The de Rham decomposition, isometries and affine transformations in Riemannian spaces

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Introduction.

The main purpose of the present paper is to show how the affine transformation group $A(M)$ and the isometry group $I(M)$ of a Riemannian space $M$ are decomposed according as the de Rham decomposition (de Rham [1]).

We define the de Rham decomposition in a way slightly different from the usual one. Namely the de Rham decomposition of $M$, or of the tangent space $T=T(x)$ at a point $x \in M$ is the direct sum decomposition $T=T_0+T_1+\cdots+T_r$ such that

1) $T_a$ is orthogonal to $T_{\beta}$ for $a \neq \beta$ ($a, \beta=0, 1, 2, \cdots, r$),
2) the homogeneous holonomy group $H$ leaves invariant $T_a$,
3) $H$ is irreducible on $T_a$ ($0 < a$),
4) $T_0$ is the maximal subspace on which $H$ acts as a discrete group.

Usually the condition 4) is replaced by the condition: $H$ is trivial on $T_0$. Unlike the usual one, the de Rham decomposition defined here will turn out to be unique up to the order of $T_a$'s.

Carrying $T_a$ at $x$ by parallel displacement along curves in $M$ to every point of $M$, we obtain a distribution, which we denote also by $T_a$, and a tensor field $P_a$ which assigns to a point $y \in M$ the orthogonal projection of $T(y)$ onto $T_a(y)$.

Let $A^I=A^I(M)$, $A^L=A^L(M)$, $I^I=I^I(M)$ and $I^L=I^L(M)$ be respectively the Lie algebra of infinitesimal affine transformations, that of the affine transformation group, that of Killing fields and that of the isometry group.

Then, one of our main results states that, assigning $P_a v$ to each $v \in A^I$, we get a homomorphism of $A^I$ onto a subalgebra $A_a$ and furthermore $A^I$ is isomorphic onto the direct sum $\sum A_a$ of these subalgebras. Similar facts for other algebras will be also proved. We shall consider, instead of $A_a$ and $I_a$, also $A^I(M_a)$ and $I^I(M_a)$, where $M_a$ is an integral manifold of $T_a$, and will obtain corresponding decomposition theorems. When $M$ is complete, more detailed results will be obtained. All these results will be found in Section 3.

A Lie algebra of infinitesimal transformations such as $A^I$ is of course less connected with transformations of the space $M$ than the Lie algebra of the group such as $A^L$ is. Nevertheless it can yet have some geometric meaning from both local and global standpoints. To illustrate this, we like to cite here two known facts: 1) $M$ is locally flat if and only if $M$ is covered with neighborhoods $U_i$ such that $A^I(U_i)$ is isomorphic onto $A^I(E)$, $E$ being the Euclidean space of the same dimension as $M$, and 2) if $A^I$ is transitive in the sense of Section 3, then the Lie algebra of $H$ is contained in the Lie algebra generated by the Kostant operators $pv$ evaluated at a point of all $v \in A^I(M)$ (Kostant [1]).

The present paper is based on Nagano [1] and is its improvement. A recent
Historical notes.

In 1952, one of the present authors proved (Yano [2]) that:

In a compact orientable Riemannian space an infinitesimal affine transformation is necessarily a motion.

In general, the group $I(M)$ of isometries of $M$ is a closed subgroup of the group $A(M)$ of affine transformations of $M$, and the above result shows that, if $M$ is compact (and orientable), then the connected component $A^0(M)$ of the identity of $A(M)$ is contained in $I(M)$.

Nomizu [2], [3], [4] studied the relations between $A(M)$ and $I(M)$ in more general Riemannian spaces and obtained the following results:

*Let $M$ be an irreducible Riemannian manifold. Then the following subgroups of $A(M)$ are all contained in the group of isometries $I(M)$:
  1) every compact subgroup of $A(M)$,
  2) every connected semi-simple Lie subgroup of $A(M)$,
  3) the commutator subgroup $[A(M), A(M)]$.*

If $M$ is a simply connected complete Riemannian manifold whose Euclidean part is of dimension $\leq 1$, then the following subgroups of $A(M)$ are all contained in $I(M)$:

*1) every compact subgroup of $A^0(M)$,
2) every connected semi-simple Lie subgroup of $A^0(M)$,
3) the commutator subgroup $[A^0(M), A^0(M)]$.*

Kobayashi [2] studied also this problem for the case where $M$ is irreducible and complete and obtained a more refined result:

*If $M$ is an irreducible and complete Riemannian manifold, then $A(M)$ is equal to $I(M)$, except the case $M$ is the one-dimensional Euclidean space.*

The same theorem as this was also proved by one of the present authors (Nagano [1]) for the connected component of the identity of $A(M)$.

One of the important lemmas in the studies of Nomizu and Kobayashi is:

*Let $M$ be irreducible and $\phi$ an affine transformation of $M$, then there exists a positive constant $c$ such that $\phi(ds) = c \cdot ds$ for the metric $ds$ of $M$.*

This lemma goes back to T. Y. Thomas [1] and Mogi [1].

