Algebras of some differentiable functions on Riemannian manifolds

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Introduction

Let R be a *Riemannian manifold* of class C^m ($1 \leq m \leq \infty$) with a metric tensor $(g_{ij}; i, j=1, 2, \dots, n)$ of class C^{m-1} , where n is the dimension of R . Here we do not require the connectedness or the second countability axiom for R .

Let $A^m(R)$ be the totality of real-valued C^m -functions f on R with the finite norms: $\|f\| < \infty$, where the norm $\|f\|$ of f is defined by the following:

$$(1) \quad \|f\| = \sup_R |f| + \sup_R \left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{1/2}$$

Here and hereafter we follow the convention in tensor analysis that the repeated indices imply summation.

It is easily seen that $A^m(R)$ is a *normed algebra* over the real number field with the norm defined by (1). In the case when R is compact, $A^m(R)$ coincides with the totality $C^m(R)$ of C^m -functions on R . Concerning this normed algebra S. B. Myers [4] proved that the structure of any *compact* Riemannian manifold R of class C^m ($1 \leq m < \infty$) is completely determined by the structure of the normed algebra $A^m(R)$ ($= C^m(R)$).

The aim of this note is to extend the above result to an arbitrary (compact or non-compact) Riemannian manifold of class C^m ($1 \leq m \leq \infty$), namely, we shall prove that the structure of R as the Riemannian manifold of class C^m ($1 \leq m \leq \infty$) is completely determined by the structure of $A^m(R)$ as the normed algebra. We state this as follows:

THEOREM. *Two Riemannian manifolds R and $'R$ of class C^m ($1 \leq m \leq \infty$) are equivalent as Riemannian manifolds if and only if $A^m(R)$ and $A^m('R)$ are equivalent as normed algebras.*

More precisely any isometry T of R onto $'R$ induces an isometric isomorphism $f \rightarrow f \circ T^{-1}$ of $A^m(R)$ onto $A^m('R)$ and conversely any isometric isomorphism of $A^m(R)$ onto $A^m('R)$: $f \rightarrow f^\sigma$ is induced by an isometry T of R onto $'R$, that is $f^\sigma = f \circ T^{-1}$.¹⁾ Here an *isometry* T of R onto $'R$ means the following:

- (i) T is a homeomorphism of R onto $'R$,
- (ii) T and T^{-1} are C^m -mappings,
- (iii) $g^{ij}(\partial y^k / \partial x^i)(\partial y^l / \partial x^j) = 'g^{kl}$,

where $y^k = y^k(x^1, x^2, \dots, x^n)$ ($k=1, 2, \dots, n$) is the local representation of T in terms

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1) Although the dimensions of R and $'R$ necessarily coincide but we need not assume that they coincide when we consider the isometric isomorphism of $A^m(R)$ onto $A^m('R)$.

of local coordinates and, (g_{ij}) and $({}'g_{ij})$ are metric tensors of R and $'R$ respectively. Hence if T is an isometry of R onto $'R$ then T^{-1} is an isometry of $'R$ onto R . We also mean by an *isometric isomorphism* σ of $A^m(R)$ onto $A^m('R)$ is a correspondence satisfying:

(i) σ is an isomorphism of $A^m(R)$ onto $A^m('R)$ as algebras over the real number field,

(ii) σ preserves the norm defined by (1), i. e., $\|f^*\| = \|f\|$.

For the proof of the theorem the essential part is to find a homeomorphism of R onto $'R$ which induces a given isometric isomorphism of $A^m(R)$ onto $A^m('R)$. In the case R and $'R$ are compact this follows at once from the Gelfand theory. But in the non-compact case we need some technics. Concerning merely differentiable structure of (not necessarily compact) R and the algebra $C^m(R)$, the similar investigations are found in Pursell [5] and Akizuki [1] in the manners different from ours. Our method is based on the concept of the non-removable singularity, that is, we make use of the fact that there exists a function f in $A^m(R-p)$ but not in $A^m(R)$ where p is an arbitrary point in R . Such a concept seems very useful in the study of various function algebras (cf. Hewitt [2], Kakutani [3], Rudin [6]) and considerably general treatments based on this concept can be performed, which will be published elsewhere.

