^h = A_{ji}^h + \phi^h_i A_{ji}^h \phi_a^h = 0, \]

(10.12) \[ 2\Psi A_{ji}^h = A_{ji}^h + \phi^h_i A_{ji}^h \phi_a^h = 0. \]
From (10.10), (10.11) and (10.12) it follows
\[ A_{ij}^{h} = -\phi_{j}^{a}A_{j}^{a}a_{i}^{b} = -\phi_{j}^{a}A_{a}^{a}i_{i}^{b}, \]
from which and (10.12), we see
\[ A_{ji}^{h} = -\phi_{j}^{a}A_{j}^{a}a_{i}^{b} = \phi_{j}^{a}\phi_{i}^{b}A_{a}^{a}i_{i}^{b} \]
\[ = -\phi_{j}^{a}\phi_{i}^{b}A_{a}^{a}i_{i}^{b} = \phi_{j}^{a}A_{a}^{a}i_{i}^{b} \]
\[ = -A_{ji}^{h}. \]

Thus we have \( A_{ji}^{h} = 0 \), i.e. \( \Gamma_{ji}^{a} = \Gamma_{j}^{a}. \)

In the almost complex case we have obtained the \( \phi \)-connection \( \Gamma_{ji}^{a} \) with the torsion tensor \( \hat{S}_{ji}^{a} = \frac{1}{2}N_{ji}^{a}(\phi) \) (Theorem 7.4). In the quaternion case we can also get the \( (\phi, \psi) \)-connection with the torsion tensor closely connected with the Nijenhuis tensors \( N_{ji}^{a}(\phi), N_{ji}^{a}(\psi) \) and \( N_{ji}^{a}(\kappa) \).

Now let \( \Gamma_{ji}^{a} \) be a \( (\phi, \psi) \)-connection. Then from Theorem 7.3 we have the relations
\[ (10.13) \]
\[ N_{ji}^{a}(\psi) = 2\Phi_{j}^{a}\Phi_{j}^{b}S_{ji}^{a}, \ N_{ji}^{a}(\psi) = 2\Psi_{j}^{a}\Psi_{j}^{b}S_{ji}^{a}, \ N_{ji}^{a}(\kappa) = 2K_{j}^{a}K_{j}^{b}S_{ji}^{a}. \]

**Lemma 10.1** Let \( S_{ji}^{a} \) be the torsion tensor of a \( \phi \)-connection, then
\[ \phi_{j}^{a}S_{ji}^{a}a_{i}^{b} = N_{ji}^{a}(\phi) - \Phi_{j}^{a}S_{ji}^{a}. \]

**Proof.**
\[ \phi_{j}^{a}S_{ji}^{a}a_{i}^{b} = \frac{1}{2}(\phi_{j}^{a}S_{ji}^{a}a_{i}^{b} - \phi_{j}^{a}S_{ji}^{a}a_{i}^{b}) \]
\[ = \frac{1}{2}\phi_{j}^{a}(S_{ji}^{b} - \phi_{j}^{a}S_{ji}^{a}a_{i}^{b})a_{i}^{b} \]
\[ = \phi_{j}^{a}(\Phi_{j}^{a}S_{ji}^{a})a_{i}^{b} = 2\Phi_{j}^{a}\Phi_{j}^{b}S_{ji}^{a} - \Phi_{j}^{a}S_{ji}^{a} \]
\[ = N_{ji}^{a}(\phi) - \Phi_{j}^{a}S_{ji}^{a}. \]

**Lemma 10.2** \( N_{ji}^{a}(\phi) = 0 \) implies \( \Psi_{j}^{a}N_{ji}^{a}(\kappa) = K_{j}^{a}N_{ji}^{a}(\psi) \).

**Proof.** Since \( N_{ji}^{a}(\phi) = 0 \), we may use a symmetric \( \phi \)-connection \( \Gamma_{ji}^{a} \), whose existence has been proved in Theorem 7.5. \( N_{ji}^{a}(\psi) \) and \( N_{ji}^{a}(\kappa) \) are written as follows:
\[ N_{ji}^{a}(\psi) = -\frac{1}{2}(\psi_{j}^{a}\nabla_{ij}\psi_{ji}^{b} - \psi_{j}^{a}\nabla_{ij}\psi_{ji}^{b}) = -\Psi_{j}^{a}\psi_{ji}^{a}\nabla_{ij}\psi_{ji}^{b}, \]
\[ N_{ji}^{a}(\kappa) = -K_{j}^{a}\psi_{ji}^{a}\nabla_{ij}\psi_{ji}^{a}. \]

But since we have
\[ \kappa_{j}^{a}\nabla_{ij}\kappa_{ji}^{b} = \phi_{j}^{a}\psi_{ji}^{a}\nabla_{ij}\psi_{ji}^{b} = \phi_{j}^{a}\nabla_{ij}\psi_{ji}^{b}, \]
it follows
\[ N_{ji}^{a}(\kappa) = -K_{j}^{a}\psi_{ji}^{a}\nabla_{ij}\psi_{ji}^{a}. \]

Thus we have
\[ \Psi_{j}^{a}N_{ji}^{a}(\kappa) = -\Psi_{j}^{a}K_{j}^{a}\psi_{ji}^{a}\nabla_{ij}\psi_{ji}^{b} = K_{j}^{a}N_{ji}^{a}(\psi). \]

**Theorem 10.4** In a quaternion manifold, there exist three \( (\phi, \psi) \)-connections \( \Gamma_{ji}^{a}, \Gamma_{j}^{a}, \Gamma_{ji}^{a} \) such that
\[ (10.14) \]
\[ S_{ji}^{a} = \Phi_{j}^{a}(N_{ji}^{a}(\psi) + N_{ji}^{a}(\kappa)), \]
\[ (10.15) \]
\[ \hat{S}_{ji}^{a} = \Psi_{j}^{a}(N_{ji}^{a}(\kappa) + N_{ji}^{a}(\phi)), \]
\[ (10.16) \]
\[ \hat{S}_{ji}^{a} = K_{j}^{a}(N_{ji}^{a}(\phi) + N_{ji}^{a}(\psi)), \]
where \( S_{ji}^{a}, \hat{S}_{ji}^{a} \) and \( \hat{S}_{ji}^{a} \) are the torsion tensors of \( \Gamma_{ji}^{a}, \Gamma_{j}^{a} \) and \( \Gamma_{ji}^{a} \) respectively.
PROOF. We introduce an arbitrary \((\phi, \psi)\)-connection \(\Gamma^h_{ji}\) with the torsion tensor \(S_{ji}^h\) and then construct the affine connections

\[
\begin{align*}
\Gamma^1_{ji} &= \Gamma^2_{ji} - 4\Phi_1 \Phi_2 S_{ji}^h, \\
\Gamma^2_{ji} &= \Gamma^3_{ji} - 4K_2 \Phi_3 S_{ji}^h, \\
\Gamma^3_{ji} &= \Gamma^4_{ji} - 4\Phi_4 K_4 S_{ji}^h.
\end{align*}
\]

We shall show that these are the required ones.

