Cluster sets of pseudo-analytic functions

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(Received August 22, 1959)

1. Let $w=T(z)$ be a quasiconformal mapping, with bounded dilatation, of $|z|<1$ onto $|w|<1$. Then, $w=T(z)$ can be extended to a topological mapping of the closed disc $|z|\leq 1$ onto $|w|\leq 1$ (cf. Ahlfors [2], Mori [1]). It had been an open question *whether the extended T is necessarily absolutely continuous on $|z|=1$* . This question has been settled by Beurling-Ahlfors [1]: The answer is in the *negative*. This is a striking result, from the view-point of the theory of cluster sets. It is well-known that a large part of this theory can be discussed systematically by applying the theory of functions of class (U) in Seidel's sense [1] (see, for example, Tsuji [2]). However, by the result of Beurling-Ahlfors, it is easily seen that we cannot extend the theory of functions of class (U) to the case of pseudo-analytic functions without any additional condition.

The main purpose of this paper is to give some supplements to the writer's earlier unified account on cluster sets (Noshiro [5]), by discussing that to what extent important results on cluster sets of analytic functions can be extended to the case of pseudo-analytic functions, with the aid of some new methods.¹⁾

2. For the sake of completeness, we state some definitions on quasiconformality, and pseudo-analyticity, and some basic known results.²⁾ Let $w=T(z)=u(x, y)+iv(x, y)$ be a topological mapping of a domain D in the z -plane onto a domain Δ in the w -plane. Suppose that $w=T(z)$ belongs to class C^1 and its Jacobian $J(z)=u_x v_y - u_y v_x$ is positive at every point z in D . Then, any infinitesimal circle with center z and of radius ε is transformed into an infinitesimal ellipse with center $w=T(z)$ and of the major axis $a\varepsilon$ and the minor axis $b\varepsilon$. If the dilatation $d(z)=a/b$ is bounded in D :

$$(1) \quad d(z) = \frac{a}{b} \leq K,$$

where K is a constant ≥ 1 , then $w=T(z)$ is called a *differentiable* quasiconformal mapping of parameter K . The condition (1) is equivalent to each of the following conditions

$$(\alpha) \quad \max_{0 \leq \theta \leq 2\pi} |w_x \cos \theta + w_y \sin \theta|^2 = \max_{\theta} \frac{|dw|^2}{|dz|^2} \leq KJ(z);$$

$$(\beta) \quad u_x^2 + u_y^2 + v_x^2 + v_y^2 \leq (K + K^{-1})J(z);$$

$$(\gamma) \quad \frac{|w_x + iw_y|}{|w_x - iw_y|} \leq \frac{K-1}{K+1}$$

(cf. Bers [1]).

1) The writer is interested in methods introduced in this paper and does not claim that this paper contains some new results.

2) As to the recent development of the theory of quasiconformal mappings, see Künzi[1].

Let Q be a simply connected domain in the z -plane bounded by a Jordan curve ∂Q and let z_1, z_2, z_3, z_4 be four distinct boundary points of Q , which lie in this order on the positively oriented boundary curve. We call such a configuration a *quadrilateral* and denote it by $Q(z_1, z_2, z_3, z_4)$. Map the domain Q conformally onto a rectangle: $0 < \xi < M, 0 < \eta < 1$, in the ζ -plane ($\zeta = \xi + i\eta$), in such a manner that z_1, z_2, z_3, z_4 correspond to the vertices $\zeta = 0, M, M+i, i$ respectively. We call the positive number M the *modulus* (or *module*) of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ or simply $\text{mod } Q$. The following geometric definition is due to Pfluger [2], Ahlfors [2] and Mori [1]. A sense-preserving topological mapping $w = T(z)$ of a plane domain D onto another such domain Δ is called *K-quasiconformal* or, simply, a *KQC mapping*, provided that for any quadrilateral Q contained in D together with its boundary,

$$(2) \quad \text{mod } T(Q) \leq K \text{ mod } Q,$$

where $T(Q)$ is the image of Q and K is a constant ≥ 1 . It is a simple matter to show that if $w = T(z)$ is *KQC*, then its inverse is also *KQC* and that the composition of a *KQC* mapping and a conformal mapping is a *KQC* mapping. Starting from the geometric definition of quasiconformality in the sense of Pfluger-Ahlfors-Mori, we can deduce some important analytic properties.