In § 1 we consider the conjugate space of some subalgebra of $A^m(R)$ which is an appropriate version of Myers [4]. In § 2 we consider a subspace, called the character space, of the conjugate space of $A^m(R)$. In § 3 the proof of the theorem is given.

1. Tangent vectors in conjugate spaces.

Let R be an n -dimensional Riemannian manifold of class C^m ($1 \leq m \leq \infty$) provided with a metric tensor $(g_{ij}; i, j=1, 2, \dots, n)$ of class C^{m-1} . We denote the totality of functions in $A^m(R)$ with compact carriers by $A_0^m(R)$, i. e., f is in $A_0^m(R)$ if f is in $A^m(R)$ and the carrier $\text{CL}\{p \in R, f(p) \neq 0\}$ is compact, where $\text{CL}S$ denote the closure of the set S . Then $A_0^m(R)$ is a normed subalgebra of $A^m(R)$ and *a fortiori* a normed space over the real number field under the norm defined by (1).

We denote by $A_0^m(R)^*$ the *conjugate space* of $A_0^m(R)$, that is, the totality of bounded linear functionals on $A_0^m(R)$.

We denote by $\mathfrak{L}_p(R)$ the *tangent vector space* at $p \in R$ and set $\mathfrak{L}(R) = \cup_p \mathfrak{L}_p(R)$. The *length* $|L|$ of L in $\mathfrak{L}_p(R)$ is given by

$$(2) \quad |L| = (g_{ij}L(x^i)L(x^j))^{1/2},$$

where $(x^1, x^2, \dots, x^n)$ is an admissible coordinate system with the origin at p . We denote $\tilde{\mathfrak{L}}_p(R)$ the totality of L in $\mathfrak{L}_p(R)$ with $|L|=1$ and $\tilde{\mathfrak{L}}(R) = \cup_p \tilde{\mathfrak{L}}_p(R)$.

For f in $A_0^m(R)$ and L in $\mathfrak{L}_p(R)$, it holds, by definition, that

$$(3) \quad L(f) = \frac{\partial f}{\partial x^i} L(x^i)$$

and from this we see that

$$(4) \quad |L(f)| \leq |L| \|f\|.$$

In fact,

$$|L(f)| = |L| |(L/|L|)(f)| \leq |L| \sup_{M \in \tilde{\mathfrak{L}}_p(R)} |M(f)|$$

$$\leq |L| \sup_{M \in \mathfrak{L}(R)} |M(f)| = |L| \|f\|.^{2)}$$

This shows that $\mathfrak{L}(R) \subset A_0^m(R)^*$ and so the norm

$$\|L\| = \sup_{\|f\|=1} |L(f)|$$

of L in $A_0^m(R)^*$ satisfies $\|L\| \leq |L|$. We shall show that the inverse inequality holds. For the aim we state

LEMMA 1. *For any tangent vector L in $\mathfrak{L}_p(R)$, there exists a family $\{f_c; 0 < c < c_0\}$ of functions in $A_0^m(R)$ satisfying the following two conditions:*

$$(5) \quad L(f_c) = |L|,$$

$$(6) \quad \lim_{c \downarrow 0} \|f_c\| = 1.$$

*Proof.*³⁾ First we choose an admissible coordinate system $(x^1, x^2, \dots, x^n)$ with the origin at p for which $g_{ij}(0) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Using this coordinate system, we set

$$f_c(x) = \begin{cases} |L|^{-1} \sum_{i=1}^n L(x^i) x^i \exp\left(\sum_{i=1}^n (1/((x^i)^2 - c^2) + 1/c^2)\right) & (|x^i| < c), \\ 0 & (\text{elsewhere on } R) \end{cases}$$

for $0 < c < c_0$, where c_0 is chosen so small that the set $(|x^i| < c_0)$ is contained in the coordinate neighborhood of x at p . The family $\{f_c; 0 < c < c_0\}$ thus defined enjoys the required properties. To see this, it is enough to ascertain that the function $g_c(x)$ of one real variable x defined by

$$g_c(x) = \begin{cases} x \exp(1/(x^2 - c^2) + 1/c^2) & (|x| < c), \\ 0 & (|x| \geq c) \end{cases}$$

satisfies the following three conditions:

- (i) $g_c(x)$ is in the class C^∞ and $\lim_{c \downarrow 0} (\sup_x |g_c(x)|) = 0$;
- (ii) $g_c'(0) = 1$;
- (iii) $\lim_{c \downarrow 0} (\sup_x |g_c'(x)|) = 1$.