First they are obviously \((\phi, \psi)\)-connections. We have

\[
4\Phi_1 \Phi_2 \Phi_3 S_{ji}^h = \frac{1}{2} (S_{ij}^h - \phi_i^h S_{ji}^h \phi_a^h - \psi_i^h S_{ji}^h \psi_a^h - \kappa_i^h S_{ji}^h \kappa_a^h
\]

\[+ \phi_j^i \phi_i^h S_{ai}^h + \phi_j^i \phi_i^h S_{ci}^h \phi_a^h + \phi_j^i \phi_i^h \phi_i^h S_{bi}^h \phi_a^h + \phi_j^i \phi_i^h \kappa_i^h \kappa_a^h),
\]

from which we get as the torsion tensor \(S_{ji}^h\) of \(\Gamma^1_{ji}\)

\[
\begin{align*}
S_{ji}^h &= \Gamma^1_{ji} - \Phi_1 S_{ji}^h + \Phi_2 (\phi_i^h S_{ji}^h \phi_a^h) + \Phi_3 (\kappa_i^h S_{ji}^h \kappa_a^h)
\end{align*}
\]

\[= \Phi_1 S_{ji}^h + \Phi_2 (N_{ji}^h(\phi) - \Phi_2 S_{ji}^h) + \Phi_3 (N_{ji}^h(\kappa) - K_2 S_{ji}^h)
\]

\[= (\Phi_1 + \Phi_2) S_{ji}^h + \Phi_3 (N_{ji}^h(\phi) + N_{ji}^h(\kappa))
\]

\[= \Phi_3 (N_{ji}^h(\phi) + N_{ji}^h(\kappa))
\]

by virtue of (9.10) and Lemma 10.1. In the quite same way we get

\[
\begin{align*}
\hat{S}_{ji}^h &= \Gamma^2_{ji} - \Phi_2 S_{ji}^h + \Phi_3 (\kappa_i^h S_{ji}^h \kappa_a^h) + \Phi_4 (\kappa_i^h S_{ji}^h \kappa_a^h)
\end{align*}
\]

\[= \Phi_2 S_{ji}^h + \Phi_3 (N_{ji}^h(\kappa) - \Phi_3 S_{ji}^h) + \Phi_4 (N_{ji}^h(\kappa) + K_3 S_{ji}^h)
\]

\[= (\Phi_2 + \Phi_3) S_{ji}^h + \Phi_4 (N_{ji}^h(\kappa) + N_{ji}^h(\kappa))
\]

\[= \Phi_4 (N_{ji}^h(\kappa) + N_{ji}^h(\kappa))
\]

for the \((\phi, \psi)\)-connections \(\Gamma^1_{ji}\) and \(\Gamma^2_{ji}\) respectively.

**Corollary 1.** \(N_{ji}^h(\phi) = 2\Phi_2 \Phi_3 (N_{ji}^h(\psi) + N_{ji}^h(\kappa))\)

\(N_{ji}^h(\psi) = 2\Phi_2 \Phi_3 (N_{ji}^h(\kappa) + N_{ji}^h(\phi))\)

\(N_{ji}^h(\kappa) = 2\Phi_2 \Phi_3 (N_{ji}^h(\phi) + N_{ji}^h(\kappa)).\)

**Proof.** Since the affine connection \(\Gamma^1_{ji}\) is a \(\phi\)-connection, from (10.13) and (10.14) it follows

\[N_{ji}^h(\phi) = 2\Phi_2 \Phi_3 S_{ji}^h = 2\Phi_2 \Phi_3 (N_{ji}^h(\psi) + N_{ji}^h(\kappa))\]

As an immediate consequence of Corollary 1 we have

**Corollary 2.** \(N_{ji}^h(\psi), N_{ji}^h(\phi)\) and \(N_{ji}^h(\kappa)\) vanish identically if any two of them vanish identically.

**Corollary 3.** If \(N_{ji}^h(\phi) = 0\), the \((\phi, \psi)\)-connection \(\Gamma^0_{ji}\) is a \(\phi\)-connection such that the affine connection \(\Gamma^0_{ji} = \Gamma^1_{ji}\) is also a \(\phi\)-connection.

**Proof.** If \(N_{ji}^h(\phi) = 0\), we have

\[\Phi_1 S_{ji}^h = \Phi_2 \Phi_3 (N_{ji}^h(\psi) + N_{ji}^h(\kappa)) = \frac{1}{2} N_{ji}^h(\phi) = 0.
\]

Then by Theorem 7.6, Corollary 3 is established.

Now if \(N_{ji}^h(\phi) = N_{ji}^h(\psi) = N_{ji}^h(\kappa) = 0\), then three \((\phi, \psi)\)-connections in Theorem 10.2 are all symmetric. Thus we have

**Theorem 10.5** In a quaternion manifold in order that the Nijenhuis tensors \(N_{ji}^h(\phi), N_{ji}^h(\psi)\) and \(N_{ji}^h(\kappa)\) simultaneously vanish identically, it is necessary and
sufficient that it be possible to introduce in the manifold a symmetric \((\phi, \psi)\)-connection.

It is to be remarked that if there exists a symmetric \((\phi, \psi)\)-connection, then by Theorem 10.3 it is unique.

Now given a tensor field \(S_{ji}^h\) such that

\[(10.17) \quad \text{it is skew-symmetric in its lower indices},\]
\[(10.18) \quad 2\partial_j\phi S_{ji}^h = N_{ji}^h(\phi), \quad 2\partial_j\phi_2 S_{ji}^h = N_{ji}^h(\psi), \quad 2K_j K_{ji}^h = N_{ji}^h(\kappa),\]

we see from the proof of Theorem 10.4 that

\[\Gamma_j^h = \Gamma_j^h + 4\partial_j\phi_4 S_{ji}^h\]

is the \((\phi, \psi)\)-connection whose torsion tensor is \(S_{ji}^h\) itself, where \(\Gamma_j^h\) is the \((\phi, \psi)\)-connection given in Theorem 10.4. Thus we have

**Theorem 10.6** Let \(S_{ji}^h\) be a tensor field in a quaternion manifold. In order that \(S_{ji}^h\) be a torsion tensor of a \((\phi, \psi)\)-connection it is necessary and sufficient that \(S_{ji}^h\) be skew-symmetric in its lower indices and satisfy

\[(10.17) \quad 2\partial_j\phi S_{ji}^h = N_{ji}^h(\phi), \quad 2\partial_j\phi_2 S_{ji}^h = N_{ji}^h(\psi), \quad 2K_j K_{ji}^h = N_{ji}^h(\kappa).\]

From Theorem 10.3 and Theorem 10.6 it follows that there exists a one-to-one correspondence between the set of all \((\phi, \psi)\)-connections on a quaternion manifold and the set of all tensors satisfying \((10.17)\) and \((10.18)\).

