On the number of anticonformal automorphisms of a Riemann surface

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Let R be an orientable Riemann surface of genus g with k boundary components, N the number of conformal mappings of R onto itself and $\bar{N}$ the number of anticonformal mappings of R onto itself. If k is not zero, we set the following restriction; Let R be a finite subsurface of another Riemann surface and its relative boundary consist of k analytic curves. In this paper, we shall prove the next theorem.

THEOREM A. *For any given g and k we have*

$$\text{Max } N = \text{Max } \bar{N}$$

where maxima are taken over all orientable Riemann surfaces respectively.

We first prepare two lemmas.

LEMMA 1. *For each R , $\bar{N}$ is equal to either N or 0.*

Proof. Let $\mathfrak{G}$ be the group of all conformal mappings of R onto itself and $\mathfrak{A}$ the set of all anticonformal mappings of R onto itself. If $\mathfrak{A}$ is not empty, there is at least one $A \in \mathfrak{A}$. For each element $T_i \in \mathfrak{G}$, there corresponds $AT_i \in \mathfrak{A}$. Conversely, for each $A_j \in \mathfrak{A}$, there corresponds $A^{-1}A_j \in \mathfrak{G}$. This correspondence is surely one-to-one, so that $\bar{N}$ is equal to N unless $\mathfrak{A}$ is empty.

LEMMA 2. *There exists at least one compact Riemann surface of genus g which has an anticonformal mapping with one fixed line.*

Proof. The algebraic genus, i. e. the genus of the double of a Riemann surface with m boundary components, h handles and c cross-caps is given by

$$G = 2h + m + c - 1.$$

For $m=1$, the equation $g=2h+c$ has always non-negative integral solutions, whatever g may be. Thus the desired surface can be obtained as the double of another surface.

Next, we state two results which have been established by A. Hurwitz and K. Oikawa.

Let R_0 be a compact Riemann surface of genus g . We fix k points $p_1, \dots, p_k$ on R_0 . Let N_0 be the number of all conformal mappings of $R_0 - \{p_1, \dots, p_k\}$ onto itself. For fixed g and k , $\text{Max } N = \text{Max } N_0$. This has been proved by K. Oikawa [2].

Let R be a compact Riemann surface of genus g which has the group of conformal mappings of maximal order. Let $\mathfrak{G}$ be the group of all conformal mappings of R onto itself. By identifying the points congruent with respect to $\mathfrak{G}$,

we get a new surface R_1 . Let π be the genus of R_1 , and $a_1, \dots, a_w$ be the branch points on R .

We connect $a_1, \dots, a_w$ with an inner point p of R_1 by non-intersecting paths, and cut R_1 along these paths and homology bases of R_1 . Thus, we get a simply-connected region R_2 . R may be regarded as an N -sheeted covering of R_2 and is constructed from N copies of R_2 by a suitable patching-work. The reconstruction depends only on the abstract structure of $\mathfrak{G}$ as well as the multiplicities of branch-points.

Now, we consider another simply-connected region R_2' . We take arbitrary w points $a_1', \dots, a_w'$ on R_2' . By the same way as was used in reconstructing R from R_2 , we get a new compact Riemann surface R' from R_2' . The group $\mathfrak{G}'$ of all conformal mappings of R' onto itself has a subgroup which is isomorphic to $\mathfrak{G}$.

Above discussion was shown in detail by A. Hurwitz [1], and this construction is essential for our present purpose. Now we proceed to prove the theorem A, by showing constructively the existence of an anticonformal mapping.

Let R_0 be a Riemann surface of genus g and with k boundary components which has the maximal number of conformal mappings. From Oikawa's work, we may take, instead of R_0 , a compact Riemann surface R of genus g and with k invariant points. From the conformal mapping group $\mathfrak{G}$ of R , we can determine, following Hurwitz's idea stated above, R_1 , π , R_2 , $a_1, \dots, a_w$ and their multiplicities. k points $p_1, \dots, p_k$ may lie on $a_1, \dots, a_w$ or not. Let the projections of these points on R_1 be $q_1, \dots, q_s$.

By lemma 2, there exists a compact Riemann surface R_1' of genus π which has one fixed line. Since it can be regarded as the double of another finite Riemann surface, there exists a symmetric homology basis. On the fixed line we set w points $a_1', \dots, a_w'$ and s points $q_1', \dots, q_s'$. These s points are to be situated in the same manner as on R_1 , that is, q_i' is to be situated on a_j' when q_i is situated on a_j and conversely.

We choose on the fixed line a point p' which coincides with neither q_i' nor a_j' . We connect p' with each of a_j' by non-intersecting paths and cut R_1' along these and symmetric homology basis.

We thus get a new simply-connected region R_2' . We construct a new surface R' according to Hurwitz's method of reconstruction. Then, R' is a Riemann surface of genus g and with k fixed points, and the group $\mathfrak{G}'$ of conformal mappings of R' onto itself has a subgroup which is isomorphic to $\mathfrak{G}$. But we have supposed that for a given pair of g and k , R has the conformal mapping group of maximal order. Hence the order of $\mathfrak{G}'$ is equal to N .

It remains to show the existence of an anticonformal mapping of R' onto itself. If the existence is assured, the proof of our theorem will be completed by means of Lemma 1.

We denote a point of R' by $T_i(p)$, where T_i is an element of $\mathfrak{G}$ and p is a point of R_2' . Except boundary of R_2' , this notation is allowed, because R' was constructed from R_2' by means of the abstract structure of $\mathfrak{G}$. Since R_1' has an anticonformal mapping with one fixed line, R_2' has also an anticonformal mapping A_0 with fixed lines. There corresponds $T_i(p)$ to $T_i^{-1}\{A_0(p)\}$. This correspondence is an anticonformal mapping in every $T_i(R_2')$. Concerning boundary of R_2' , $T_j^{-1}\{A_0(R_2')\}$ and $T_i^{-1}\{A_0(R_2')\}$ are adjacent whenever $T_j(R_2')$ and $T_i(R_2')$ are adjacent. Hence the above anticonformal mapping is continuable across the boun-

dary of R_2' . Thus the existence of an anticonformal mapping of R' onto itself is assured, and our theorem has been proved.

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References

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- [2] Oikawa, K., A supplement to “Note on conformal mappings of a Riemann surface onto itself”. Kōdai Math. Sem. Rep. **8** (1951), 115–116.