The condition (i) is evident. To see (ii) and (iii), we compute as follows:

$$g_c'(x) = \begin{cases} (1 - 2x^2/(x^2 - c^2)^2) \exp(1/(x^2 - c^2) + 1/c^2) & (|x| < c), \\ 0 & (|x| \geq c), \end{cases}$$

from which (ii) is clear. For the proof of (iii), we calculate as follows:

$$g_c''(x) = \begin{cases} (4/(x^2 - c^2)^4) \exp(1/(x^2 - c^2) + 1/c^2) x h_c(x) & (|x| < c), \\ 0 & (|x| \geq c), \end{cases}$$

where

$$h_c(x) = (x^2 - (3/2)c^4) + 2^{-1}(x^4 + 2c^2x^2).$$

Let the totality of zeros of $g_c''(x)$ in $|x| < c$ be denoted by Z_c . Then Z_c contains at most five elements and 0 is contained in Z_c .

Take an arbitrary non-zero element z in Z_c . Then z is a zero of $h_c(x)$ and hence $z^2 - (3/2)c^4 = -2^{-1}(z^4 + 2c^2z^2)$, from which

2) Here we use $(g^{ij}(\partial f / \partial x^i)(\partial f / \partial x^j))^{1/2} = \sup_{M \in \mathfrak{L}_p(R)} |M(f)|$ at p .

3) The construction of the family $\{f_c; 0 < c < c_0\}$ for the case $1 \leq m < \infty$ is found in Myers [4]. The present construction admits the case $m = \infty$.

$$|z^2 - (3/2)c^4| \leq 2^{-1}z^4 + c^2z^2.$$

As $|z| < c$, we get $|z^2 - (3/2)c^4| < (3/2)c^4$ or $z^2 < 3c^4$ and finally we have

$$|z^2 - (3/2)c^4| = O(c^6).$$

Thus we may set $z^2 = (3/2)c^4 + O(c^6)$ and from this

$$g_c'(z) = (-2 + O(c^2)) \exp(-3/2 + O(c^2)).$$

On the other hand, $g_c'(0) = 1$. From these, we get

$$\sup_x |g_c'(x)| = \max_{z \in Z_c} |g_c'(z)| = \max(1, (2 + O(c^2)) \exp(-3/2 + O(c^2))).$$

Hence

$$\lim_{c \downarrow 0} (\sup_x |g_c'(x)|) = \max(1, 2 \exp(-3/2)) = 1,$$

which proves (iii). This completes the proof.

LEMMA 2. $|L| = \|L\|$ for L in $\mathfrak{L}_p(R)$.

Proof. Take the family $\{f_c; 0 < c < c_0\}$ constructed for L in Lemma 1. Then we get

$$|L| = L(f) \leq \|L\| \|f_c\|.$$

Making $c \downarrow 0$, we have $|L| \leq \|L\|$. The converse inequality is already obtained. Thus $|L| = \|L\|$.

LEMMA 3. If T is a topological mapping of a Riemannian manifold R of class C^m ($1 \leq m \leq \infty$) onto another Riemannian manifold $'R$ of class C^m such that $f \rightarrow f^\sigma = f \circ T^{-1}$ is an algebraic isomorphism of $A_0^m(R)$ onto $A_0^m('R)$, then the adjoint isomorphism σ^* (of σ): $A_0^m('R)^* \rightarrow A_0^m(R)^*$, carries $\mathfrak{L}_{Tp}('R)$ onto $\mathfrak{L}_p(R)$.