Using the method of Nomizu, Hano [1] proved the following theorems:

*Let $M$ be a simply connected complete Riemannian mainfold and $M=M_0 \times M_1 \times \cdots \times M_r$ be the de Rham decomposition of $M$. Then the group $A^0(M)$ is isomorphic to the direct product $A^0(M_0) \times A^0(M_1) \times \cdots \times A^0(M_r)$ and the group $I^0(M)$ isomorphic to the direct product $I^0(M_0) \times I^0(M_1) \times \cdots \times I^0(M_r)$, where $A^0(M_i)$ (resp. $I^0(M)$ and $I^0(M_i)$) is the connected component of the identity in $A(M_i)$ (resp. $I(M)$ and $I(M_i)$) respectively.*

*Let $M$ be a complete Riemannian manifold. If the length of an infitnesimal affine transformation $v$ is bounded on $M$, then $v$ is a Killing vector field.*
Ishihara and Obata [1] also studied these problems and obtained the following theorems:

Let $M$ be a connected irreducible Riemannian manifold and $\phi$ an affine transformation in $M$ which is not isometry. Then $\phi$ has no fixed point.

Let $M$ be a connected irreducible complete Riemannian manifold, then $\mathcal{A}(M) = \mathcal{I}(M)$.

Let $M$ be a connected Riemannian manifold. If $M$ is complete and has no locally flat part, then $\mathcal{A}_0(M) = \mathcal{I}_0(M)$.

Let $M$ be a locally flat Riemannian manifold which is connected and complete. Then in order that an infinitesimal affine transformation $\nu$ be a motion, it is necessary and sufficient that the length of $\nu$ be bounded.

The last two theorems cover the theorem stated at the beginning. They gave some examples which show that the assumption in the above theorem cannot be made weaker in some sense.

Lichnerowicz [1] gave interesting comments on these results.

### Notations and definitions.

Let $M$ be a connected $n$-dimensional Riemannian manifold of differentiability class $\mathcal{C}^\infty$. As usual, $g$, $K$ and $\mathcal{F}$ denote respectively the metric tensor of $M$, the curvature tensor of $M$ and the operator of covariant differentiation with respect to the Levi-Civita connection $L$ defined as a connection without torsion satisfying $\mathcal{T}_g = 0$.

An affine field $\nu$ on $M$ is by definition a vector field on $M$ satisfying $\mathcal{L}_\nu L = 0$, that is, a vector field $\nu$ such that the Lie derivative (Yano [3]) of $L$ with respect to $\nu$ vanishes. The totality of affine fields on $M$ will be denoted by $\mathcal{A}(M)$ or simply by $\mathcal{A}$. The $\mathcal{A}$ is a Lie algebra over the real field $\mathbb{R}$ with respect to the usual bracket product. A vector field is said to be complete when it generates a one-parameter group of transformations. The affine fields which are complete form a subalgebra $\mathcal{A}^c(M)$ or $\mathcal{A}^c$ of $\mathcal{A}$ (Nomizu [1]). A homothetic field $\nu$ is by definition a vector field $\nu$ satisfying $\mathcal{L}_\nu g = 2cg$ for some constant $c \in \mathbb{R}$, (Yano [1]). The constant $c$ is then called the homothetic constant. A homothetic field $\nu$ belongs to $\mathcal{A}^c$, because $\mathcal{L}_\nu g = 2cg$ implies $\mathcal{L}_\nu L = 0$ (Yano [1]). When $c = 0$ for a homothetic field $\nu$, the field $\nu$ is an isometric field. The set of isometric fields will be denoted by $\mathcal{I}(M)$ or $\mathcal{I}$. We put $\mathcal{I}^c = \mathcal{I}(M) \cap \mathcal{A}^c$.

A diffeomorphism $\alpha$ of $M$ into another Riemannian manifold $N$ is said to be homothetic when there exists a constant $c$ such that $|d\alpha(X)| = c|X|$ for any vector $X$ on $M$, $|X|$ denoting the length of the vector $X$.

From now on, $R$ will be identified with the one-dimensional Euclidean space. Given a vector field $\nu$ on $M$, we can define a local one-parameter group $G$ consisting of diffeomorphisms $\text{Exp} tv$ of an open set $U_t$ of $M$ into $M$, $t \in \mathbb{R}$. The set may be empty for some $t \in \mathbb{R}$. The field $\nu$ is complete when $U_t = M$ for all $t \in \mathbb{R}$. A trajectory $\gamma$ of $\nu$ is the orbit of a point $x \in M$ under $G$, that is, the curve $\gamma = \gamma(t) = (\text{Exp} tv)(x)$, $t$ being defined in an interval containing 0. In terms of local coordinates $(x^i)$ ($h, i, j, \cdots = 1, 2, \cdots, n$) the orbit $\gamma$ is given by the solution of the ordinary differential equations $Dx^i = v^i$, $D$ denoting the differentiation $d/dt$ with respect to the canonical parameter $t$. Along the trajectory $\gamma$, we can consider another parameter $s$, that is,
the arc length of the curve $\gamma$ which is given by $|v|^2 = (Ds)^2$.

