Some theorems on analytic functions in an annulus

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1. Preliminaries.

At the beginning of this paper we introduce some notations which are necessary for the discussion in the sequel.

(i) Let a and b be any two points in the unit circle $|w| < 1$ in the w -plane, then we denote by $[a, b]$ the hyperbolic distance [7] of these two points, i. e.,

$$(1) \quad [a, b] = \left| \frac{a-b}{1-\bar{a}b} \right|.$$

(ii) We denote the ring-domain bounded by $|w|=1$ and the rectilinear slit $<0, 1/P>$ ($P > 1$) by G_P which is called the *Grötzsch's extremal domain* [1] and whose Riemann modulus is denoted by $\Phi(P)$. It is well-known that $\Phi(P)$ is a strictly increasing function for $P (> 1)$. Let P be uniquely determined for a given positive constant Q so as to satisfy the condition

$$(2) \quad \frac{1}{Q} = \Phi(P),$$

then the annulus $D: Q < |z| < 1$ can be mapped univalently onto G_P laid on the w -plane by the function $w=F(z; Q)$ which is called the *Grötzsch's extremal function*.

(iii) We denote by $\mathfrak{F}$ a class of the functions $w=f(z)$ which are single-valued and analytic (*not necessarily univalent*) in the annulus D , and whose ranges of values D_f for D lie in the circle $|w| < 1$ and have the boundary component $|w|=1$ as the image of outer circle $|z|=1$ of D .

The object of this paper is to establish some theorems on the ranges of values taken by the analytic functions which are single-valued in D and satisfy various conditions. The proofs are based on the following fundamental theorem [4] which is regarded as an extension of the theorem of Hayman (Theorem II formulated in his paper [2]).

FUNDAMENTAL THEOREM. *Let e be a closed set of real numbers which is contained in the interval $0 \leq x < 1$ and whose Lebesgue measure is at least $1/P$ ($1/Q=\Phi(P)$). For each $x \in e$, let be associated a closed set of points $C(x)$ contained entirely in the circle $|w| < 1$ such that, if w_1, w_2 be any points on $C(x_1), C(x_2)$ respectively, then the inequality for the hyperbolic distance*

$$(3) \quad [w_1, w_2] \geq [x_1, x_2]$$

always holds. Then the range of values D_f taken by a function $w=f(z) \in \mathfrak{F}$ in the annulus D contains at least one of the sets $\{C(x)\}$ ($x \in e$), except possibly when e is an interval of length $1/P$ and $w=f(z)$ takes the form

$$(4) \quad f(z) \equiv S\{F(z; Q)\} \in \mathfrak{F},$$

S denoting any linear transformation mapping the unit circular disk $|w| < 1$ onto itself.

2. Range of values of a function $\in \mathfrak{S}$.

Using the fundamental theorem we shall prove the following

THEOREM 1. *Let $\mathfrak{S}$ be a class of the functions $\zeta=\varphi(z)$ which are single-valued and analytic (not necessarily univalent) in the annulus D : $Q < |z| < 1$, and whose ranges of values D_ζ for D lie in the strip T : $0 < \text{Im } \zeta < 1$ and have the boundary component $\text{Im } \zeta = 0, 1$ as the image of outer circle $|z|=1$ of D . Suppose that a function $\zeta=\varphi(z) \in \mathfrak{S}$. Then, for each component E_j ($j=1, 2, \dots$) of the complement of D_ζ with respect to T , there holds the inequality*

$$(5) \quad \text{Osc}_{\zeta \in E_j}(\text{Re } \zeta) \leq \frac{1}{\pi} \log \frac{P+1}{P-1}, \quad \left(\frac{1}{Q} = \Phi(P) \right).$$

The result is best possible.

Proof. Let ζ_0 be a point maximizing $\text{Re } \zeta$ ($\zeta \in E_j$) for a fixed component E_j of the complement of D_ζ with respect to T . We map the strip T univalently onto the unit circle $|w| < 1$ in such a way that the point ζ_0 is carried into the origin $w=0$ and the rectilinear segment through the point ζ_0 : $\text{Re } \zeta = \text{Re } \zeta_0$, $0 \leq \text{Im } \zeta \leq 1$ into the diameter $\langle -i, i \rangle$ of the circle $|w|=1$. Suppose that the points $w=a$ ($=e^{i\theta}$) and $w=b$ ($=-e^{-i\theta}$) (θ : real) correspond to $\zeta=-\infty$ and $\zeta=+\infty$, respectively. Such a univalent mapping is carried out by the inverse function $w=g(\zeta)$ of the function

$$(6) \quad \zeta = \frac{1}{\pi} \log \frac{w-a}{w-b} + \text{Re } \zeta_0 + i\mu \quad (\mu: \text{real const.}).$$

It is clear that the composite function $w=g\{\varphi(z)\} \in \mathfrak{F}$. Denote by D_w the range of values taken by $w=g\{\varphi(z)\}$ in D .