11. Affine connections in complex analytic manifolds with quaternion structure

We consider in this section, a quaternion manifold defined by \((\phi^h, \psi^h)\) such that \(\phi^h\) represents the complex analytic structure and take the complex analytic coordinate system \((z^c, \bar{z}^c)\). Then \(\phi^h\) takes the form \(\phi_i^h = I^h_i\) and \(\psi_i^h\) and \(\kappa_i^h\) have components of the forms

\[
(\psi_i^h) = \begin{pmatrix}
0 & \psi_i^{\bar{k}} \\
\bar{\psi}_k & 0
\end{pmatrix} \quad (\kappa_i^h) = \begin{pmatrix}
0 & -i\psi_i^{\bar{k}} \\
i\bar{\psi}_k & 0
\end{pmatrix}.
\]

A \(\phi\)-connection \(I_{ji}^h\) is such one that \(I_{ji}^h = I_{ji}^h = 0\). The covariant derivatives of \(\psi_i^h\):

\[
\nabla_j \psi_i^h = \partial_j \psi_i^h + \psi_i^k \Gamma_{ji}^h I_{j\lambda}^h = 0.
\]

if written out fully, are

\[
\nabla_j \psi_i^{\bar{k}} = 0; \quad \text{conj.},
\]
\[
\nabla_j \psi_i^{\bar{k}} = \partial_j \psi_i^{\bar{k}} + \psi_i^{\bar{k}} \Gamma_{ji}^h I_{j\lambda}^h = \bar{\nabla}_j \psi_i^{\bar{k}}; \quad \text{conj.},
\]
\[
\nabla_j \psi_i^{\bar{k}} = \partial_j \psi_i^{\bar{k}} + \psi_i^{\bar{k}} \Gamma_{ji}^h I_{j\lambda}^h = \bar{\nabla}_j \psi_i^{\bar{k}}; \quad \text{conj.}
\]

These equations show that in order to have \(\nabla_i \psi_i^h = 0\) it is necessary and sufficient that

\[
\nabla_i \psi_i^{\bar{k}} = \nabla_i \psi_i^{\bar{k}} = 0,
\]

which is equivalent to
It is easily verified that $\Gamma_{\mu\lambda}^\kappa$ in (11.1) are components of an affine connection. Since, if $\Gamma_{\mu\lambda}^\kappa$ is a $\phi$-connection, $(\Gamma_{\mu\lambda}^\kappa, \Gamma_{\mu\lambda}^\kappa)$ are components of a mixed tensor, we have

**Theorem 11.1** In a complex analytic manifold with quaternion structure $(\phi_i^h, \psi_i^h)$, the $(\phi, \psi)$-connection $\Gamma_{\mu\lambda}^\kappa$ is determined if the manifold is given a tensor field $T_{j\lambda}^h$ of type $(1, 2)$:

(11.2)

$$\Gamma_{\mu\lambda}^\kappa = - (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa; \text{ conj.}$$

If we give a tensor $T_{j\lambda}^h = 0$, we get the $(\phi, \psi)$-connection $\Gamma_{\mu\lambda}^\kappa = - (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa; \text{ conj.}$ and others being zero.

**Corollary 1.** In a complex analytic manifold with quaternion structure $(\phi_i^h, \psi_i^h)$, where $\phi_i^h$ gives the complex analytic structure, there exists the unique connection $\Gamma_{\mu\lambda}^\kappa = (\Gamma_{\mu\lambda}^\kappa, \Gamma_{\mu\lambda}^\kappa)$.

Such a connection $\Gamma_{\mu\lambda}^\kappa = - (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa$ has the torsion tensor $S_{j\mu}^h = (S_{\mu\lambda}^\kappa, S_{\lambda\mu}^\kappa)$, where $S_{\mu\lambda}^\kappa = - (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa$. Since we know by Theorem 3.1 that $N_{j\mu}^h(\psi)$ vanishes identically if and only if $\partial_\mu \psi_i^\lambda = 0$, we have

**Corollary 2.** In a complex analytic manifold with quaternion structure $(\phi_i^h, \psi_i^h)$, where $\phi_i^h$ gives the complex analytic structure, in order that the Nijenhuis tensor $N_{j\mu}^h(\psi)$ vanish identically it is necessary and sufficient that it be possible to introduce a symmetric $(\phi, \psi)$-connection in the manifold.

For the $(\phi, \psi)$-connection $\Gamma_{\mu\lambda}^\kappa = (\Gamma_{\mu\lambda}^\kappa, \Gamma_{\mu\lambda}^\kappa)$, where $\Gamma_{\mu\lambda}^\kappa = - (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa$, the curvature tensor $R_{\mu\lambda}^\kappa$ is given by

$$R_{\mu\lambda}^\kappa = - R_{\mu\lambda}^\kappa = \partial_\mu \Gamma_{\mu\lambda}^\kappa; \text{ conj.}$$

others being zero. On the other hand we have defined in (4.1) the tensor $M_{j\mu}^h = (M_{\mu\lambda}^\kappa, M_{\lambda\mu}^\kappa)$:

$$M_{\mu\lambda}^\kappa = \partial_\mu ((\partial_\mu \psi_i^\lambda) \psi_i^\kappa); \text{ conj.}$$

Thus we have

$$M_{\mu\lambda}^\kappa = - R_{\mu\lambda}^\kappa; \text{ conj.}$$

From this and Theorem 5.1 we have

**Theorem 11.2** In order that a quaternion structure $(\phi_i^h, \psi_i^h)$ of class $C^\omega$ in a manifold of class $C^\omega$ be integrable, it is necessary and sufficient that it be possible to introduce a symmetric $(\phi, \psi)$-connection which is locally flat.

Now given a $\phi$-connection $\tilde{\Gamma}_{\mu\lambda}^\kappa = (\tilde{\Gamma}_{\mu\lambda}^\kappa, \tilde{\Gamma}_{\mu\lambda}^\kappa, \tilde{\Gamma}_{\mu\lambda}^\kappa, \tilde{\Gamma}_{\mu\lambda}^\kappa)$ then we have

\[
(\tilde{\nabla}_\mu \psi_i^\lambda) \tilde{\psi}_i^\mu \psi_i^\kappa = (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa + \tilde{\Gamma}_{\mu\lambda}^\kappa \tilde{\psi}_i^\mu \psi_i^\kappa + \tilde{\Gamma}_{\mu\lambda}^\kappa
\]

\[
(\tilde{\nabla}_\mu \psi_i^\lambda) \tilde{\psi}_i^\mu \psi_i^\kappa = (\partial_\mu \psi_i^\lambda, \partial_\lambda \psi_i^\mu) \psi_i^\kappa + \psi_i^\lambda \tilde{\Gamma}_{\mu\lambda}^\kappa \tilde{\psi}_i^\mu \psi_i^\kappa + \tilde{\Gamma}_{\mu\lambda}^\kappa
\]

from which we have $\Gamma_{\mu\lambda}^\kappa = \mathcal{V} \Gamma_{\mu\lambda}^\kappa$ as follows:

$$\Gamma_{\mu\lambda}^\kappa = \tilde{\Gamma}_{\mu\lambda}^\kappa - \frac{1}{2} (\nabla_\mu \psi_i^\lambda) \tilde{\psi}_i^\mu \psi_i^\kappa$$
Afine connections on manifolds with almost complex, quaternion or Hermitian structure.