The de Rham decomposition $T = \sum_{s} T_s$ of the tangent space $T = T(x)$ at $x \in M$ is defined as stated in the Introduction. It is given by the system $(P_s)$ of the orthogonal projections $P_s$ of $T$ onto $T_s$. Each $P_s$ will be called a de Rham projection. An orthogonal projection $P$ of the tangent space $T$ into a subspace of $T$ is admissible when $P$ is invariant by the homogeneous holonomy group $H$ and $P(T) \cap T_0$ is equal to $T_0$ or $\{0\}$. A parallel tensor field of $(1,1)$-type will also be said to be admissible when it assigns an admissible projection to each point.

The identity component of $H$ will be denoted by $H^0$. $P_s$ will be identified with the distribution $x \mapsto T_s(x)$ in the expression “integral manifold of $P_s$”.

The linear endomorphism $P$ of $T$ into itself naturally extends to that of the tensor product of any finite number of copies of $T$ and its dual space. This generalized projection will be denoted also by $P$. When $Q$ is a tensor field of $(1,1)$-type, $P \otimes Q$ denotes the tensor field thus obtained, while $P \circ Q$ denotes the tensor field such that $P \circ Q(x)$ is the composition of the endomorphisms $P(x)$ and $Q(x)$.

§ 1. Homothetic transformations.

**Proposition 1.** If a homothetic field $v$ vanishes at a point $p \in M$, then there exists a neighborhood $V$ of $p$ such that $\text{Exp}(tv)$ is defined in $V$ for the interval $ct \leq 0$, $c$ being the homothetic constant.

**Proof.** We may assume that $c$ is non-positive. Let $V$ be an open sphere with $p$ as center whose radius $r$ is so small that $V$ is relatively compact. Then there exists a positive number $e$ such that $\text{Exp}(tv)$ is defined on $V$ for $0 \leq t \leq e$.

Any point $q$ in $V$ can be joined to $p$ by a geodesic $\gamma$ whose arc length $\beta$ is equal to the distance between $p$ and $q$. The length of $\text{Exp}(tv)\beta$ is $\text{Exp} ct$ multiplied by the length $|\beta|$ of $\beta$. Thus $|\text{Exp}(tv)\beta|$ is not greater than $|\beta| < r$ and consequently, the point $\text{Exp}(tv)\gamma(q)$ lies in $V$. It follows from this that $\text{Exp}(tv)$ is defined on $V$ for $0 \leq t$, that is, for $ct \leq 0$.

**Proposition 2.** Let $v$ be a non-isometric homothetic field in $M$. i) Assume that there is a connected open subset $U$ such that any trajectory $\gamma$ in $U$ has on its closure a point $p \in U$ at which $v$ vanishes. Then $U$ is flat, and simply connected. ii) If $M$ is complete, then $M$ may be taken as $U$; so $M$ is isometric to the Euclidean space. (See Kobayashi [2]).

**Proof.** Our original proof (Nagano [1]) of this proposition was similar to that in Kobayashi [2]. Hence we shall give here a proof different from it. Let $|K|$ and $|v|$ be the length of $K$ and $v$ respectively:

$$|K|^2 = g^{ir} g^{ia} g_{bp} K_{kj}\ K_{pq}, \quad |v|^2 = g_{ij} v^i v^j.$$  

Then, $c$ denoting the homothetic constant, we have

$$\mathcal{L}_v |K|^2 = -4c |K|^2 \quad \text{and} \quad \mathcal{L}_v |v|^2 = 2c |v|^2,$$

from which

$$\mathcal{L}_v |K|^2 |v|^4 = 0.$$  

It follows from this that $|K|^2 |v|^4$ is constant along any trajectory $\gamma$ of $v$. When $v$ vanishes at a point $p \in M$ and $p$ adheres to $\gamma$, $|K|^2 |v|^4$ is zero along the $\gamma$. Thus by our assumption $|K|^2 |v|^4$ is always zero in $U$. But the points at which $v$ vanishes.
form a nowhere dense subset of $U$ and consequently we conclude that $K$ is identically zero in $U$. Thus $U$ is flat.

Now we may assume that $c$ is negative. For an arbitrary point $x$ in $U$, the trajectory $\gamma$ passing through $x$ has a point $q$ in $V$ defined in the Proposition 1, because there exists a point $p$ on $\gamma$ at which $v$ vanishes. We may assume that $|v(q)| < |v(x)|$, that is, the number $t_0$ satisfying $(\text{Exp} t_0 v)(x) = q$ is positive. From Proposition 1, we deduce that $(\text{Exp} tv)$ is defined on $U$, for $0 \leq t$.

Given a closed curve $\beta$ in $U$ issuing from $p$, $(\text{Exp} tv) \beta$, $0 < t$, is a curve in $U$ issuing from $p$ and of length $(\text{Exp} ct)|\beta|$. Thus $(\text{Exp} tv) \beta$ converges to the point $p$ when $t \to \infty$. Therefore $\beta$ is null-homotopic. $U$ is thus simply connected and the part i) of Proposition 2 is proved.

From the equality $\mathbf{L}_v |v|^2 = 2c |v|^2$, or $\mathbf{L}_v |v| = c |v|$, it follows by virtue of $|v|^2 = (Ds)^2$ that $b = cs - |v|$ is constant along an arbitrary trajectory $\gamma$, $s$ being the arc length of $\gamma$.