THEOREM 1. *Let $w = T(z) = u(x, y) + iv(x, y)$ be a KQC mapping (in the sense of Pfluger-Ahlfors-Mori) of a plane domain D in the z -plane ($z = x + iy$) onto a domain Δ in the w -plane ($w = u + iv$). Then*

- (i) $w = T(z)$ is totally differentiable almost everywhere in D (Mori [1]);
- (ii) for each point z , at which $w = T(z)$ is totally differentiable,

$$\max_{\theta} \frac{|dw|^2}{|dz|^2} \leq KJ(z)$$

(Mori [1]);

- (iii) for almost every $y = y_0$, $w = T(x, y_0)$ is absolutely continuous in x on any closed interval in the intersection of $y = y_0$ and D (Mori [1], Strebel [1], Pfluger [3]);

- (iv) the partial derivatives of $w = T(z)$ are locally square integrable in D (Bers [1]);

- (v) $w = T(z)$ is a measurable mapping; more precisely, for every measurable set $e \subset D$, the image $T(e)$ is measurable and has measure

$$\iint_e J(z) dx dy \quad (\text{Morrey [1], Jenkins [1]).}^{3)}$$

In virtue of these properties (i)–(v), we can extend almost all known results on differentiable quasiconformal mappings to the class of *KQC* mappings (in the sense of Pfluger-Ahlfors-Mori), by simple recapitulation of the original proofs. For example, we can prove: *Let $w = T(z)$ be a topological mapping of a plane domain D onto another such domain. If $w = T(z)$ is KQC in a neighborhood of each point of D , then it is KQC in D (Ahlfors [1], Mori [1]).*

Let F be an open Riemann surface and $w = f(p)$ be a single-valued complex-valued function defined on F . If $w = f(p)$ is an interior transformation on F in the sense of Stoilow [3], then all the elements $q = [p, f(p)]$ form a covering surface

3) Ahlfors gave a simple proof for (v) in his lecture at Osaka University in 1955.

Φ of the w -plane. We call $w=f(p)$ a K -pseudo-analytic function or simply a KPA function on F , provided that either $w=f(p)$ is identically equal to a constant or the mapping $\varphi: p \rightarrow q=[p, f(p)]$ is a KQC mapping of F onto Φ . Let Ψ be another Riemann surface conformally equivalent to Φ and let $q=\phi(r)$ be a conformal mapping of Ψ onto Φ . Then, the composed mapping $T: p \rightarrow r$ is a KQC mapping of F onto Ψ . Thus we obtain the following representation

$$(3) \quad w=f(p) \equiv \phi[T(p)],$$

where $w=\phi(r) \equiv f(\varphi^{-1}(\phi(r)))$ is a single-valued analytic function on Ψ .

3. *Cluster sets of pseudo-analytic functions.*⁴⁾ At the beginnig, we give a proof for Ohtsuka's extension [2] of Nevanlinna's theorem [1].

THEOREM 2. Let E be a compact set of logarithmic capacity zero, D_1 a domain containing E completely in its interior and $D=D_1-E$. Suppose that $w=f(z)$ is a KPA function in D . Then, the cluster set $C_D(f, z_0)$ at each point z_0 of E is either a single point or the whole w -plane.

Proof. Without loss of generality, we may assume that D_1 is the unit disc: $|z|<1$. Let Φ be the covering surface generated by $w=f(z)$ on D , of the w -plane. Then, by the uniformization theory, we can map Φ conformally onto a plane domain $\mathcal{A}$ in the ζ -plane. We may assume that the boundary component of $\mathcal{A}$, which corresponds to $|z|=1$, is $|\zeta|=1$ and $\mathcal{A}$ is contained in the unit disc $|\zeta|<1$. Denote by $w=\phi(\zeta)$ the function which maps $\mathcal{A}$ conformally onto Φ . Then, by (3), we have $w=f(z)=\phi[T(z)]$ where $\zeta=T(z)$ is a KQC mapping of D onto $\mathcal{A}$ such that the boundary component $|z|=1$ of D corresponds to $|\zeta|=1$. By reflections with respect to $|z|=1$ and $|\zeta|=1$, we can extend T to be a KQC mapping of the double $\hat{D}$ onto the double $\hat{\mathcal{A}}$ by Ahlfors' theorem⁵⁾. Since $\hat{D}$ is of parabolic type, $\hat{\mathcal{A}}$ is also of parabolic type by Pfluger's theorem [1]. Hence the complement E_ζ of $\mathcal{A}$ with respect to $|\zeta|<1$ is a compact set of (logarithmic) capacity zero. Obviously, we can continue $w=T(z)$ to be a topological mapping of $|z|<1$ onto $|\zeta|<1$. By Nevanlinna's theorem [1], the cluster set $C_{\mathcal{A}}(\phi, \zeta_0)$ at each point ζ_0 of E_ζ is either a single point or the whole w -plane. From this fact, our assertion follows immediately.

This theorem shows the possibility of extending the results on cluster sets of single-valued analytic functions to the case of pseudo-analytic functions.

