On the harmonic boundary of an open Riemann surface, I

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Every open Riemann surface R can be represented as a dense subset of a compact Hausdorff space R^* considered by Gelfand and Royden (cf. [3], [4], [10]). The ideal boundary of R is then realized as a closed non-dense subset Γ of R^* , which is, by definition, the set of maximal ideals containing the ideal K of BD -functions with compact carriers. The set Δ of maximal ideals containing $\bar{K}$ (closure of K by BD -topology) constitutes an essential part of Γ and is called the harmonic boundary of R .

In this article we shall study some properties on harmonic boundaries and non-compact subregions on R . By a non-compact subregion G on R we shall understand in the sequel a non-compact domain on R whose relative boundary ∂G consists of an at most countable number of disjoint analytic Jordan curves not accumulating to any point of R .

1. LEMMA 1. (Mori-Ota [8]) *Every HBD-function on R attains its maximum and minimum on Δ .*

LEMMA 2. *Let M be a point of $\Delta=\Delta(R)$ and E be a closed subset of Δ which is non-empty and disjoint with M . Then there exists a function $u_M \in HBD$ on R such that $u_M(M)=1$ and $u_M=0$ on E .*

Proof. From the topology (cf. [4]) of R^* we see that there is a function $u \in BD$ which belongs to the ideal $\cap_{N \in E} N$, but not to M . Let $u = u_0 + \varphi$ be the orthogonal decomposition such that $u_0 = \text{proj. } u$ (projection of u to HBD -space) $\in HBD$ and $\varphi \in \bar{K}$ (Royden [9]). Since $\varphi=0$ on Δ , $u_0(M)=u(M) \neq 0$ and $u_0(N)=u(N)=0$ for any point $N \in E$. Hence $u_M = u_0/u_0(M)$ fulfils our conditions, q. e. d.

LEMMA 3. *Suppose that Δ_1 and Δ_2 be two non-empty closed subsets of Δ such that $\Delta_1 \cup \Delta_2 = \Delta$ and they are mutually disjoint. Then there exists a (non-negative) function $u \in HBD$ on R such that $u=1$ on Δ_1 and $u=0$ on Δ_2 .*

Proof. For each point $M \in \Delta_1$, we consider the above function $u_M \in HBD$ on R such that $u_M(M)=1$ and $u_M=0$ on Δ_2 , and associate a neighborhood $\{p; u_M(p) > 1/2\}$. Then, from the compactness of Δ_1 , it can be covered by a finite number of such neighborhoods $\{p; u_{M_i}(p) > 1/2\}$ ($M_i \in \Delta_1$, $i=1, 2, \dots, k$). Then $v_1 = \text{proj.}(\sum_{i=1}^k u_{M_i}^2) \in HBD$ vanishes on Δ_2 while $v_1 > 1/4$ on Δ_1 . Analogously there exists a function $v_2 \in HBD$ such that $v_2=0$ on Δ_1 and $v_2 > 1/4$ on Δ_2 . Thus an HBD -function $v = v_1 + v_2$ is greater than $1/4$ throughout R^* by Lemma 1. Hence there exists a BD -function $1/v$ on R . The projection u of BD -function v_1/v to HBD -space obviously fulfils our conditions, which completes the proof.

Let G be a non-compact subregion on R and $\hat{G}$ denotes the double of G with

respect to ∂G . Then we note that if $\varphi \in \bar{K} = \bar{K}(R)$, the symmetric extension $\hat{\varphi}$ of φ on $\hat{G}$ belongs to $\bar{K}(\hat{G})$. Indeed, let $\{\varphi_n\}$ be a sequence of elements of $K(R)$ such that $\varphi_n \rightarrow \varphi$ (in BD) on R . Then $\hat{\varphi}_n \in K(\hat{G})$, that is, the carriers are compact in $\hat{G}$ and the Dirichlet integrals $D_{\hat{G}}[\hat{\varphi}_n]$ over $\hat{G}$ are finite. Obviously $D_{\hat{G}}[\hat{\varphi} - \hat{\varphi}_n] \leq 2D_R[\varphi - \varphi_n]$. Thus $\hat{\varphi}_n$ are uniformly bounded and $\hat{\varphi}_n \rightarrow \hat{\varphi}$ (in BD) on $\hat{G}$, which imply $\hat{\varphi} \in \bar{K}(\hat{G})$.