When $M$ is complete, we can make $s$ tend to $b/c$ and find that there exists a point $p$ at which $v(p) = 0$. Hence $M$ can be taken as $U$ in i). Being complete, simply connected and flat, $M$ is isometric to the Euclidean space. Thus the part ii) of Proposition 2 is proved.

Without proof, we state an analogous proposition: if $\alpha$ is a non-isometric homothetic map of an open subset $U$ of $M$ into $U$ and if there exists a fixed point of $\alpha$ in $U$, then $U$ is flat and simply connected. If $\alpha$ is onto, then $U$ is complete; therefore $M$ coincides with $U$ and is isometric to the Euclidean space (Cf. Kobayashi [2]; Ishihara and Obata [1], lemma 1).

**Proposition 3.**

i) Let $v$ be a complete non-isometric homothetic field on $M$. The field $v$ admits a fixed point $p$, $v(p) = 0$, if and only if $M$ is isometric to the Euclidean space. ii) If $v$ does not admit a fixed point, then $M$ is not complete and is the product of $R$ and a hypersurface $N$ as a manifold. iii) Conversely, given such a manifold $R \times N$, $N$ being an arbitrary connected manifold of class $C^\infty$, a non-complete Riemannian metric is given to it in such a way that there exists a complete non-isometric homothetic field without fixed points (Nagano [1]).

**Proof.** For the proof of i), we put $M = U$ and apply Proposition 2. Note that any Cauchy sequence in $M$ can be brought into an arbitrary compact neighborhood of $p$ by some homothetic transformation $\text{Exp} tv$. To establish ii), we put $N = \{x \in M; |v(x)| = 1\}$ and consider the map $\rho$ of $R \times N$ onto $M$ defined by $\rho(r, x) = (\text{Exp} r v)(x)$. It is easy to see that $M$ is not complete and that $\rho$ is a diffeomorphism. The proof of iii) is omitted.

**Proposition 4.** When $M$ is complete, every homothetic field $v$ is complete.

**Proof.** From $\mathbf{L}_v |v| = c |v|$, we have $D^2 s = c Ds$ and consequently $s - ke^{st}$ is constant along the trajectory, where $k$ is a constant $\neq 0$. Since $M$ is complete, $\sup s$ and $\inf s$ are infinity or the value for which $v$ vanishes. Hence $\text{Exp} tv$ is defined for any $t \in R$.

§ 2. The de Rham decomposition (de Rham [1]).

We recall that a de Rham decomposition of $M$ or of tangent space $T$ at a point of
M is the direct sum decomposition \( T = T_0 + T_1 + \cdots + T_r \) such that
1) each \( T_\alpha (\alpha, \beta, \cdots = 0, 1, 2, \cdots, r) \) is orthogonal to \( T_\beta \) for \( \alpha \neq \beta \),
2) \( H \) leaves invariant each \( T_\alpha \),
3) \( H \) is irreducible on \( T_\alpha \) \( (0 < \alpha) \),
4) \( H^0 \) acts trivially only on \( T_0 \).

**Proposition 5.** The condition 4) above is equivalent to 4'): \( H^0 \) leaves invariant a vector \( X \in T \) if and only if \( X \) belongs to \( T_0 \).

**Proof.** The condition 4) obviously follows from 4'). Assume 4) and also that a vector \( X \in T \) which does not belong to \( T_0 \) is left invariant by \( H^0 \). We may suppose that \( X \neq 0 \) belongs to \( T_\alpha \) for some positive \( \alpha \). Then for any \( \lambda \in H \), \( H^0 \) leaves \( \lambda(X) \) invariant because \( H^0 \) is a normal subgroup of \( H \). But \( H \) being irreducible on \( T_\alpha \), \( H(X) \) spans \( T_\alpha \). This shows that \( H^0 \) is trivial on \( T_\alpha \), contrary to 4).

Fix a de Rham decomposition \( T(x) = \sum T_\alpha(x) \) of the tangent space \( T(x) \) at a point \( x \) and define a de Rham decomposition \( T(y) = \sum T_\alpha(y) \) of the tangent space \( T(y) \) at an arbitrary point \( y \) by the parallel displacement of \( T(x) = \sum T_\alpha(x) \) along any curve joining \( x \) to \( y \). Then the de Rham projection \( P_\alpha(y): T(y) \rightarrow T_\alpha(y) \) forms a parallel tensor field \( P_\alpha \) of type \((1, 1)\).

Clearly we have \( \sum P_\alpha = E \) and \( P_\alpha \circ P_\beta = \delta_\beta \circ P_\beta \) (no summation convention is applied with respect to Greek indices), that is, \( \sum P_\alpha^i_j = \delta^i_j \) and \( P_\alpha^i_j P_\beta^j_k = \delta_\beta \circ P_\beta^i_k \), where \( E \) is the identity and \( P_\alpha \circ P_\beta \) is defined by \( (P_\alpha \circ P_\beta)(x) = P_\alpha(x) \circ P_\beta(x) \).