Proof. First we note that T and T^{-1} are m -fold continuously differentiable, as is easily shown. Now let L be in $\mathfrak{L}_{Tp}('R)$. We have to prove $L^{\sigma*}$ is in $\mathfrak{L}_p(R)$. By the definition of σ^* ,

$$L^{\sigma*}(f) = L(f^\sigma) = L(f \circ T^{-1})$$

for f in $A_0^m(R)$. Let the local representation of T^{-1} at p , in terms of local coordinate system $(x^1, x^2, \dots, x^n)$ at p and $y = (y^1, y^2, \dots, y^n)$ at Tp respectively with $x(p) = y(Tp) = 0$, be $x^i = x^i(y^1, y^2, \dots, y^n)$ ($i = 1, 2, \dots, n$). Then

$$L^{\sigma*}(f) = L(f^\sigma) = \frac{\partial}{\partial y^i} f(x(0)) \cdot L(y^i) = \frac{\partial}{\partial x^j} f(0) \frac{\partial x^j}{\partial y^i} L(y^i).$$

Hence if we set $L^{\sigma*}(x^j) = L(y^i) \frac{\partial x^j}{\partial y^i}$, we get

$$L^{\sigma*}(f) = \frac{\partial f}{\partial x^j} L^{\sigma*}(x^j),$$

which shows that $L^{\sigma*}$ is in $\mathfrak{L}_p(R)$. The converse is similarly verified.

LEMMA 4. If T is a topological mapping of a Riemannian manifold R of class C^m ($1 \leq m \leq \infty$) onto another Riemannian manifold $'R$ of class C^m such that $f \rightarrow f^\sigma = f \circ T^{-1}$ is an isomorphism of $A_0^m(R)$ onto $A_0^m('R)$, then T is an isometry.

Proof. We use the same notations as in the proof of Lemma 3. First we note

that the isometricity of σ implies the same for σ^* . Hence by Lemmas 2 and 3, we conclude that

$$L^{\sigma^*} \text{ is in } \mathfrak{L}_p(R) \text{ and } |L^{\sigma^*}| = |L|$$

for L in $\mathfrak{L}_{Tp}('R)$. Thus by (2)

$$g_{ij}L^{\sigma^*}(x^i)L^{\sigma^*}(x^j) = 'g_{kl}L(y^k)L(y^l),$$

where (g_{ij}) and $('g_{ij})$ is the metric tensors in R and $'R$ respectively. Using the relation $L^{\sigma^*}(x^j) = ((\partial x^j)/(\partial y^i))L(y^i)$ obtained in the proof of Lemma 3, we get

$$g_{ij} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} L(y^k)L(y^l) = 'g_{kl}L(y^k)L(y^l)$$

for any L in $\mathfrak{L}_{Tp}('R)$, which implies $g_{ij}((\partial x^i)/(\partial y^k))((\partial x^j)/(\partial y^l)) = 'g_{kl}$ which is equivalent to

$$'g^{kl} = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} g^{ij}.$$

Thus T is an isometry of R onto $'R$.

2. Compactification of R .

As before, let R be a Riemannian manifold of class C^m ($1 \leq m \leq \infty$). We consider a functional χ on $A^m(R)$ called a *character* satisfying the following:

(7) χ is an algebraic homomorphism of $A^m(R)$ onto the real number field with $\chi(1) = 1$.

Let R^* be the totality of characters χ on $A^m(R)$. First we note that

$$(8) \quad \|\chi\| = 1 \text{ and } |\chi(f)| \leq \sup_R |f|.$$

To see this we note that if g is in $A^m(R)$ and $g \geq c > 0$ on R , then $\sqrt{g}$ is in $A^m(R)$. Therefore we may write $g = (\sqrt{g})^2$ in $A^m(R)$ and hence $\chi(g) = (\chi(\sqrt{g}))^2$ which is non-negative. If $g \geq 0$, then $g + c \geq c > 0$ and so $\chi(g + c) = \chi(g) + c \geq 0$ and making $c \downarrow 0$ we get $\chi(g) \geq 0$.

Let $f \in A^m(R)$ and $a = \sup_R |f|$. Then $a \pm f \geq 0$ and so $\chi(a \pm f) \geq 0$ or $|\chi(f)| \leq a \leq \|f\|$. Therefore $\|\chi\| \leq 1$. On the other hand $|\chi(1)| \leq \|\chi\| \|1\|$ implies $\|\chi\| \geq 1$. Hence (8) is proved.