2. In the following we shall denote by $\sigma(R)$ the number of harmonic boundary points of a Riemann surface R . Obviously $\sigma(R) = 0$ if and only if $\bar{K} = BD$, i. e. $1 \in \bar{K}$. It is known ([9]) that $1 \in \bar{K}$ is equivalent to $R \in O_G$. But we shall give now another simple proof. First if the harmonic measure of Γ is zero, we know immediately that $1 \in \bar{K}$. Conversely, if $1 \in \bar{K}$, any $u \in HBD$ on R belongs to $\bar{K}$; hence there is a sequence $\{\varphi_n\}$ such that $\varphi_n \in K$, $\varphi_n \rightarrow u$ (in BD) on R . Since $|D_R(u, u - \varphi_n)|^2 \leq D_R(u)D_R(u - \varphi_n) \rightarrow 0$ for $n \rightarrow \infty$, it follows that $D_R(u) = \lim_{n \rightarrow \infty} D_R(u, \varphi_n) = 0$ by Green's formula, hence $u \equiv \text{const}$. Now consider the harmonic measure $\omega = \omega(p, \Gamma, R - \bar{R}_0)$ on $R - \bar{R}_0$ (R_0 is a compact domain) and its anti-symmetric extension $\hat{\omega}$ on the double $(R - \bar{R}_0)^\wedge$, then $\hat{\omega}$ is an HBD -function on $(R - \bar{R}_0)^\wedge$. Since the symmetric extension 1 of $1 \in \bar{K}$ belongs to $\bar{K}(\hat{G})$, $\hat{\omega}$ must reduce to a constant, i. e. $\omega \equiv \text{const}$, which proves our assertion. From the proof we see also that if G is a non-compact subregion on $R \in O_G$, then $\hat{G}$ belongs to O_G .

PROPOSITION 1. Let G be a non-compact subregion on an open Riemann surface R . If the double $\hat{G}$ of G with respect to ∂G is of hyperbolic type, the $\bar{G}$ (closure of G in R^*) contains harmonic boundary points of R .

Proof. Since $\hat{G} \notin O_G$, $\sigma(\hat{G}) > 0$ and hence for any non-negative $\hat{\varphi} \in \bar{K}(\hat{G})$ we have $\inf_G \hat{\varphi} = 0$, in particular by a remark in sec. 1

$$(*) \quad \inf_G \varphi = 0 \text{ for any non-negative } \varphi \in \bar{K} = \bar{K}(R).$$

This shows also that the original surface R is of hyperbolic type. Now suppose that $\bar{G} \cap \Gamma$ have no points of $\mathcal{A}$, then for each point $M \in \bar{G} \cap \Gamma$ we can construct a non-negative function $f_M \in \bar{K}$ on R such that $f_M(M) > 0$. For example, take a function $g_M \in \bar{K}$ which does not belong to the maximal ideal M , then g_M^2 would fulfil the conditions. Since $\bar{G} \cap \Gamma$ is compact, by means of above functions f_M we can easily get a function $\phi \in \bar{K}$ such that $\phi \geq 0$ on $\bar{G} \cap \Gamma$. Next consider a Green function $g(p, q)$ of R and

$$g_r(p) = \min[g(p, q), r] \in \bar{K}$$

where r is a positive number such that the level curve $g = r$ is compact. Then $\varphi = g_r + \phi \in \bar{K}$ is continuous and positive on $\bar{G}$, which contradicts (*), q. e. d.

COROLLARY (Kuramochi [5]). Let R_λ be a hyperbolic Riemann surface and $g(p, q)$ be its Green function, then the double of domain $G_\lambda = \{p; g(p, q) > \lambda > 0\}$ with respect to ∂G_λ is a parabolic Riemann surface, unless it is compact.

PROPOSITION 2. Let G be a non-compact subregion on an open Riemann surface R such that $G \notin SO_{HD}$, that is, there exists a non-constant HD -function on G which

vanishes on ∂G . Then $\bar{G} - \bar{\partial G}(R^*)$ contains some harmonic boundary points of R .

Proof. First we note that the double $\hat{G}$ of G does not belong to $O_{HD} \supset O_G$, hence $\bar{K}$ contains harmonic boundary points A_1 of R . According to A. Mori [7] there exists a non-constant *HBD*-function u_0 on G which vanishes on ∂G . Without loss of generality we may assume that $M = \sup_G u_0 > 0$. Then $F = \{p; u_0(p) > M - \varepsilon > 0\}$ is a non-compact subregion on G and we find that $\bar{F}$ contains also harmonic boundary points $A_2 (\subset A_1)$ of R and that $A_2 \subset \bar{G} - \bar{\partial G}$.

PROPOSITION 3.1. $\sigma(R) = 0$ if and only if $R \in O_G$,
 $\sigma(R) = 1$ if and only if $R \in O_{HD} - O_G$.