Nextly, by the function $W=(w-a)/(w-b)$, we transform the right half Γ of the system of Apollonius' circles with limit points a and b into the system of concentric circles $|W|=r$ ($0 < r \leq 1$) in the W -plane. We denote by Γ' the system of circular arcs which lie on the circles belonging to Γ and are intercepted by two circular arcs K and K' connecting the points a and b inside the unit circle $|w|=1$. We may take K and K' sufficiently near to the circular arcs on $|w|=1$: $\widehat{ab}$ and $\widehat{ba}$, respectively, such that $K \cup K'$ encloses the complement of D_w .

We now take the interval $0 \leq x \leq 1/P$ for the set e and the circular arc of Γ' corresponding to $|W|=r$ by a relation

$$(7) \quad x = \frac{1-r}{1+r}$$

for each set $C(x)$ ($x \in e$), and apply the fundamental theorem to the function $w=g\{\varphi(z)\}$. Then the condition (3) for $C(x)$ is satisfied. In fact, if we denote by $W_j = r_j e^{i\theta_j}$ ($j=1, 2$) the point in the W -plane corresponding to any point on $w_j \in C(x_j)$ ($x_j \in e$, $j=1, 2$), we obtain the inequality

$$(8) \quad [w_1, w_2] = \left| \frac{w_1 - w_2}{1 - \overline{w_1} w_2} \right| = \left| \frac{W_1 - W_2}{\overline{W_1} + e^{-2i\theta} W_2} \right| \\ \geq \left| \frac{r_1 - r_2}{r_1 + r_2} \right| = \left| \frac{x_1 - x_2}{1 - x_1 x_2} \right| = [x_1, x_2],$$

using (7). Hence, from the fundamental theorem, the range of values D_w taken by the function $w=g\{\varphi(z)\}$ contains at least one of the sets $\{C(x)\}$ ($x \in \langle 0, 1/P \rangle$), unless

the function takes the form $w=S\{F(z; Q)\}$. Therefore, the range of values D_ζ taken by the function $\zeta=\varphi(z)$ contains at least one of the segments:

$$(9) \quad \operatorname{Re} \zeta = k \left(\frac{1}{\pi} \log \frac{P-1}{P+1} + \operatorname{Re} \zeta_0 \leq k \leq \operatorname{Re} \zeta_0 \right), \quad 0 < \operatorname{Im} \zeta < 1,$$

except possibly when $\varphi(z) \equiv g^{-1}[S\{F(z; Q)\}]$. Thus, we have the inequality

$$\operatorname{Osc}_{\zeta \in E_j}(\operatorname{Re} \zeta) < \frac{1}{\pi} \log \frac{P+1}{P-1}.$$

Finally, restricting ourselves to the exceptional case when $\varphi(z) \equiv g^{-1}[S\{F(z; Q)\}]$, we obtain the result that the equality sign in (5) holds if and only if $\zeta=\varphi(z) \in \mathfrak{S}$ maps the annulus D univalently onto the strip T slit along the rectilinear segment: $\operatorname{Im} \zeta = 1/2$. The details of calculations will be omitted here; cf. [5].

3. A generalization of Szegő's theorem.

In 1922, Szegő [6] proved the following theorem:

Let $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent in $E: |z|<1$. Let l be the length of any principal chord of the image domain D_f for E . Then

$$l \geq 1.$$

The result is best possible.

In this section we consider the following class $\mathfrak{R}$ of functions: (i) the functions $w=f(z)$ are single-valued and analytic in an annulus $D: 1<|z|<R$, (ii) the ranges of values D_f for D lie in the domain $|w|>1$ and have the boundary component $|w|=1$ as the image of inner circle $|z|=1$ of D .

We now attempt to extend the above result to the case of any analytic function $\in \mathfrak{R}$ and prove the following

THEOREM 2. *Let $w=f(z) \in \mathfrak{R}$ and let l be the length of any principal chord of the simply-connected domain $\tilde{D}_f$ which is bounded by the outer boundary of D_f and contains D_f . Then there holds the inequality*

$$(10) \quad l \geq 2(P + \sqrt{P^2 - 1}),$$

P being uniquely determined for R by the relation $R=\Phi(P)$. The result is best possible.