Locally, \( M \) is the Riemannian product of the integral manifolds (with the induced Riemannian metric) of the distributions \( T_\alpha \): \( y \rightarrow T_\alpha(y) \). It is seen that \( P_\alpha \) being extended as explained above, we have
\[
g = \sum P_\alpha^\alpha, \quad K = \sum P_\alpha K
\]
and
\[
\mathcal{L}_{P_\alpha} P_{\beta} g = 0 \quad (\alpha \neq \beta)
\]
for any vector field \( \nu \), as will be easily verified by choosing suitable local coordinates.

We shall now prove the following key lemma.

**Proposition 6.** Let \( Q \) be an endomorphism of \( T(x) \) such that all eigenvalues of \( Q \) are real and \( Q \) commutes with any element in \( H \). Then \( Q \) is written in the form \( Q = \sum a_\alpha P_\alpha \), where \( a_0 \) is an endomorphism of \( T_0 \) and \( a_\alpha \in \mathbb{R} \) for \( 0 < \alpha \).

**Proof.** As an endomorphism of \( T \), \( Q \) is written in the form \( Q = \sum_{0 \leq \alpha; \beta \leq r} Q_{\alpha \beta} \), where \( Q_{\alpha \beta} \) is an endomorphism of \( T \) satisfying \( Q_{\alpha \beta} = P_\alpha \circ Q_{\alpha \beta} \circ P_\beta \). Any element \( A \) of \( H \) can be written as the sum of endomorphisms \( A_\alpha \), where \( A_\alpha = P_\alpha \circ A_\alpha \circ P_\alpha \) because of \( A(T_\alpha) = T_\alpha \).

By the assumption we have \( A_\alpha \circ Q_{\alpha \beta} = Q_{\alpha \beta} \circ A_\beta \). Since \( H \) is irreducible on \( T_\gamma \), \( 0 < \gamma \), we find that the image of the linear map \( Q_{\alpha \beta} | T_{\beta} \) (=restriction of \( Q_{\alpha \beta} \) to \( T_{\beta} \)): \( T_{\gamma} \rightarrow T_{\alpha} \) is trivial (i.e. = 0 or \( T_{\alpha} \)) if \( \alpha \neq 0 \), and the kernel of \( Q_{\alpha \beta} \) is trivial if \( \beta \neq 0 \). Hence in case \( 0 < \alpha = \beta \), we have \( Q_{\alpha \alpha} = a_\alpha P_\alpha, 0 < \alpha \), for some \( a_\alpha \in \mathbb{R} \) by Schur's lemma.

In case \( 0 < \alpha, \beta \) with \( \alpha \neq \beta \), \( Q_{\alpha \beta} \) is zero or regular. In the latter case, we have \( Q_{\alpha \beta}^{-1} = A_\alpha \circ Q_{\alpha \beta} = A_\beta \). Consequently \( A_\alpha = E \) implies \( A_\beta = E \). But we may suppose that \( P_\beta K \neq 0 \) at \( x \) by carrying \( Q \) to an endomorphism of \( T(y) \) for some \( y \) by the parallel displacement along a curve if necessary. The parallel displacement along a curve in an integral manifold containing \( x \) of \( P_\beta \) defines an element \( A \in H \) such that \( A_\alpha = E \) and \( A_\beta \neq E \) which leads to a contradiction. Thus \( Q_{\alpha \beta} \) vanishes in this case.
It remains to investigate the case where at least one of $\alpha$ and $\beta$ is zero. Assume that $\alpha=0, \beta\neq 0$ and $Q_{\alpha\beta} \neq 0$. The kernel of $Q_{\alpha\beta}|T_\beta$ is zero. By (4') there exists an element $A$ of $H^0$ such that $A_0=E$ and $A_\alpha \neq E$. For any $X$ in $T_\beta$, we have $Q_{\alpha\beta}(A_\alpha X - X) = A_\alpha Q_{\alpha\beta} X_\alpha - Q_{\alpha\beta} X_\alpha = 0$. Hence $A_\beta X = X$ which leads to a contradiction. Therefore $Q_{\alpha\beta} = 0$ ($\beta \neq 0$). The case $Q_{\alpha\beta} (\alpha \neq 0)$ is treated in a similar way or by considering the dual map $\bar{Q}_{\alpha\beta}$.

When $Q$ is an admissible projection, $Q^2 = Q$ implies that $a_0 = E$ or $0$ and $a_1, \ldots, a_r$ are $1$ or $0$. Thus an admissible projection $P$ is the sum of some $P_\alpha$'s (without repetition).

**Proposition 7.** $v \in A^I$ implies $\xi_v P = 0$ for any admissible projection $P$.

**Proof.** Since, if $v \in A^I$, then $\xi_v P$ commutes with $P$ (Yano [3]), $\xi_v P$ is parallel and so invariant by $H$. The eigenvalues of $\xi_v P$ are all real. It is sufficient to consider the case where $P$ is equal to some de Rham projection $P_\alpha$.

By Proposition 6, $\xi_v P_\alpha$ is of the form $\xi_v P_\alpha = \sum a_\alpha P_\alpha$. On the other hand, $P_\alpha = P_\alpha \xi_v P_\alpha$ implies $\xi_v P_\alpha = (\xi_v P_\alpha) P_\alpha + P_\alpha (\xi_v P_\alpha) = \alpha P_\alpha + P_\alpha a_\alpha P_\alpha = 2a_\alpha P_\alpha$. Therefore $a_\alpha = 0$ for all $\beta$.