We have then

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} \left( \partial_\mu \phi_\lambda - \partial_\nu \phi_\lambda + \phi_\mu \Gamma^\lambda_{\nu\lambda} - \phi_\nu \Gamma^\lambda_{\mu\lambda} \right); \text{ conj.} \]

\[ \Gamma^e_{\mu\lambda} = \frac{1}{2} \left( \partial_\mu \phi_\lambda - \partial_\nu \phi_\lambda - \phi_\mu \Gamma^\lambda_{\nu\lambda} + \phi_\nu \Gamma^\lambda_{\mu\lambda} \right); \text{ conj.} \]

We have then

\[ -\left( \partial_\mu \phi_\lambda \right) \phi^\lambda_a - \phi_\mu \Gamma^\lambda_{\mu\lambda} = \left( \partial_\nu \phi_\lambda \right) \phi^\lambda_a - \phi_\nu \Gamma^\lambda_{\nu\lambda} - \frac{1}{2} \left( \phi_\mu \Gamma^\lambda_{\nu\lambda} + \phi_\nu \Gamma^\lambda_{\mu\lambda} - \Gamma^\lambda_{\mu\lambda} \right) \]

\[ = \frac{1}{2} \left( \partial_\mu \phi_a - \partial_\nu \phi_a - \phi_a \Gamma^\lambda_{\mu\lambda} - \phi_a \Gamma^\lambda_{\nu\lambda} \right) \Gamma^\lambda_{\mu\lambda}. \]

This shows that to construct \( \mathcal{A}_{Try} \) from a \( \phi \)-connection \( \mathcal{A}_{Try} \) is to construct the affine connection in Theorem 11.1 in the complex analytic case.

Next we consider a \( (\phi, \delta) \)-connection (11.2), and construct the \( (\phi, \delta) \)-connection

\[ \Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - 4\mathcal{A}_{Try} \phi_\mu S_{j\nu}^{\lambda}, \]

where \( S_{j\nu}^{\lambda} \) is the torsion tensor of \( \Gamma^\mu_{\nu\lambda} \). As is easily seen, we have

\[ (\phi_\mu \phi_\nu S_{j\nu}^{\lambda}) = \left( S_{\nu\lambda}^{\mu}, S_{\lambda\mu}^{\nu} \right), \]

\[ S_{\nu\lambda}^{\mu} = \frac{1}{2} \Gamma^\mu_{\nu\lambda}. \]

Thus we have

\[ \Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + 2\phi_a \Gamma^\lambda_{\mu\lambda} = \frac{1}{2} \Gamma^\mu_{\nu\lambda}, \]

\[ \Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - 2S_{\nu\lambda}^{\mu} = \Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\nu\lambda} = 0. \]

This shows that to construct \( \mathcal{A}_{Try} = \mathcal{A}_{Try} - 4\mathcal{A}_{Try} \phi_\mu S_{j\nu}^{\lambda} \) is nothing but to construct the connection in Corollary 1 to Theorem 11.1 in the complex analytic case.

**CHAPTER IV Almost Hermitian structure**

**12. Algebraic lemmas III**

Let \( g_{\lambda\nu} \) be a Riemannian metric and let its inverse be written \( g^{\lambda\nu} \). Using this \( g_{\lambda\nu} \) we define two linear operators \( A_1, A_2 \) operating on mixed tensors:

\[ A_1 P_{\mu}^{\nu} = \frac{1}{2} (P_{\mu}^{\nu} - g^{\lambda\alpha} g_{\lambda
u} P_{\mu}^{\alpha}), \]

\[ A_2 P_{\mu}^{\nu} = \frac{1}{2} (P_{\mu}^{\nu} + g^{\lambda\alpha} g_{\lambda
u} P_{\mu}^{\alpha}). \]

Then the propositions analogous to (6.1)-(6.5) hold good. From the definition of \( A_1, A_2 \), we see that

\[ A_1 P_{\mu}^{\nu} = P_{(\lambda\alpha}\nu\lambda) g^{\alpha\nu} = \frac{1}{2} g^{\alpha\nu} (P_{\alpha\nu} - P_{\alpha\nu}), \]

\[ A_2 P_{\mu}^{\nu} = P_{(\lambda\alpha}\nu\lambda) g^{\alpha\nu} = \frac{1}{2} g^{\alpha\nu} (P_{\alpha\nu} + P_{\alpha\nu}), \]

where
Now let $\phi^b$ be an almost complex structure which is orthogonal with respect to $g_{ih}$:
\[(12.5)\]
If we put
\[(12.6)\]
the condition $\phi_i^a \phi_a^h = -\delta_i^h$ is written as
\[(12.7)\]
and the condition (12.5) is written as
\[(12.8)\]
Among the operations $A_1$, $A_2$, $\phi_1$, $\phi_2$, there are the following relations.
\[(12.9)\]
For instance, for any tensor $P_{j}^{ih}$ we have
\[(12.10)\]
In order for a tensor $P_{j}^{ih}$ to satisfy $\phi_2 P_{j}^{ih} = A_2 P_{j}^{ih} = 0$, it is necessary and sufficient that there exist a tensor $Q_{i}^{ih}$ such that
\[(12.11)\]
13. Metric connections
Let us consider a differentiable manifold with a metric tensor $g_{ih}$. An affine connection $\Gamma_{j}^{ih}$ is said to be metric (with respect to $g_{ih}$) if
\[(13.1)\]
Let $\tilde{\Gamma}_{j}^{ih}$ be an arbitrary but fixed affine connection and $\Gamma_{j}^{ih} = \tilde{\Gamma}_{j}^{ih} + Q_{j}^{ih}$ be another affine connection. We have then
\[(13.2)\]
In order to have $\nabla_{j} g_{ih} = 0$ it is necessary and sufficient that
\[(13.3)\]
which is equivalent to
\[(13.4)\]
If we put $P_{j}^{ih} = \frac{1}{2} g^{ha} \nabla_{j} g_{ia}$, then (13.4) is written in the form
\[(13.5)\]
But we have
\[(13.6)\]
Then given an affine connection \( \hat{\Gamma}^{b}_{ji} \), there always exists a tensor \( Q^{b}_{ji} \) satisfying (13.5). Thus in order to have \( \nabla_{j}g_{ia} = 0 \) it is necessary and sufficient that \( Q^{b}_{ji} \) be of the form

\[
Q^{b}_{ji} = P^{b}_{ji} + A^{b}_{ji},
\]

where \( A^{b}_{ji} \) is arbitrary. Thus we have

**Theorem 13.1**\(^{26}\) Let \( \hat{\Gamma}^{b}_{ji} \) be an arbitrary but fixed affine connection in a differentiable manifold with metric tensor \( g_{ia} \). Then in order that an affine connection \( \Gamma^{b}_{ji} \) be metric with respect to \( g_{ia} \), it is necessary and sufficient that \( \Gamma^{b}_{ji} \) be written in the form

\[
\Gamma^{b}_{ji} = \hat{\Gamma}^{b}_{ji} + \frac{1}{2} g^{ha} \nabla_{j}g_{ia} + \frac{1}{2} (A^{b}_{ji} - g^{hb}g_{ja}A^{b}_{ja}),
\]

\( A^{b}_{ji} \) being a tensor.