For a point p in R , we set for f in $A^m(R)$

$$\chi_p(f) = f(p).$$

Then clearly (7) is satisfied by χ_p . Hence χ_p is in R^* . Moreover $p \neq q$ implies $\chi_p \neq \chi_q$ because $A^m(R)$ separates points, i. e., there exists a function f in $A^m(R)$ such that $f(p) \neq f(q)$ for any $p \neq q$ in R . Thus $p \rightarrow \chi_p$ gives a one to one correspondence of R onto a subset $\tilde{R}$ in R^* . We call $\chi_p \in \tilde{R}$ the *point character*.

Now we introduce a topology in R^* by the relative topology of the weak topology as functionals in $A^m(R)^*$. We remark that the convergence in R^* is then given as follows: a directed sequence $\{\chi_i\}$ in R^* converges to χ if and only if $\chi_i(f)$ converges to $\chi(f)$ as the sequence of real numbers for every f in $A^m(R)$.

LEMMA 5 R is homeomorphic to $\tilde{R}$ in R^* .

Proof. The topology in R is defined by the following: a directed sequence

$\{p_i\}$ in R converges to p if and only if $f(p_i)$ converges to $f(p)$ for every f in $A^m(R)$. From this and the above remark follows Lemma 5.

LEMMA 6. R^* is a compact Hausdorff space and $\tilde{R}$ is a dense open subset of R^* .

Proof. R^* is in the unit sphere of $A^m(R)^*$ and clearly closed in $A^m(R)^*$ with respect to the weak topology as functionals. Hence, by the Tychonoff-Kakutani theorem, R^* is a compact subset of the Hausdorff space $A^m(R)^*$ with respect to the weak topology as functionals. Thus the topology introduced in R^* makes R^* a compact Hausdorff space.

Next we consider the function f^* on R^* defined by $f^*(\chi) = \chi(f)$. Then f^* is continuous on R^* by the definition of the topology of R^* . Let A be the totality $\{f^*\}$ of f^* . By the correspondence $f \rightarrow f^*$, $A^m(R)$ and A is isomorphic as algebras. Now it is clear that A separates points on R^* , i. e., there exists a function f^* in A such that $f^*(\chi) \neq f^*(\chi')$ if $\chi \neq \chi'$ in R^* . Together with the fact that 1 is contained in A , the Stone-Weierstrass theorem tells that A is dense in the totality $C(R^*)$ of continuous functions on R^* with respect to the uniform convergence.

If R is not dense in R^* , there exists a continuous function g on R^* such that $g \equiv 0$ on R and $g \neq 0$ on R^* . Let f_n^* in A be a sequence approximating g uniformly on R^* . We may assume $|f_n^*(\chi_p)| \leq 1/n$ on R . Then $f_n^*(\chi_p) = f_n(p)$ shows that $\sup |f_n| \leq 1/n$ on R and from (8) we conclude that $|f_n^*| \leq 1/n$ on R^* and so $0 = \lim_n f_n^*(\chi) = g(\chi)$, which is a contradiction. Hence $\tilde{R}$ is dense in R^* .

