A criterion for a set to be of 1-dimensional measure zero

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1. Let F be a domain in the complex z -plane Ω and let E be the boundary of F . We assume that the point $z=\infty$ is an interior point of F . Consider a finite number of doubly connected domains $R^{(k)} (k=1, \dots, \nu)$ satisfying the followings:

- i) the closure $\overline{R^{(k)}}$ of $R^{(k)}$ is contained in F ,
- ii) the boundary of $R^{(k)}$ consists of two analytic closed curves,
- iii) the complementary set of each $\overline{R^{(k)}}$ consists of two domains the one $F^{(k)}$ of which contains the point $z=\infty$ and the other $G^{(k)}$ of which has at least a point common with the set E ,
- iv) any point of the set E is contained in some $G^{(k)}$

and

- v) if $i \neq j$, then $R^{(i)}$ lies in $F^{(j)}$.

We shall say that the system $R = \{R^{(k)}\}_{k=1}^{\nu}$ of such doubly connected domains separates, in F , the point $z=\infty$ from E . If we put $F_0 = \bigcap_{k=1}^{\nu} F^{(k)}$, we can see that F_0 is a domain compact relative to F and that the complementary set of the set $\bigcup_{k=1}^{\nu} \overline{R^{(k)}} \cup F_0$ is the union $\bigcup_{k=1}^{\nu} G^{(k)}$.

We can always take an enumerable number of systems $R_n = \{R_n^{(k)}\}_{k=1}^{\nu(n)}$ ($n=1, 2, \dots$) of doubly connected domains such that each system R_n separates, in F , the point $z=\infty$ from the boundary of F and that each domain $R_n^{(k)}$ of the system R_{n+1} lies in a domain $G_n^{(i)}$ being a connected component of the complementary set of some $\overline{R_n^{(i)}}$ and not containing the point $z=\infty$. In the following, systems R_n ($n=1, 2, \dots$) separating, in F , $z=\infty$ from E means such a sequence of systems as above.

Now, denote by $\log \mu_n^k$ the modulus of the domain $R_n^{(k)}$ and put

$$\log \mu_n = \min_{1 \leq k \leq \nu(n)} \log \mu_n^k.$$

Considering a boundary component γ of F in the sense of Kerékjártó-Stoilow, we see that there exists a sequence of domains $R_n^{(k_n)}$ ($n=1, 2, \dots$) such that the domain $G_n^{(k_n)}$ contains γ and $R_n^{(k_m)}$ for any $m (> n)$. If the series

$$(1) \quad \sum_{n=1}^{\infty} \log \mu_n$$

diverges, then the series

$$\sum_{n=1}^{\infty} \log \mu_n^{k_n}$$

is also divergent and, hence, by Savage's theorem [6], the boundary component γ is weak in the sense of Sario [5]. Therefore, we have easily the following proposition from Sario's theorem [5].

PROPOSITION 1. *If the series (1) diverges, then the boundary set E of F belongs to $N_{\mathfrak{B}}$ in the sense of Ahlfors-Beurling [1].*

Of course, the above proposition is a direct corollary of the following due to Sario [4].

PROPOSITION 2. *If the series (1) diverges, then the set E belongs to $N_{\mathfrak{D}}$.*

Further, by Pfluger-Mori's result [2], [3], we can easily verify the following

PROPOSITION 3. *If*

$$\limsup_{m \rightarrow \infty} \left[\sum_{n=1}^m \log \mu_n - \frac{1}{2} \log \nu(m) \right] = +\infty,$$

then the set E belongs to $N_{\mathfrak{B}}$.

2. Here we shall give a criterion for a linear set E to belong to $N_{\mathfrak{B}}$ which depends on moduli $\log \mu_n$ but is independent of the number $\nu(n)$. We use the same notations as in § 1.

THEOREM. *Let E be a closed set lying on a straight line L . If there exist systems R_n ($n=1, 2, \dots$) separating $z=\infty$ from E such that*

a) *each $R_n^{(k)}$ ($1 \leq k \leq \nu(n)$, $n=1, 2, \dots$) is bounded by two circles symmetric about L and that*

b) *the series (1) diverges,*

then the set E is of 1-dimensional measure zero and hence belongs to $N_{\mathfrak{B}}$.

To establish this, we use the following elementary lemma.