Now let \( I^{b}_{ji} \) be an affine connection defined by

\[
I^{b}_{ji} = \hat{\Gamma}^{b}_{ji} + A^{b}_{ji},
\]

We have then

\[
g^{ha} \nabla_{j}g_{ia} = g^{ha} \nabla_{j}g_{ia} - g^{ha}A^{b}_{ji}g_{ba} - g^{ha}A^{b}_{ja}g_{ba} - g^{ha}A^{b}_{ja}g_{ba} = g^{ha} \nabla_{j}g_{ia} - A^{b}_{ji} - g^{hb}g_{ia}A^{b}_{ja}.
\]

If we write

\[
I^{b}_{ji} = M^{b}_{ji} = I^{b}_{ji} + \frac{1}{2} g^{ha} \nabla_{j}g_{ia},
\]

we get

\[
M^{b}_{ji} = M^{b}_{ji} + \frac{1}{2} g^{ha} \nabla_{j}g_{ia}.
\]

Thus we have

**Theorem 13.2** In a differentiable manifold with a metric \( g_{ia} \), an affine connection \( \Gamma^{b}_{ji} \) is metric with respect to \( g_{ia} \) if and only if there exists an affine connection \( \hat{\Gamma}^{b}_{ji} \) such that \( \Gamma^{b}_{ji} = M^{b}_{ji} \).

If we denote by \( S^{b}_{ji} \) the torsion tensor of a metric connection \( \Gamma^{b}_{ji} \), then, as is well known\(^{27}\), the so-called Christoffel symbol belonging to \( g_{ia} \) is given by

\[
\{j^{b}_{i}\} = \Gamma^{b}_{ji} - g^{ha}(S_{jai} - S_{iaj} + S_{aji}),
\]

where

\[
S^{b}_{ji} = S^{b}_{ji} g_{ab}.
\]

Furthermore, it is known that any two metric connections which have the same torsion tensor coincide with each other. This follows from the fact that if a

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\(^{26}\) Cf. Kawaguchi [14].

\(^{27}\) For instance, see Schouten [23].
tensor $T_{jih}$ is symmetric in $ji$ and skew-symmetric in $ih$, then $T_{jih}$ is the zero tensor.

14. Almost Hermitian structure

We consider an almost complex manifold defined by $\phi_i^h$. If a Riemannian metric $g_{ih}$ on the manifold satisfies

$$g_{ih} = \phi_i^h \phi_h^a g_{ag},$$

or equivalently

$$\phi_a g_{ih} = 0 \text{ or } \Lambda a \phi_i^h = 0,$$

due to the metric $g_{ih}$ is called an *almost Hermitian metric*. As is easily seen, (14.2) is equivalent with the condition that the tensor $\phi_i^{ag} g_{ah}$ is anti-symmetric.

We call an *almost Hermitian manifold* an almost complex manifold endowed with an almost Hermitian structure. In an almost Hermitian manifold the tensor

$$\phi_{jih} = \partial_{[j} \phi_{ih]}$$

is anti-symmetric in its covariant indices. If the tensor $\phi_{jih}$ vanishes identically, the manifold is called an *almost Kählerian manifold* and the metric is called the *almost Kählerian metric*. Furthermore if the tensor $N_{i^h}g(a)$ vanishes identically, the manifold is called a *pseudo-Kählerian manifold*.

**Theorem 14.1** 28) In an almost complex manifold there always exists an almost Hermitian metric.

**Proof.** Let $\gamma_{ih}$ be an arbitrary Riemannian metric in the manifold. Since $\phi_i^h \phi_h^a \gamma_{ah}$ is also positive definite and symmetric, i.e., Riemannian, so also is $g_{ih} = \phi_i^h \gamma_{ih}$. Furthermore we have clearly $\phi_a g_{ih} = 0$. Since $\gamma_{ih}$ always exists 29), the existence of an almost Hermitian metric is established.

If a manifold admits a quaternion structure $(\phi_i^h, \phi_i^{h'})$, starting from an arbitrary Riemannian metric $\gamma_{ih}$, we can construct a Riemannian metric $g_{ih} = \phi_i^h \gamma_{ih}$, which is an almost Hermitian metric with respect to both $\phi_i^h$ and $\phi_i^{h'}$. Thus we have

**Theorem 14.2** If a manifold admits a quaternion structure $(\phi_i^h, \phi_i^{h'})$, there always exists an almost Hermitian metric with respect to both $\phi_i^h$ and $\phi_i^{h'}$.

15. Affine connections in almost Hermitian manifolds 29)

We consider in this section an almost Hermitian manifold defined by $\phi_i^h$ and $g_{ih}$ and give a method of obtaining a metric $\phi$-connection starting from any affine connection in the manifold.

**Lemma 15.1** In an almost Hermitian manifold for any affine connection $\Gamma^h_{ji}$ we have

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28) Cf. Frölicher [9], Lichnerowicz [17].

29) Cf. Libermann [15], Lichnerowicz [17, 18], Patterson [22], Schouten and Yano [25], Yano [30].
Affine connections on manifolds with almost complex, quaternion or Hermitian structure.

PROOF. By the definitions we have

\[ \Phi \Gamma^h_{\beta i} = \Gamma^h_{\beta i} + \frac{1}{4} (\nabla_j g_{ia}) g^{ah} - (\nabla_j \phi^a_i) \phi^h_a - (\nabla_j \phi^a_i) \phi^{ah}. \]

Now on account of \( \phi_{ia} = \phi^h_b g_{ba} \) we get

\[ \nabla_j \phi_{ia} = \nabla_j \phi^h_a g_{ba} + \phi^h_b \nabla_j g_{ba}, \]

from which

\[ \phi^h_b (\nabla_j g_{ba}) \phi^{ah} = (\nabla_j \phi^h_a) g_{ba} + \phi^h_b \nabla_j g_{ba}, \]

Thus we have

\[ \Phi \Gamma^h_{\beta i} = \Gamma^h_{\beta i} - \frac{1}{2} (\nabla_j \phi^a_i) \phi^h_a + \frac{1}{4} (\nabla_j g_{ia}) g^{ah} - \frac{1}{4} (\nabla_j \phi^a_i) \phi^h_a + \frac{1}{4} (\nabla_j \phi^a_i) \phi^{ah}. \]

In the same way we have

\[ A \Phi \Gamma^h_{\beta i} = A(\Gamma^h_{\beta i} - \frac{1}{2} (\nabla_j \phi^a_i) \phi^h_a) \]

\[ = \nabla_j \phi^h_a g_{ba} + \phi^h_b \nabla_j g_{ba} + \frac{1}{2} (\nabla_j g_{ia}) g^{ah} - (\nabla_j \phi^a_i) \phi^h_a - (\nabla_j \phi^a_i) \phi^{ah}. \]

By the same arguments as in Theorem 10.1 we have

**Theorem 15.1** In an almost Hermitian manifold an affine connection \( \Gamma^h_{\beta i} \) is a metric \( \Phi \)-connection, i.e. an affine connection in which the almost complex structure and the almost Hermitian metric are covariant constant, if and only if there exists an affine connection \( \tilde{\Gamma}^h_{\beta i} \) such that \( \Gamma^h_{\beta i} = \Phi \tilde{\Gamma}^h_{\beta i} \).