**Proposition 8.** For each $v \in A^I$, we have $\forall P \xi_v P = \xi_v P \forall$ for any admissible projection $P$.

**Proof.** This follows easily from Proposition 7.

**Proposition 9.** $v \in A^I$ implies i) $Pv \in A^I$ and ii) $\xi_v P g = \xi_v P g = P \xi_v g$, where $P$ is an arbitrary admissible projection.

**Proof.** To prove i), we have only to verify the vanishing of

$$(\xi_v P_L)_{jk} = P_{jk}(\xi_v g) + K_{jk} P_{ijk}(\xi_v g)$$

(Yano [3]). By virtue of Proposition 8 and $K = \sum P_\alpha K$, the right hand side of the above equation is equal to $(P \xi_v L)_{jk}$ which is zero by assumption. Thus $Pv$ belongs to $A^I$.

We next proceed to the proof of ii). From $\xi_v P g = 0$ ($\alpha \neq \beta$), we have

$$\xi_v P g = \xi_v \sum P_\alpha g = \xi_v P g.$$

By Proposition 7 and i) just proved above, this is equal to $P \xi_v P g$ and consequently

$$\xi_v P g = P \xi_v P g.$$

Using this relation, we have

$$\xi_v P g = P \xi_v P g = P \sum P_\alpha \xi_v P_\alpha g = P \sum \xi_v P_\alpha g = P \xi_v \sum P_\alpha g = P \xi_v g,$$

which proves ii).

As a corollary to Proposition 9, we have

**Proposition 10.** $v \in I^I$ implies $Pv \in I^I$ for any admissible projection $P$.

**Proposition 11.** If $u, v \in A^I$, then

$$\xi_v P' u = \xi_v P (\xi_v v)$$

for any admissible projections $P$ and $P'$.

**Proof.** On account of Propositions 7 and 9, this is proved as follows:
As corollaries to this, we have

**Proposition 12.** An admissible projection is an endomorphism of $A^I$ and $I^I$.

**Proposition 13.** For each $v \in A^I$ and a positive $\alpha$, $P^\alpha v$ is homothetic on an arbitrary integral manifold of $P_\alpha$. When $H$ is irreducible on $T_0$, this holds also for $\alpha=0$ (Mogi [1], Hiramatu [1]).

**Proposition 14.** The de Rham decomposition is unique up to the order.

**Proof.** Given a subspace of the tangent space which is left invariant by $H$ and on which $H$ is irreducible, we denote by $P$ the corresponding projection. Applying Proposition 6 to $P$, we find $P=\sum a_\alpha P_\alpha$ for some $a_\alpha$, where all $a_\alpha$'s belong to $R$. Here we have

$a_\alpha^2=a_\alpha$ together with $\sum a_\alpha=1$,

which shows that one of $a_\alpha$'s is equal to 1, the others being all zero. Thus $P$ must coincide with one of $P_\alpha$'s.

§ 3. Theorems.

**Theorem 1.** i) $A^I$ is isomorphic to the direct sum of subalgebras $A_\alpha^I=\{P_\alpha v; v \in A^I\}$. ii) $A^I_\alpha$, $0<\alpha$, consists of homothetic fields if $P_\alpha v \in A^I_\alpha$ is restricted on integral manifold of $P_\alpha$. This holds also for $\alpha=0$ when $H_\alpha$ is irreducible (Nagano [1]).

**Proof.** By Proposition 12, $A_\alpha^I$ is a subalgebra. Clearly $A^I$ is the direct sum of vector subspaces $A_\alpha^I$. On the other hand, by Proposition 11, we have $[A^I_\alpha, A^I_\beta]=0$ for $\alpha \neq \beta$, which completes the proof of part i). The part ii) is nothing but Proposition 13.

**Remark.** It is obvious that $v \in A^I$ belongs to $I^I$ if and only if $P_\alpha v \in I^I$ for any $\alpha$.

An analogous theorem holds for $I^I$:

**Theorem 2.** i) $I^I$ is isomorphic to the direct sum of subalgebras $I^I_\alpha=\{P_\alpha v; v \in I^I\}$. ii) $I^I_\alpha$, $0<\alpha$, is the ideal of $A^I_\alpha$ with $\dim A^I_\alpha-\dim I^I_\alpha \leq 1$ (Nagano [1]).

**Proof.** Part i) follows from Proposition 12 and Theorem 1. The part ii) is obvious.

**Theorem 3.** If $M$ is the Riemannian product of integral manifolds $M_\alpha$ of the de Rham projections $P_\alpha$, then $A^I_\alpha \subset A^I(M)$ is isomorphic onto $A^I(M_\alpha)$. Analogous theorems hold for $I^I(M)$, $A^I(M)$, and $I^I(M)$.