It remains to prove the latter. Suppose that $R \notin O_{HD}$, i. e. there is a non-constant *HD*-function u on R , then $G_\lambda = \{p; u(p) > \lambda, \inf_R u < \lambda < \sup_R u\}$ and $G_\lambda^c = \{p; u(p) < \lambda\}$ are disjoint non-compact subregions on R . Since G_λ and G_λ^c do not belong to SO_{HD} , $\sigma(R) \geq 2$ by Prop. 2. Thus if $\sigma(R) = 1$, we see that $R \in O_{HD} - O_G$. Next if $\sigma(R) \geq 2$, there would exist a non-constant *HBD*-function of R by Lemma 2, which proves the converse.

3. Let $O_{HD_n} (n=1, 2, \dots)$ be the classes of Riemann surfaces whose ideal boundaries consist of at most n maximal *HD*-indivisible sets (Constantinescu-Cornea [2]). Then we have

PROPOSITION 3.2. The following three statements are equivalent:

- (i) $R \in O_{HD_n}$; (ii) $\sigma(R) \leq n$;
- (iii) The dimension of *HBD*-space on R is $\leq n$.

Hence we have $\sigma(R) = n$ if and only if $R \in O_{HD_n} - O_{HD_{n-1}} (n \geq 2)$.

Since we can readily see from [2] the relation between (i) and (iii), we shall here note the equivalence of (ii) and (iii) only. Suppose $\sigma(R) \geq n+1$ then for $n+1$ points $\{M_i\}$ of A we would have, by Lemma 2, $n+1$ linearly independent *HBD*-functions u_i on R such that $u_i(M_i) = 1$ and $u_i(M_j) = 0 (i \neq j)$. To see the converse, take above functions $u_1, u_2, \dots, u_m (m = \sigma(R) \leq n)$. Suppose that there are at least $n+1 (\geq 2)$ linearly independent functions $\omega_i \in \text{HBD}$, then $\omega_i - \sum_{j=1}^m \omega_i(M_j) u_j (i=1, 2, \dots, n+1)$ vanish on A , hence these are reduced to zero identically on R by Lemma 1. Thus ω_i would become linearly dependent, which is absurd.

PROPOSITION 4. If $1 \leq \sigma(R) < +\infty$, then the genus of R is infinite.

Proof. Let $\{R_n\}_{n=0,1,\dots}$ be an exhaustion of R . Since R is of hyperbolic type by Prop. 3.1, the harmonic measure $\omega = \omega(p, \Gamma, R - \bar{R}_0)$ is not identically zero. Suppose the genus of R is finite, then for large n , $R - R_n$ become of planar character and a subregion G such as $\{p; \omega(p) > m, m = \sup_{R_n - R_0} \omega < 1\}$ lies in $R - R_n$. Since a non-constant function $\omega - m \in \text{HBD}$ vanishes on ∂G , G (on $R \in O_{HD_n} - O_G$) must belong to O_{AD} (Kuramochi [6]). This is absurd, because there exists a function which maps G conformally onto a plane region with finite area, q. e. d.

As an extension of Bader-Parreau's theorem [1] ($m=2$) we have the following

PROPOSITION 5. Let $G_1, G_2, \dots, G_m$ be non-compact subregions on R which are disjoint each other. If there exist non-constant *HD*-functions U_i on $G_i + \partial G_i$ which

vanish on $\partial G_i (i=1, 2, \dots, m)$, then $\sigma(R) \geq m$, i. e. $R \notin O_{HD_{m-1}}$. The converse is also true.

Proof. Since each $G_i \notin SO_{HD}$, by Prop. 2 $\overline{G_i} - \partial \overline{G_i}(R^*)$ contains at least one point of A , hence $\sigma(R) \geq m$. Conversely, let $M_1, M_2, \dots, M_m$ be m points of A , then since these points are mutually disjoint by the topology of R^* (cf. [4]), there are m functions $u_i \in HBD$ on R such that $u_i(M_i) = 1$ and $u_i(M_j) = 0$ ($i \neq j$). Hence we find easily that, for suitable constants λ_i , m domains $G_i^* = \{u_i > \lambda_i\}$ become disjoint on $R^* - \Gamma$. It suffices now to transform G_i^* into R and set $U_i = u_i - \lambda_i$.

4. PROPOSITION 6. Let G be a non-compact subregion of R such that $G \in NO_{HD}$, that is, there does not exist any non-constant HD -functions on G with vanishing normal derivatives on ∂G . If the double $\hat{G}$ of G with respect to ∂G does not belong to O_{HD} , then $\sigma(\hat{G}) = 2$.