**Corollary 1.** In an almost Hermitian manifold there always exists a metric \( \Phi \)-connection.

**Corollary 2.** Let \( \Gamma^h_{\beta i} \) be a metric \( \Phi \)-connection in an almost Hermitian manifold, then \( \tilde{\Gamma}^h_{\beta i} = \Gamma^h_{\beta i} + A_{ji}^h \), \( A_{ji}^h \) being a tensor field, is a metric \( \Phi \)-connection if and only if \( \Phi_{3} A_{ji}^h = A_{3} A_{ji}^h = 0 \) or equivalently there exists a tensor field \( B_{ji} \) such that \( A_{ji}^h = \phi^h_a A_{ai} B_{ji}^h \).

It is to be noted that two metric \( \Phi \)-connections coincide with each other if and only if their torsion tensors coincide with each other.

Now consider a differentiable manifold admitting a quaternion structure \( (\phi^a_i, \phi^h_i) \) and an almost Hermitian metric \( g_{ia} \) with respect to both \( \phi^a_i \) and \( \phi^h_i \). Since \( \phi^h_i \) is also orthogonal with respect to \( g_{ia} \), i.e. \( \nabla_{j} g_{ia} = 0 \), we have

\[ \Phi A \Gamma^a_{\beta i} = A \Psi \Gamma^a_{\beta i} \]
for any affine connection. It follows that the operations $\Phi$, $\Psi$, $A$ commute with each other. Thus we have

**Theorem 15.2** In a manifold admitting a quaternion structure $(\varphi^h, \varphi^h)$ and an almost Hermitian metric $g_{ih}$ with respect to both $\varphi^h$ and $\varphi^h$, there always exists a metric $(\varphi, \varphi)$-connection. Such a connection and only such one is written in the form $\Phi \Lambda \Gamma^h_j$ for some affine connection $\Gamma^h_j$.

Again we consider an almost Hermitian manifold endowed with a metric $\varphi$-connection $\Gamma^a_{ji}$. Denoting by $\nabla_j$ the operation of covariant differentiation with respect to $\Gamma^a_{ji}$ we have

$$\nabla_j \varphi_{ih} = \partial_j \varphi_{ih} - \Gamma^a_{ja} \varphi_{ah} - \Gamma^a_{ja} \varphi_{ah} = 0,$$

from which

$$\varphi_{jih} = \frac{1}{3} (\partial_j \varphi_{ih} + \partial_j \varphi_{hj} + \partial_h \varphi_{ji})$$

$$= \frac{1}{3} (\Gamma^a_{ja} \varphi_{ih} + \Gamma^a_{ja} \varphi_{hj} + \Gamma^a_{ja} \varphi_{hi} + \Gamma^a_{ja} \varphi_{ja} + \Gamma^a_{ja} \varphi_{ih})$$

$$= \frac{1}{3} \{(\Gamma^a_{ja} - \Gamma^a_{ja}) \varphi_{ah} + (\Gamma^a_{j} - \Gamma^a_{j} \varphi_{aj} + (\Gamma^a_{j} - \Gamma^a_{j} \varphi_{aj}) \varphi_{aj} \}$$

$$= -\frac{2}{3} (S_{jih} \varphi_{ah} + S_{jih} \varphi_{aj} + S_{jih} \varphi_{ai}) = -2S_{jih} \varphi_{ah}.$$

If we put $S_{jih} = S_{jih} g_{ah}$ we get

(15.1) $$\varphi_{jih} = -\frac{2}{3} (\varphi_{h} S_{jih} + \varphi_{h} S_{jih} + \varphi_{h} S_{jih}) = -\varphi_{jih} S_{jih},$$

from which

$$\varphi_{h} \varphi_{jih} = \frac{2}{3} (S_{jih} - \phi_j S_{jih} - \phi_j S_{jih}) = \phi_{jih} S_{jih},$$

$$\varphi_{h} \varphi_{jih} = \frac{2}{3} (S_{jih} - \phi_j S_{jih} + \phi_j S_{jih}) = \phi_{jih} S_{jih},$$

$$\varphi_{h} \varphi_{jih} = \frac{2}{3} (S_{jih} - \phi_j S_{jih} + \phi_j S_{jih}) = \phi_{jih} S_{jih},$$

Furthermore since $\Gamma^a_{ji}$ is a $\varphi$-connection, the Nijenhuis tensor $N_{jih}^a(\varphi)$ and the torsion tensor $S_{jih}^b$ of $\Gamma^a_{ji}$ are related by (7.3), which are written in the form with covariant indices

(15.3) $$N_{jih}^a(\varphi) = N_{jih}^a(\varphi) g_{ah} = 2(\varphi_{jih} S_{jih}) g_{ah}$$

$$= \frac{1}{2} (S_{jih} - \phi_j S_{jih} - \phi_j S_{jih} - \phi_j S_{jih}).$$

Now we construct an affine connection

(15.4) $$\Gamma^b_{ji} = \Gamma^a_{ji} - 4A_i \Phi_j \Phi_i S_{jih},$$

which is obviously a metric $\varphi$-connection by Theorem 15.1. On putting

$$T^b_{ji} = 4A_i \Phi_j \Phi_i S_{jih} \quad \text{and} \quad T_{jih} = T_{jih} g_{ah},$$
we have by straight calculations
\[ T_{jkh} = \frac{1}{2} (S_{jkh} + \phi_j^a \phi_k^b S_{cbh} + \phi_k^b \phi_h^a S_{jba} - \phi_h^a \phi_j^b S_{cia}) + S_{hjk} + \phi_h^a \phi_j^b S_{act} + \phi_j^b \phi_h^a S_{a,jb} - \phi_j^b \phi_k^a S_{hoc}). \]

Then the torsion tensor \( S_{jkh} \) of \( I_{jkh} \) is written as
\[ S_{jkh} = S_{jkh} - T_{jkh}. \]

or with covariant indices
\[ 4S_{jkh} = 4S_{jkh} - 4T_{jkh}. \]

From (15.2) and (15.3) we get
\[ (15.5) \]
\[ \text{THEOREM 15.3} \quad \text{In an almost Hermitian manifold it is possible to introduce a metric} \phi-\text{connection whose torsion tensor is given by (15.5).} \]