LEMMA. *Let Δ be a doubly connected domain bounded by two circles $|z|=\rho$ and $|z-z_0|=r$, where z_0 is real and $\text{Max}(|z_0+r|, |z_0-r|) < \rho$. Then the modulus of Δ is dominated by $\log(\rho/r)$ from the above.*

Proof. It is easy to see that the circle $|z-z_0|=r$ lies in $|z| < \rho$. By a suitable linear transformation

$$\zeta = \zeta(z) = \rho \frac{z - \alpha \rho}{\rho - \alpha z} \quad (-1 < \alpha < 1),$$

we can map Δ in the concentric ring domain $\delta < |\zeta| < \rho$ one to one conformally such that the images of points $z_1 = -\rho$, $z_2 = z_0 - r$, $z_3 = z_0 + r$ and $z_4 = \rho$ are the points $\zeta_1 = -\rho$, $\zeta_2 = -\delta$, $\zeta_3 = \delta$ and $\zeta_4 = \rho$, respectively, where $\delta > 0$. It is evident that the cross ratio (z_1, z_2, z_3, z_4) equals the cross ratio $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$. From this, we have

$$\frac{(\rho + \delta)^2}{\delta} = \frac{(r + \rho + z_0)(r + \rho - z_0)}{r}.$$

Hence we get the inequality

$$\frac{\rho^2}{\delta} + \delta \leq \frac{\rho^2}{r} + r.$$

Since the function $(\rho^2/x) + x$ is monotonously decreasing in $0 < x < \rho$, we obtain the inequality $\delta \geq r$. The modulus of Δ equals $\log(\rho/\delta)$, so we have our lemma.

Proof of Theorem. We denote by $C_n^{(k)}$ and $\Gamma_n^{(k)}$ the interior circle and the exterior circle bounding $R_n^{(k)}$, respectively. Let $r_n^{(k)}$ and $\rho_n^{(k)}$ be the radius of $C_n^{(k)}$ and $\Gamma_n^{(k)}$, respectively, and put

$$r_n = \sum_{k=1}^{\nu(n)} r_n^{(k)}, \quad \rho_n = \sum_{k=1}^{\nu(n)} \rho_n^{(k)}.$$

It is evident that $r_n \geq \rho_{n+1}$. Since the closed discs $(\Gamma_n^{(k)})$ bounded by $\Gamma_n^{(k)}$ ($k=1, \dots, \nu(n)$) cover the set E , it is sufficient to show that $\lim_{n \rightarrow \infty} \rho_n = 0$. By our lemma, we have

$$\log \mu_n = \min_{1 \leq k \leq \nu(n)} \log \mu_n^k \leq \log \rho_n^{(k)} - \log r_n^{(k)},$$

or

$$\mu_n r_n^{(k)} \leq \rho_n^{(k)} \quad (k=1, \dots, \nu(n)).$$

Summing up these, we get

$$\mu_n r_n \leq \rho_n,$$

which gives

$$\mu_n \leq \frac{\rho_n}{r_n} \leq \frac{\rho_n}{\rho_{n+1}}.$$

Therefore, it holds

$$\prod_{n=1}^N \mu_n \leq \frac{\rho_1}{\rho_{N+1}}.$$

By the assumption b), we have $\lim_{n \rightarrow \infty} \rho_n = 0$.

3. In the case of a special domain F , we can prove the converse of our Theorem.

Consider a segment S_1 with length l_1 on the real axis of Ω . From S_1 , we delete the open segment T_1 with the length l_1/p_1 ($p_1 > 1$) such that the set $S_2 = S_1 - T_1$ consists of two closed segments $S_2^{(j)}$ ($j=1, 2$) with equal length l_2 . Evidently the total length of S_2 equals $2l_2 = l_1(1 - 1/p_1)$. In general, from the set S_m we delete the open segments $T_m^{(j)}$ ($j=1, \dots, 2^{m-1}$) such that each $T_m^{(j)}$ has length l_m/p_m ($p_m > 1$) and that $S_m - \bigcup_{j=1}^{2^{m-1}} T_m^{(j)}$ consists of closed segments $S_{m+1}^{(j)}$ ($j=1, \dots, 2^m$) with equal length l_{m+1} . It is obvious that the total length of the set S_m is $2^{m-1}l_m = 2^{m-2}l_{m-1}(1 - 1/p_{m-1}) = l_1 \prod_{k=1}^{m-1} (1 - 1/p_k)$, $S_m \supset S_{m+1}$ and $\bigcap_{m=1}^{\infty} S_m$ is a non-empty perfect closed set. We denote this set by $E(p_1, p_2, \dots)$ and call this a Cantor set. Obviously, the complementary set F of $E(p_1, p_2, \dots)$ with respect to Ω is a domain. We put two assumptions:

$$\alpha) \quad 2p_{k+1} > p_k - 1 \quad (k=1, 2, \dots)$$

and

$$\beta) \quad \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k}\right) = 0.$$

The assumption α) is not so strong. For instance, if $p_k \leq p_{k+1}$ ($k=1, 2, \dots$), then α) is satisfied. The assumption β) is equivalent to the assumption that the set $E(p_1, p_2, \dots)$ is of 1-dimensional measure zero.

We describe two concentric circles $\Gamma_{m+1}^{(j)}$ and $C_{m+1}^{(j)}$ with centre $z_{m+1}^{(j)}$ being the middle point of $S_{m+1}^{(j)}$ ($m \geq 1$) and with the radius

$$\rho_{m+1} = \frac{l_m}{4} \left(1 + \frac{1}{p_m}\right)$$

and the radius

$$r_{m+1} = \frac{l_m}{4} \left(1 + \frac{1}{p_{m+1}}\right) \left(1 - \frac{1}{p_m}\right),$$

respectively. Then, since

$$l_{m+1} = \frac{1}{2} l_m \left(1 - \frac{1}{p_m}\right),$$

we see by the assumption α)

$$l_{m+1} < 2r_{m+1} < 2\rho_{m+1} = r_m.$$

In other words, the circle $C_{m+1}^{(j)}$ lies inside $\Gamma_{m+1}^{(j)}$ and the segment $S_{m+1}^{(j)}$ lies inside $C_{m+1}^{(j)}$. If we denote by $R_m^{(j)}$ the doubly connected domain bounded by $C_{m+1}^{(j)}$ and $\Gamma_{m+1}^{(j)}$, the systems $R_m = \{R_m^{(j)}\}_{j=1}^{2^m}$ ($m=1, 2, \dots$) separates, in F , $z=\infty$ from $E(p_1, p_2, \dots)$ and satisfies the condition a) of Theorem.

Next, the modulus $\log \mu_m^j$ of $R_m^{(j)}$ ($j=1, \dots, 2^m$) is equal to

$$\log \frac{\rho_{m+1}}{r_{m+1}} = \log \left(1 + \frac{1}{p_m}\right) - \log \left(1 + \frac{1}{p_{m+1}}\right) \left(1 - \frac{1}{p_m}\right).$$

Thus we get

$$\begin{aligned} \sum_{m=1}^N \log \mu_m &= \log \prod_{m=1}^N \frac{\left(1 + \frac{1}{p_m}\right)}{\left(1 + \frac{1}{p_{m+1}}\right) \left(1 - \frac{1}{p_m}\right)} \\ &= \log \frac{1 + \frac{1}{p_1}}{1 + \frac{1}{p_{N+1}}} \frac{1}{\prod_{m=1}^N \left(1 - \frac{1}{p_m}\right)} \geq \log \frac{1}{2 \prod_{m=1}^N \left(1 - \frac{1}{p_m}\right)}. \end{aligned}$$

By the assumption β), the right hand side of this inequality diverges as N tends to infinity and hence we obtain the condition b) of Theorem. Thus we have

PROPOSITION 4. *For a Cantor set $E(p_1, p_2, \dots)$ satisfying α) to be of 1-dimensional measure zero, it is necessary and sufficient that there exist systems $R_n = \{R_n^{(k)}\}_{k=1}^{2^{(n)}}$ ($n=1, 2, \dots$) separating, in $F = \Omega - E(p_1, p_2, \dots)$, the point $z=\infty$ from $E(p_1, p_2, \dots)$ and satisfying the conditions a) and b) in our Theorem.*

REMARK. An analogue of Proposition 4 is obtained for some special 2-dimensional Cantor set $E^2(p_1, p_2, \dots)$ (Cf. Sario [4]). Proposition 4 and this analogue are closely related to results of Sario [4].

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