To see that $\tilde{R}$ is open in R^* , we observe that $\chi \in R^* - \tilde{R}$ if and only if $f^*(\chi) = 0$ for all f in $A_0^m(R)$. (If this can be shown, then $R^* - \tilde{R}$ is a closed set as it is the intersection of zero sets of continuous functions. Hence $\tilde{R}$ is open.) The "if part" of the above assertion is clear. So we have only to prove $\chi(f) = 0$ if $\chi \in R^* - \tilde{R}$ and $f \in A_0^m(R)$. Suppose that there exists f_0 in $A_0^m(R)$ such that $\chi(f_0) \neq 0$. We may assume $f_0 \geq 0$. Let $S = \{p \in R; f_0(p) > 0\}$ and $Z = \{f \in A_0^m(R), \chi(f) = 0, \{p \in R; f(p) \neq 0\} \subset \text{CL}. S\}$. For each point p in S we can find f_p in Z such that $f_p(p) \neq 0$. If this is not the case, $(g - \chi(g))f_0$ is in Z for any g in $A^m(R)$ and so $(g(p) - \chi(g))f_0(p) = 0$. Thus $\chi(g) = g(p)$ and hence $\chi = \chi_p \in R$, which is a contradiction. Thus we can find $f_p(p) \neq 0$ in Z . Clearly we may assume $f_p \geq 0$ and $f_p(p) = 1$. Let $e_0 = (f_0 - \chi(f_0))^2 / \chi(f_0)^2$. Then $\chi(e_0) = 0$ and $e_0 \geq 0$. Moreover $e_0(p) \geq 1/2$ on $R - S'$, where S' is a compact set in S . We can find a finite number of points p_i in S' and f_{p_i} in Z such that $\sum_i f_{p_i} \geq 1/2$ on S' . Then $e = e_0 + \sum_i f_{p_i} \geq 1/2$ and $\chi(e) \geq 1/2$ and on the other hand $\chi(e) = \chi(e_0) + \sum_i \chi(f_{p_i}) = 0$. Thus we arrive at a contradiction. Therefore $\tilde{R}$ is open in R^* . This completes the proof of Lemma 6.

Thus we may identify $\tilde{R}$ with R . Then R^* is a compact topological continuation of R and R is an open dense subset of R^* . We shall refer R^* to the compactification of R with respect to $A^m(R)$.⁴⁾

3. Proof of Theorem.

Let T be an isometry of R onto $'R$. Then $f \rightarrow f \circ T^{-1}$ is clearly an isometric isomorphism of $A^m(R)$ onto $A^m('R)$. We shall prove the converse of this. Let

4) Needless to say, $R = R^*$ if R is compact.

$f \rightarrow f^\sigma$ be an isometric isomorphism of $A^m(R)$ onto $A^m('R)$. We take the compactifications R^* and $'R^*$ of R and $'R$ with respect to $A^m(R)$ and $A^m('R)$ respectively. For a point (i. e., a character) χ in R^* , we define $T\chi$ by $T\chi(f^\sigma) = \chi(f)$ for all f in $A^m(R)$. Then it is easily seen that $Tx \in 'R^*$ and we may write

$$(9) \quad f^\sigma(T\chi) = f(\chi).$$

Then $\chi \rightarrow T\chi$ gives a homeomorphic correspondence of R^* onto $'R^*$. We shall prove that if p_0 is in R then Tp_0 is in $'R$. Contrary to the assertion assume that $Tp_0 = \chi_0$ is in $'R^* - 'R$.

Let U be an open neighborhood of p_0 with compact closure in R . Then $'U = TU$ is an open neighborhood of χ_0 in $'R^*$. Let $'V = 'U \cap 'R$ and $V = T^{-1}'V$. Then V and $'V$ are open subsets of R and $'R$ respectively and V is dense in U because $'R$ is dense in $'R^*$ and so $'V$ is dense in $'U$.

Now we choose an admissible coordinate system $x = (x^1, x^2, \dots, x^n)$ with origin at p_0 . Let A_1 and A_2 ($A_1 \subset \text{CL}.A_1 \subset A_2$) be a neighborhood of p_0 contained with its closure both in the coordinate neighborhood W of x and U . We consider the function g on W defined by

$$g(x) = \left(\sum_{i=1}^n (x^i)^2 \right) \sin \left(\sum_{i=1}^n (x^i)^2 \right)^{1/2}.$$

Then g is a C^∞ -function in $W - p_0$ and its first order derivatives are all bounded by 2 in absolute value sufficiently near p_0 but not continuous at p_0 . Hence if we choose φ in $A^m(R)$ to be 1 on A_1 and 0 outside A_2 , then $f_{p_0} = g\varphi$ is in $A^m(R - p_0)$ but not in $A^m(R)$, i. e., f_{p_0} cannot be continued to p_0 so as to be in $A^m(R)$. We observe that f_{p_0} vanishes on a neighborhood of the boundary of U relative to R .