Now if the manifold is almost Kählerian, i.e. if \( \phi_{jkh} = 0 \), we have
\[ (15.6) \]
\[ \text{Finally if the manifold is pseudo-Kählerian, i.e. if} \ N_{jkh} = 0, \ \phi_{jkh} = 0, \ \text{we have} \]
\[ (15.8) \]
\[ \text{and the connection becomes Riemannian. Conversely if the Riemannian connection is a metric} \phi-\text{connection in an almost Hermitian manifold, then it is easy to see that} \]
\[ N_{jkh} = 0 \quad \text{and} \ \phi_{jkh} = 0, \]
\[ \text{so the manifold is pseudo-Kählerian. Thus we have} \]
\[ \text{THEOREM 15.4} \quad \text{In order that an almost Hermitian manifold be pseudo-Kählerian, it is necessary and sufficient that the Riemannian connection be a} \phi-\text{connection.} \]

Since the connection \( I_{jkh} \) is a metric \( \phi \)-connection, in an almost Hermitian manifold, the torsion tensor \( S_{jkh} \) satisfies the relation (15.1). Substituting (15.5) into (15.1) and simplifying, we find
\[ (15.9) \quad \frac{1}{4} (\phi_{jkh} - \phi_j^a \phi_k^b \phi cbh - \phi_h^a \phi_j^b \phi cia - \phi_i^b \phi_h^a \phi_{jba}) = - N_{jkh} \phi_{jkh} = - N_{jkh} \phi_{jkh}. \]
which means that the pure components of $\phi_{j\ell h}$ coincide with $-\phi_{j\ell}^f N_{h\beta}(\phi)$. Thus we have, for an almost Kählerian space:

$$\phi_{j\ell}^f N_{h\beta}(\phi) = 0,$$

and for a pseudo-Hermitian space:

$$\phi_{j\ell h} - \phi_{j}^f \phi_{h\beta} - \phi_{h}^f \phi_{j \ell a} - \phi_{j}^h \phi_{h \beta a} = 0,$$

which means that the pure components of $\phi_{j\ell h}$ vanish identically.

Now it is easily seen that the proof of (15.5) is based only on (7.3) and (15.1). Therefore in an almost Hermitian manifold given a tensor field $S_{j\ell \beta}$ which is anti-symmetric in its covariant indices and satisfies (7.3) and (15.1), the affine connection

$$\Gamma_{j\ell}^h = \tilde{\Gamma}_{j\ell}^h + 4 A_{j} \phi_{\ell} S_{j\ell \beta}$$

is a metric $\phi$-connection whose torsion tensor is $S_{j\ell \beta}$, where $\tilde{\Gamma}_{j\ell}^h$ is one given in (15.4) and with the torsion tensor (15.5). Thus we have

**Theorem 15.5** Let $S_{j\ell \beta}$ be a tensor field in an almost Hermitian manifold. In order that $S_{j\ell \beta}$ be a torsion tensor of a metric $\phi$-connection it is necessary and sufficient that $S_{j\ell \beta}$ be anti-symmetric in its lower indices and satisfy

$$2\phi_{j}^f S_{j\ell \beta} = N_{j\ell \beta}(\phi), \quad 2 S_{j\ell \beta} \phi_{h \beta a} = -\phi_{j\ell h}.$$

16. Affine connections in Hermitian manifolds

We consider a complex analytic manifold of $n$ complex dimensions endowed with a Hermitian metric $g_{i\lambda h}$, where the Riemannian metric tensor $g_{i\lambda}(z, \bar{z})$ is self-adjoint and satisfies $\phi_{\alpha} g_{i\lambda h} = 0$, or equivalently

$$g_{\lambda \kappa} = g_{\bar{\kappa} \bar{\lambda}} = 0.$$

It follows that

$$g_{\lambda \kappa} = g_{\bar{\kappa} \bar{\lambda}} = 0, \quad g_{\lambda \kappa} = g_{\bar{\kappa} \bar{\lambda}} = g_{\bar{\kappa} \bar{\lambda}}.$$

If there is given a $\phi$-connection $\Gamma_{j\ell}^h = (\tilde{\Gamma}_{j\ell}^h, \Gamma_{j\ell}^\alpha)$, the covariant derivatives of $g_{i\lambda}$:

$$\nabla_{j\ell} g_{i\lambda} = \partial_{j\ell} g_{i\lambda} - \Gamma_{j\ell}^\alpha g_{i\alpha} - \Gamma_{j\ell}^\lambda g_{i\alpha}$$

are

$$\nabla_{j} g_{\lambda \kappa} = \nabla_{j} g_{\bar{\lambda} \bar{\kappa}} = 0, \quad \nabla_{j} g_{\lambda \kappa} = \partial_{j} g_{\lambda \kappa} - \Gamma_{j}^\alpha g_{\lambda \alpha} - \Gamma_{j}^\lambda g_{\lambda \alpha}; \quad \text{conj}.$$

the others being given by symmetry and self-adjointness of $g_{i\lambda}$. These equations show that in order to have $\nabla_{j\ell} g_{i\lambda} = 0$ it is necessary and sufficient that $\Gamma_{j\ell}^h$ is such that

$$\Gamma_{j\ell}^\kappa = g_{\lambda \kappa} \partial_{j} g_{\lambda \bar{\beta}} - g_{\bar{\lambda} \bar{\kappa}} g_{\bar{\beta} \bar{\lambda}} \Gamma_{j\ell}^\beta; \quad \text{conj},$$

$$(\Gamma_{j\ell}^\kappa, \Gamma_{j\ell}^\bar{\kappa})$$

being arbitrary.

On the other hand, the Christoffel symbols have the values

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30) Cf. the footnote 29) and also cf. Bochner [2], Chern [4, 5], Guggenheimer [11], Lichnerowicz [17, 18], Schouten and van Dantzig [24], Schouten and Yano [25], Yano [29, 30].
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(16.4) \{\mu, \nu\} = g^{\alpha \beta} \partial_{\alpha} g_{\mu \nu} \partial_{\beta}, \quad \{\mu, \kappa\} = g^{\alpha \beta} \partial_{\alpha} \partial_{\kappa} g_{\mu \nu}, \quad \{\mu, \kappa\} = g^{\alpha \beta} \partial_{\alpha} \partial_{\mu} g_{\kappa \nu}; \quad \text{conj.}

These show that the metric \( \phi \)-connection \( \Gamma_{\mu \kappa}^{\alpha} = \phi \{\mu, \kappa\} \) have values

(16.5) \Gamma_{\mu \kappa}^{\alpha} = \{\mu, \kappa\}, \quad \Gamma_{\mu \kappa}^{\alpha} = \{\mu, \kappa\}; \quad \text{conj.},

the others being zero, and also \( \Gamma_{\mu \kappa}^{\alpha} \) is obtainable by putting \( \Gamma_{\mu \kappa}^{\alpha} = g^{\alpha \beta} \partial_{\beta} g_{\mu \nu} \) in (16.3).