Proof. First of all, we note that any function of HBD on $\hat{G}$ is decomposed into a constant and an anti-symmetric function. In fact, for any $u \in HBD$ on $\hat{G}$ we write

$$u = u_1 + u_2,$$

$$u_1(p) = (u(p) + u(\tilde{p}))/2, \quad u_2(p) = (u(p) - u(\tilde{p}))/2,$$

where $\tilde{p}$ denotes the symmetric point of p with respect to ∂G . Since the normal derivative of u_1 vanishes on ∂G , u_1 must be a constant. In the following we denote by u_0 the anti-symmetric part of an HBD -function u on $\hat{G}$. Now since $\hat{G} \notin O_{HD} = O_{HBD}$, there exists a non-constant function $u \in HBD$ on $\hat{G}$. Consider the orthogonal decomposition of u_0^2 ($\neq \text{const}$) such that

$$u_0^2 = v + \varphi = v_0 + k + \varphi$$

where $v \in HBD(\hat{G})$, $\varphi \in \overline{K}(\hat{G})$ and k is a constant. It follows $\varphi = -k$ on ∂G . We note that k is positive. In fact, consider an exhaustion $\{R_n\}$ of R and harmonic functions v_n on $(R_n \cap G)'$ with boundary values u_0^2 , then $v_n - u_0^2 \geq 0$ on $(R_n \cap G)$ since u_0^2 is subharmonic. It follows that $v \geq u_0^2$ for $n \rightarrow \infty$, where the equality never holds. Otherwise $v \equiv u_0^2$ by minimum principle, that is, u_0^2 becomes harmonic. This is possible only if $u_0 \equiv \text{const}$. Now let $\tilde{\varphi}$ be a BD -function on $\hat{G}$ such that $\tilde{\varphi} = \varphi^2$ on G and $\tilde{\varphi} = k^2$ on $\tilde{G} = \hat{G} - G$. Then $\tilde{\varphi} = 0$ on $\overline{G} \cap A(\hat{G})$, while $\tilde{\varphi} = k^2$ ($\neq 0$) on $\tilde{\tilde{G}} \cap A(\hat{G})$. Thus we find that $\partial \overline{G}$ does not contain any point of $A(\hat{G})$, that is, $A(\hat{G})$ is split into two compact sets $A_1 (\subset \overline{G} \cap A(\hat{G}))$ and $A_2 (\subset \tilde{\tilde{G}} \cap A(\hat{G}))$, which are non-empty on account of $\hat{G} \notin O_G$. Thus we know by Lemma 3 that there exists a function $w \in HBD$ on $\hat{G}$ such that $w = 1$ on A_1 and $= 0$ on A_2 . Next for any function $U \in HBD$ on $\hat{G}$ consider the orthogonal decomposition of Uw :

$$Uw = h + \phi = h_0 + \gamma + \phi$$

where $h \in HBD(\hat{G})$, $\phi \in \overline{K}(\hat{G})$ and γ is a constant. From the property of w it follows that $h = 0$ on A_2 and $h = U$ on A_1 . Hence

$$h_0 = -\gamma \text{ on } A_2 \text{ and } h_0 = -\gamma + U \text{ on } A_1.$$

Since h_0 is anti-symmetric, we find that $h_0 = \gamma$ on A_1 , hence U must be a constant (2γ) on A_1 . In fact, suppose $h_0 \neq \gamma$ on A_1 , then there is a non-compact subregion

$F = \{p; |h_0(p) - \gamma| > \delta_0 > 0\}$ such that $\bar{F}$ contains some points of A_1 . It is easily seen that the symmetric region $\tilde{F}$ of F with respect to ∂G contains also some points of A_2 . While, in $\tilde{F}$, $|h_0(\tilde{p}) + \gamma| = |-h_0(p) + \gamma| > \delta_0$, which is absurd. Thus we know similarly that U is constant on A_2 . From this we can conclude that A_1 and A_2 consists of a single point respectively, q. e. d.

From the above proof we have immediately a sufficient condition in order that $R \in O_{HD}$ (cf. [2], Satz 11).

PROPOSITION 7. *Let G be a non-compact subregion on R . If $G \in NO_{HD}$ and $\hat{G} \notin O_{HD}$ (e. g., $G \notin SO_{HD}$), then $R \in U_{HD}$, i. e., there exists a non-constant $\underline{HD}$ (or HD) minimal function on R .*

It is noted that the hypothesis in Prop. 7 is not void, i. e., there exists a non-compact subregion G such that $G \in NO_{HD}$ and $\notin SO_{HD}$. For example, $G = R - \bar{K}$ on $R \in O_{HD} - O_G$ fulfils the conditions, where K is a compact domain on R with analytic boundaries.

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