**Proof.** It is sufficient to prove $A^I_\alpha$ to be isomorphic onto $A^I(M_\alpha)$. For an arbitrary $u \in A^I_\alpha$, let $\phi(u)$ denote the restriction of $u$ to $M_\alpha$. Then $\phi(u)$ is a vector field on $M_\alpha$. Thus $\phi(u)$ belongs to $A^I(M_\alpha)$. Obviously $\phi$ is a homomorphism. $\phi$ is an isomorphism (into) because of Proposition 7. Finally we verify that $\phi$ is onto. For any $v \in A^I(M_\alpha)$, a vector field $u$ on $M$ is uniquely determined in such a way that 1) the restriction of $u$ to $M_\alpha$ coincides with $v$, 2) $P_\beta u=u$ and 3) $\mathcal{L}_u P_\beta=0$ for all $\beta$. It is not hard to prove that $u$ belongs to $A^I_\alpha$ and $\phi(u)=v$.

**Remark 1.** According to a well known theorem of de Rham [1], the hypothesis
of Theorem 3 is satisfied when $M$ is complete and simply connected. We also note that $M$ is covered by neighborhoods which are the Riemannian products of integral manifolds of $P$, the de Rham projections of $M$.

**Remark 2.** Replacing de Rham projections $P$ by admissible projections $Q$ such that $Q_\mu Q_\nu = 0$ for $\lambda \neq \mu$ and $\sum Q_\lambda = E$, we can obtain a generalization of Theorem 3.

**Theorem 4.** Let $U$ be an arbitrary open subspace of $M$ and $U_\alpha \subset U$ be a connected integral manifold of $P$. Then there exist isomorphisms

$$A^\lambda (M) \cong A^\lambda (U) \cong \Sigma A^\alpha (U_\alpha),$$

(which are not onto in general). The composition $\mu$ of these isomorphisms carries $A^\alpha$ into $A^\lambda (U)$ and for $\alpha > 0$, into the subalgebra of homothetic fields. An analogous theorem holds for $I^\lambda$. Furthermore, $\nu \in A^\lambda (M)$ belongs to $I^\lambda (M)$ if and only if $\mu (\nu) \in \Sigma I^\alpha (U_\alpha)$ for some $U$ (Nagano [1]).

**Proof.** The first isomorphism $\rho$ of $A^\lambda (M)$ into $A^\lambda (U)$ is defined by the restriction. $\rho$ is an isomorphism because an affine field $v$ is uniquely determined by $v(x)$ and $\nabla v(x)$, $x$ being an arbitrary but fixed point. The second isomorphism is defined as in Theorem 3. If $\mu (\nu) \in \Sigma I^\alpha (U_\alpha)$, then $\nu \in I^\lambda (U)$. From the fact that $\mathcal{L}_v$ is parallel for any $v \in A^\lambda (M)$, we conclude that $v \in I^\lambda (M)$.

**Remark** The isomorphisms in Theorem 3 are not onto, for example, if $U$ is locally flat and $M$ is not.

We shall now show five applications of the above theorems to some special cases. A Lie algebra $A$ of vector fields on $M$ is said to be transitive when the natural map $\nu$ of $A \times M$ into the tangent bundle of $M$ defined by $\nu(v,x) = v(x)$ is onto.

**Proposition 15.** If $M$ is of class $C^\infty$ or if $A^\lambda$ is transitive, then the isomorphism $A^\lambda (U) \rightarrow \Sigma A^\alpha (U_\alpha)$ is onto, where $U$, an open cell in $M$, is assumed to be the Riemannian product of integral manifolds $U_\alpha \subset U$ of $P$.

**Proof.** By a theorem of Nijenhuis [1], $H_0$ then coincides with the restricted homogeneous holonomy group of $U$. Restricted to $U$, $P$ is thus an admissible projection of $U$. Our Proposition follows from Theorem 3 and Remark 2.

**Proposition 16.** Assume $H_0$ fixes at most one direction. Then an affine field $v$ is isometric if and only if the length $|v|$ of $v$ is constant, (that is, $v$ is isometric on each trajectory).

**Proof.** Suppose that $|v|$ is constant on each trajectory. By Theorem 3, we have

$$0 = \mathcal{L}_v |v|^2 = \sum \mathcal{L}_{P_\alpha v} |P_\alpha v|^2$$

and

$$\mathcal{L}_{P_\alpha v} |P_\alpha v|^2 = c_\alpha \text{ (constant).}$$

For $\alpha > 0$, we get, from Theorem 1, ii),

$$\mathcal{L}_{P_\alpha v} |P_\alpha v|^2 = b_\alpha |P_\alpha v|^2 \text{ (constant).}$$

Thus, it follows that $b_\alpha = c_\alpha = 0$ for $\alpha > 0$. Hence $c_0 = 0$. From Proposition 9, ii), we have
The fact \( c_0 = 0 \) means \( P_0 \) is isometric because \( \text{rank}(P_0) \leq 1 \) by virtue of our assumption. Eventually we have
\[
\epsilon v = \epsilon P \epsilon v = 0,
\]
i.e. \( v \) is isometric. The converse is obvious.

**Proposition 17.** If \( A^I \) or \( I^I \) is simple and transitive, then so is each \( A_{\alpha}^I \) (hence \( A^I(U_\alpha) \) in Theorem 4) or \( I_{\alpha}^I \) (hence \( I^I(U_\alpha) \) in Theorem 4).