Now if we put \( \Gamma_{\mu \kappa}^{\alpha} = 0 \) in (16.3), we have a connection

(16.6) \hat{\Gamma}_{\mu \kappa}^{\alpha} = g^{\alpha \beta} \partial_{\beta} g_{\mu \nu}; \quad \text{conj.}

the others being zero. It is also verified that \( \hat{\Gamma}_{\mu \kappa}^{\alpha} \) is obtained from \( \Gamma_{\mu \kappa}^{\alpha} \) in (16.2) by the method of Theorem 15.3:

(16.7) \hat{\Gamma}_{\mu \kappa}^{\alpha} = \Gamma_{\mu \kappa}^{\alpha} - 4 \Omega_{\mu}^{\alpha} \phi_{\mu \nu} S_{\mu \nu}^{\kappa}.

In fact, only non-vanishing components of \( \phi_{\mu}^{\nu} \phi_{\mu}^{\nu} \) are \( S_{\mu \kappa}^{\mu} = \frac{1}{2} \Gamma_{\mu \kappa}^{\mu} \); \quad \text{conj.}, and consequently those of

\[ 4 \Omega_{\mu}^{\alpha} \phi_{\mu}^{\nu} S_{\mu \nu}^{\kappa} = -g^{\mu \nu} g_{\mu \nu} \Gamma_{\mu \kappa}^{\mu}, \quad 4 \Omega_{\mu}^{\alpha} \phi_{\mu}^{\nu} S_{\mu \nu}^{\kappa} = \Gamma_{\mu \kappa}^{\mu}; \quad \text{conj.}

From these it follows that (16.6) and (16.7) are identical.

17. Pseudo-Kählerian manifold with quaternion structure\(^{31}\)

We consider a pseudo-Kählerian manifold defined by \( \phi_{\mu}^{\nu} \) and \( g_{\mu \nu} \). By definition in the manifold we have

\[ N_{\mu \nu}^{\rho}(\phi) = 0, \quad \phi_{\mu}^{\nu} g_{\nu \rho} = 0, \quad \phi_{\mu \nu} = \partial_{\rho} \phi_{\mu \rho} = 0 \]

and the Christoffel symbol \( \{_{\mu \nu}^{\rho}\} \) belonging to \( g_{\mu \nu} \) is a \( \phi \)-connection by virtue of Theorem 15.4.

In the pseudo-Kählerian manifold, as is well known,\(^{32}\) from the Ricci identity

(17.1) \[ R_{\mu \nu \kappa}^{\rho} = \phi_{\mu}^{\rho} \phi_{\kappa}^{\nu} R_{\kappa \rho}, \quad \phi_{\mu}^{\rho} = \partial_{\sigma} \phi_{\mu}^{\rho} R_{\nu \rho} \]

where \( R_{\mu \nu \kappa}^{\rho} = R_{\mu \nu \kappa}^{\rho} g_{\rho \kappa} \). These show that \( R_{\mu \nu \kappa}^{\rho} \) is hybrid in \( \mu \nu \) and \( \kappa \rho \). Furthermore for the Ricci curvature we have

(17.2) \[ 2 \phi_{\mu}^{\rho} R_{\mu \rho} = \phi_{\nu}^{\rho} R_{\nu \rho} \]

Now we assume, in our pseudo-Kählerian manifold, there is given another almost complex structure \( \phi_{\mu}^{\nu} \) which constitutes a quaternion structure \( (\phi_{\mu}^{\nu}, \psi_{\mu}^{\nu}) \) together with \( \phi_{\mu}^{\nu} \). We further assume that the Riemannian connection leaves \( \phi_{\mu}^{\nu} \) invariant: \( \nabla_{\mu} \phi_{\nu}^{\rho} = 0 \). Then in the quite same way as in the case of \( \phi_{\mu}^{\nu} \) we obtain

(17.3) \[ 2 \phi_{\mu}^{\rho} R_{\mu \rho} = \phi_{\nu}^{\rho} R_{\nu \rho} \]

On the other hand, \( \phi_{\mu}^{\rho} \phi_{\nu}^{\rho} + \psi_{\mu}^{\rho} \phi_{\nu}^{\rho} = 0 \) implies

(17.4) \[ \phi_{\mu}^{\rho} + \phi_{\nu}^{\rho} \phi_{\mu}^{\nu} = 0, \]

31) There are some results of H. Wakakuwa on this subject which are not yet published.
which means that $\varphi^i_{\ell h}$ is pure in $i h$ with respect to $\phi_i^h$.

Now we have the following

**Lemma 17.1** Let $P^i_{\ell h}$ and $Q_{i h}$ be tensors. If $P^i_{\ell h}$ is pure and $Q_{i h}$ is hybrid with respect to $\phi_i^h$, then we have $P^i_{\ell h}Q_{i h} = 0$.

**Proof.** From the assumptions we have

$$P^i_{\ell h} + \phi^i_{\ell} \phi^h_{\ell} P^h_{\ell h} = 0, \quad Q_{i h} - \phi^i_{h} \phi^h_{i} Q_{i h} = 0.$$

Then we find

$$P^i_{\ell h}Q_{i h} = P^h_{\ell h}(\phi^i_{\ell} \phi^h_{\ell} Q_{i h}) = (P^i_{\ell h} \phi^h_{\ell} + \phi^h_{\ell} Q_{i h}) = -P^h_{\ell h} Q_{i h},$$

from which

$$P^i_{\ell h}Q_{i h} = 0.$$

Since $R_{dei h}$ is hybrid and $\varphi^{de} = \text{pure in } dc$ we have

$$\varphi^{de} R_{dei h} = 0,$$

from which and (17.4) we obtain $\phi^i_{h} R_{i h} = 0$. Since $\varphi^i_{h}$ has inverse, we get

$$R_{i h} = 0.$$

Thus we have

**Theorem 17.1** If a pseudo-Kählerian manifold has a quaternion structure, then the Ricci curvature vanishes identically.

This is evident if we consider the homogeneous holonomy group of the manifold. In fact, the holonomy group of the $4m$ dimensional pseudo-Kählerian manifold with a quaternion structure is contained in the unitary symplectic group $Sp(m)$ which is a subgroup of the special unitary group $SU(n)$. A pseudo-Kählerian manifold whose holonomy group is a subgroup of $SU(n)$ has the vanishing Ricci curvature.\[33\]

Now if a pseudo-Kählerian manifold has constant holomorphic curvature, then the curvature tensor has the form

$$(17.6) \quad R_{i h j k} = \frac{k}{2} (g_{i[z_1 j} g_{j h]} + \phi^{[z_1 i] h] \phi^{j]}_{i} \phi_{[i h]}, \quad k = \text{const.}$$

from which we obtain

$$(17.7) \quad R_{i h} = \frac{(n+1)k}{2} g_{i h}, \quad R = n(n+1)k.$$

**Theorem 17.2** If a pseudo-Kählerian manifold with vanishing Ricci curvature has constant holomorphic curvature, then it is of zero curvature.

**Corollary.** If a pseudo-Kählerian manifold with a quaternion structure has constant holomorphic curvature, then it is of zero curvature.

**Bibliography**


[33] Cf. Lichnerowicz [17].
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