If f is in $A_0^m(V) (\subset A^m(R))$, then (9) shows that f^σ is in $A_0^m('V)$ and $f^\sigma = f \circ T^{-1}$ on $'V$ and conversely if f^σ is in $A_0^m('V) (\subset A^m('R))$ then f is in $A_0^m(V)$ and $f = f^\sigma \circ T$. Hence for f in $A_0^m(V)$, $\|f\| = \|f^\sigma \circ T^{-1}\|$ and so T induces an isometric isomorphism of $A_0^m(V)$ onto $A_0^m('V)$. Thus by Lemma 4, T is an isometry of V onto $'V$.

As f_{p_0} is in $A^m(R - p_0)$ and as p_0 does not belong to V , f_{p_0} belongs to $A^m(V)$. Consider the function $F = f_{p_0} \circ T^{-1}$ on $'V$. Let $x = x(y)$ be the local representation of T^{-1} . Clearly

$$\sup_V |F| \leq \sup_{R - p_0} |f_{p_0}| < \infty.$$

We also have

$$\frac{\partial F}{\partial y^i} = \frac{\partial f_{p_0}}{\partial x^k} \frac{\partial x^k}{\partial y^i}$$

and by the isometricity of T^{-1} we get

$$'g^{ij} \frac{\partial F}{\partial y^i} \frac{\partial F}{\partial y^j} = 'g^{ij} \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \frac{\partial f_{p_0}}{\partial x^k} \frac{\partial f_{p_0}}{\partial x^l} = g^{kl} \frac{f_{p_0}}{x^k} \frac{f_{p_0}}{x^l}$$

Hence $\|F\|_V \leq \|f_{p_0}\|_{R - p_0}$.

Next we show that F vanishes on a neighborhood of the boundary of $'V$ relative to $'R$. To see this, take a point p in the relative boundary of $'V$ with respect to $'R$ and consider $T^{-1}p$. Then there occurs two cases:

- (i) $T^{-1}p$ is in $R^* - R$. In this case we can find a neighborhood G of p such that $T^{-1}G \cap U = \emptyset$ since $\text{CL}.U$ is compact in R . Then $F(q) = f_{p_0}(T^{-1}(q)) = 0$ if $q \in G$.
- (ii) $T^{-1}p$ is in R . In this case, $T^{-1}p$ cannot be in U , since if $T^{-1}p \subset U$, then

$p = T(T^{-1}p) \in 'U \cap 'R = 'V$, which is a contradiction. Therefore if we choose a sufficiently small neighborhood G of Tp , then $T^{-1}G$ is disjoint from the carrier of f_{p_0} and hence $F(q) = 0$ if q is in G .

In any case F vanishes on a neighborhood of each point of the relative boundary of $'V$ with respect to $'R$. Thus we see that if we extend F to the whole $'R$ by defining $F = 0$ outside $'V$ in $'R$, then F belongs to $A^m('R)$.

Hence we can find an element f in $A^m(R)$ such that $f^* = F$. By (9) we may write on $'R^*$ and *a fortiori* on $'V$

$$f(T^{-1}(p)) = F(p) \quad (p \in 'V).$$

On the other hand, $F(p) = f_{p_0}(T^{-1}(p))$. So we get $f(T^{-1}(p)) = f_{p_0}(T^{-1}(p))$ on $'V$ or

$$f(q) = f_{p_0}(q) \quad \text{on } V.$$

Both f and f_{p_0} are continuous on $R - p_0$ and V is dense in U , so we get

$$f_{p_0}(q) = f(q) \quad \text{on } U - p_0.$$

This shows that if we define $f_{p_0}(p_0) = f(p_0)$ then f_{p_0} is in $A^m(R)$ which contradicts the assumption on f_{p_0} . Thus we have proved that $TR \subset 'R$. The similar discussion implies $T^{-1}'R \subset R$ and therefore we can conclude that $TR = 'R$.

Thus we see that σ is induced by a homeomorphism T of R onto $'R$ and of course T induces an isometric isomorphism of $A_0^m(R)$ onto $A_0^m('R)$. Thus by Lemma 4 T is an isometry.

This completes the proof of the theorem.

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