**Proposition 18.** If \( A^I \) or \( I^I \) is simple and transitive, then \( H \) is discrete or irreducible.

**Proof.** Assume \( H \) is not discrete and \( A^I \) is transitive. Since each \( A_{\alpha}^I \) is not trivial by the preceding Proposition, the simpleness of \( A^I \) implies \( P_1 = E \) by virtue of i) of Theorem 1.

**Remark.** The converse is not true: even if \( A^I \) is transitive and \( H \) is irreducible, \( A^I \) is not necessarily simple. Furthermore, it is possible (Kostant [2]) that \( H \) is reducible and a simple subalgebra of \( I^I \) is transitive.

**Proposition 19.** Assume that \( M \) is of class \( C_\alpha \) or that \( A^I \) is transitive. If \( H_0 \) does not fix a direction, then any affine field \( v \) which vanishes at a point \( x \) is isometric.

**Proof.** Suppose that \( P_{\alpha}v \) is not isometric. Then \( P_{\alpha}v \) is not isometric in any open subset. By Theorem 1 and Proposition 2, a neighborhood of \( x \) in \( M_\alpha \), an integral manifold containing \( x \) of \( P_{\alpha} \), is flat. Since \( H_0 \) coincides with the restricted homogeneous holonomy group of any open subspace of \( M \), \( H_0 \) must leave fixed any tangent vector of \( M_\alpha \) at \( x \), contrary to our hypothesis.

**Theorem 5.** When \( M \) is complete, every affine field \( v \) is complete and consequently \( A^I = A^I \) and hence \( I^I = I^I \) (Nagano [1]). (This is a special case of a theorem of Kobayashi [1].)

Without loss of generality, we can assume that \( M \) is simply connected. By de Rham’s theorem, the hypothesis of Theorem 3 is satisfied. Hence we have \( A^I(M) = \Sigma A^I(M_\alpha) \). Since \( M_0 \) is isometric to a Euclidean space, we obtain \( A^I(M_0) = A^I(M_0) \). On the other hand, by Proposition 8, \( A^I(M_\alpha) \), \( 0 < \alpha \), consists of homothetic fields. From Proposition 4, it follows that \( A^I(M_\alpha) = A^I(M_\alpha) \), which proves the Theorem.

**Theorem 6.** If \( M \) is complete, then \( A^I(M) \) is isomorphic onto \( A_{\alpha}^I \sum_{0 < \alpha} I_{\alpha}^I \) and into \( A^I(M_0) + \sum_{0 < \alpha} I^I(M_\alpha) \), where \( M_\alpha \) is an arbitrary maximal connected integral manifold of \( P_{\alpha} \). Similarly \( I^I(M) \) is a subalgebra of \( \Sigma_{0 \leq \alpha} I^I(M) \) (Nagano [1]).

**Proof.** Taking account of Theorems 1 and 4 (put \( U = M \) and \( U_{\alpha} = M_\alpha \)) we have to show that
\[
A_{\alpha}^I = I_{\alpha}^I \quad (\subseteq I_{\alpha}^I), \quad 0 < \alpha
\]
and that the image of \( A_{\alpha}^I \) by the second homomorphism \( \phi \) in Theorem 4 is contained in \( I^I(M_\alpha) \), \( 0 < \alpha \). The former follows from the latter.
For any $v \in A$, $\phi(v)$ is the restriction of $v$ to $M$. By Theorem 1, $\phi(v)$ is homothetic. Since $M$ is complete, this implies that $\phi(v)$ is complete by Theorem 5 and so that $M$ is locally flat by Proposition 2 unless $\phi(v)$ is isometric. If $\phi(v)$ is not isometric, then $P_M(v)$ is not isometric on any integral manifold $M'$ of $P$. Therefore $M'$ must be locally flat. Hence $H^0$ is discrete on $T = P(T)$, contrary to the definition of de Rham decomposition. Thus $\phi(v)$ is isometric and the proof is completed.

**Corollary 1.** If $M$ is complete and simply connected, then we have $A(M) = A(M_0) + \sum_{0 < s} I(M_s)$ and $I(M) = \sum_{0 < s} I(M_s)$. (Hano [1], Ishihara-Obata [1], Lichnerowicz [1]).

**Corollary 2.** If $A(M)$ or $I(M)$ is transitive, then so is $A(M_0)$ or $I(M_0)$, for any maximal connected integral manifold $M_0$ of $P$.

**Proof.** $M$ is then complete together with $M$. Hence we can apply Theorem 6.

**Corollary 3.** If $M$ is complete and a maximal integral manifold $M_0$ of $P_0$ is compact, then we have $A(M) = \bar{I}(M)$.

This generalizes Yano’s theorem mentioned in the Historical Notes.

**Remark.** Let $N$ be a totally geodesic subspace of $M$ and $P$ be a map which assigns to a vector $X$ tangent to $M$ at a point $x \in N$ a vector $P(X)$ tangent to $N$ at $x$ such that $X - P(X)$ is normal to $N$. For any $v \in I$, we have $P(v|N) \in I(N)$, according to personal communications from S. Kobayashi and T. Takahashi. Therefore if $I(M)$ is transitive and $N$ is a maximal totally geodesic subspace of $M$, then $I(N)$ is transitive.

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**Bibliography**