Proof. Let D_f' be the image domain of D_f by a transformation $w'=1/w$ and let E_j' be some component of the complement of D_f' with respect to the open disk $|w'|<1$. Suppose that E_j' contains two points a and $-a$ ($a>0$), lying symmetrically with respect to the origin $w'=0$. Then we have

$$(11) \quad \frac{1}{a} \geq P + \sqrt{P^2 - 1}.$$

In fact, by the univalent function utilized in Sec. 2 ($a=1$, $b=-1$)

$$(12) \quad \zeta = \frac{1}{\pi} \log \frac{w'-1}{w'+1} - \frac{i}{2} + \lambda \quad (\lambda: \text{real const.}),$$

we map D_f' onto a subdomain of the strip $T: 0<\operatorname{Im} \zeta<1$, and using Theorem 1 we obtain

$$\text{Osc}_{\zeta \in E_j''}(\text{Re } \zeta) \leq \frac{1}{\pi} \log \frac{P+1}{P-1} \quad (R = \Phi(P))$$

for the image E_j'' of E_j' by the mapping (12). From the inequality we easily obtain

$$\frac{1}{\pi} \left(\log \left| \frac{a+1}{a-1} \right| - \log \left| \frac{a-1}{a+1} \right| \right) \leq \frac{1}{\pi} \log \frac{P+1}{P-1}$$

and therefore (11)

Returning to the simply-connected domain $\tilde{D}_f$ in the w -plane, let h_1 and h_2 be two end-points of any principal chord of $\tilde{D}_f$. We may assume that $h_1 > 0$ and $h_2 < 0$ without any loss of generality. Then $l = |h_1| + |h_2|$.

Firstly we consider the case when $|h_1| = |h_2|$. By transforming $\tilde{D}_f$ onto some domain $\tilde{D}_f$ by the mapping $w' = 1/w$ and by applying (11), we have

$$h_1 = -h_2 \geq P + \sqrt{P^2 - 1},$$

and accordingly

$$l \geq 2(P + \sqrt{P^2 - 1}).$$

Secondly we consider the case when $|h_1| \neq |h_2|$. Let α be a real constant satisfying the condition

$$(13) \quad \frac{h_1 - \alpha}{1 - \alpha h_1} = -\frac{h_2 - \alpha}{1 - \alpha h_2}.$$

Such a real constant α can be uniquely determined as the root with smaller absolute value of the quadratic equation

$$(14) \quad \alpha^2 - 2 \frac{1 + h_1 h_2}{h_1 + h_2} \alpha + 1 = 0.$$

Then, by a mapping $w'' = (w - \alpha)/(1 - \alpha w)$, the domain $\tilde{D}_f$ is transformed onto some domain D_f'' in the w'' -plane whose complement with respect to the domain $|w''| > 1$ contains two points a'' and $-a''$ ($a'' = |h_1 - \alpha|/|1 - \alpha h_1|$), lying symmetrically with respect to the origin $w'' = 0$. Hence there holds

$$(15) \quad |a''| \geq P + \sqrt{P^2 - 1}$$

from the result (11). On the other hand, there holds

$$(16) \quad |a''| = \left| \frac{h_1 - \alpha}{1 - \alpha h_1} \right| = \frac{|h_1| \sqrt{h_2^2 - 1} + |h_2| \sqrt{h_1^2 - 1}}{\sqrt{h_1^2 - 1} + \sqrt{h_2^2 - 1}} < \frac{1}{2}(|h_1| + |h_2|) = \frac{1}{2}l$$

from (14) and an elementary inequality

$$(a - b)(\sqrt{a^2 - 1} - \sqrt{b^2 - 1}) > 0 \quad (a, b > 1; a \neq b).$$

Thus we also obtain

$$l > 2(P + \sqrt{P^2 - 1})$$

for the case $|h_1| \neq |h_2|$, from (15) and (16). The equality sign never holds for this case.

From the above considerations it is seen that the equality sign of (10) holds only if the image domain of D in the ζ -plane by a mapping

$$\zeta = \frac{1}{\pi} \log \frac{1/w - 1}{1/w + 1} - \frac{i}{2} + \lambda \quad (w = f(z) \in \Re)$$

is schlicht and coincides with the strip T slit along the segment $\operatorname{Im} \zeta = 1/2$. Thus there occurs the equality if and only if the function $w = f(z) \in \Re$ univalently maps the annulus D onto the domain $|w| > 1$ slit along the segment on the real axis: $|w| \geq P + \sqrt{P^2 - 1}$, up to a rotation about the origin $w = 0$. Therefore the inequality (10) is best possible.

REMARK. Performing the transformations $Z = z/R$ and $W = w/R$ to the domains D and D_f respectively, we obtain Szegő's theorem from Theorem 2 as the limiting case when $R \rightarrow \infty$, because the right-hand side of (10) divided by R tends to 1 [3].

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