Let G be a circular disc: $|w-w_0|<\rho$ and e_w be a totally disconnected compact set of positive capacity in the interior of G . Let ω be the harmonic measure in $G-e_w$ with boundary values 0 on $|w-w_0|=\rho$ and 1 on e_w , $\tilde{\omega}$ its conjugate and

$$\int_{|w-w_0|=\rho} d\tilde{\omega} = d.$$

We consider

4) As to the definitions of cluster sets and range of values, see Noshiro [2]. In the present paper, we adopt new notations $C_D(f, z_0)$, $C_{D-E}(f, z_0)$, $R_D(f, z_0)$ instead of

$$S_{z_0}^{(D)}, \quad S_{z_0}^{*(D)}, \quad R_{z_0}^{(D)}.$$

5) Cf. Ahlfors [2]; Theorem 4, p. 9.

$$(4) \quad \zeta(w) = \exp \left\{ -\frac{2\pi}{d}(\omega + i\bar{\omega}) \right\}^{6)}$$

in the domain $G_1 = G - e_w$ and introduce the conformal metric

$$(5) \quad ds_w = |\zeta'(w)| |dw|$$

in G_1 .

LEMMA. Let F be a finite covering surface of the basic surface $G_1 = G - e_w$. Then, using the conformal metric (5), one has

$$(6) \quad A \leq hL,$$

where A is the area of F , L the total length of the relative boundary β of F and h is a positive constant independent of F (T. Yosida [1]).

Proof. By reflection with respect to the circle $|w - w_0| = \rho$, we construct the double $\hat{F}$ of F above the double $\hat{G}_1$ of $G_1 = G - e_w$. Let $\partial\hat{F}$ denote the boundary of $\hat{F}$. We denote by A_s and L_s the spherical area of F and the spherical length of β with respect to the metric (5). Then, by a well-known formula,

$$\begin{aligned} 2A_s &= \iint_{\hat{F}} \frac{|\zeta'(w)|^2}{(1 + |\zeta(w)|^2)^2} du dv, \quad (w = u + iv), \\ &\leq \frac{1}{2} \int_{\partial\hat{F}} \frac{|\zeta(w)|}{1 + |\zeta(w)|^2} |\zeta'(w)| |dw| \\ &\leq \frac{h_1}{2} \int_{\partial\hat{F}} \frac{|\zeta'(w)|}{1 + |\zeta(w)|^2} |dw| = 2 \cdot \frac{h_1}{2} \int_{\beta} \frac{|\zeta'(w)| |dw|}{1 + |\zeta(w)|^2} \\ &= h_1 L_s, \end{aligned}$$

since $|\zeta(w)|$ is bounded: $|\zeta(w)| \leq h_1$ in $\hat{G}_1$ for a certain positive number h_1 ; whence follows that $A_s \leq (h_1/2)L_s$ and so $A \leq hL$ where $h = 2h_1$.

THEOREM 3. Let D be an arbitrary domain, Γ its boundary, E a compact set of capacity zero on Γ and z_0 a point of E . Suppose that $w = f(z)$ is a KPA function in D . Then

(i) if $\alpha \in C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is an exceptional value of $w = f(z)$ in a neighborhood of z_0 , then either α is an asymptotic value of $f(z)$ at z_0 or there is a sequence of accessible boundary points $z_n \in E$ ($n = 1, 2, \dots$) converging to z_0 such that α is an asymptotic value of $f(z)$ at each point z_n (Noshiro [3]);

(ii) if z_0 is an accumulation point of $\Gamma - E$, i.e. $z_0 \in (\overline{\Gamma - E})$, then $\Omega = C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is an open set (Yosida [1], Ohtsuka [1]);

(iii) if $z_0 \in (\overline{\Gamma - E})$ and if Ω is not empty, then $\Omega - R_D(f, z_0)$ is at most of capacity zero (Yosida [1], Ohtsuka [1]);

(iv) if D is a simply connected domain of hyperbolic type, and if Ω is not

6) Any branch $w = w(\zeta)$ of the inverse of $\zeta = \zeta(w)$ is single-valued and regular on $|\zeta| = 1$. Continue $w = w(\zeta)$ analytically, with rational characters, along every radius starting from the point on $|\zeta| = 1$ toward $\zeta = 0$. Then, for almost every ϑ , the continuation along the radial segment S_ϑ : $\zeta = |\zeta|e^{i\vartheta}$, $e^{-2\pi/d} < |\zeta| < 1$ is possible. Thus, $w = w(\zeta)$ defines a single-valued univalent analytic function $w = \phi(\zeta)$ in the annulus $e^{-2\pi/d} < |\zeta| < 1$ with radial slits whose arguments ϑ in the interval $(0, 2\pi)$ form a set of linear measure zero. This analogue of BreLOT-Choquet's theorem [1] can be proved directly, since the irregular points for the Dirichlet problem form a set of type F_σ of capacity zero.

empty, and further if $f(z)$ is bounded in the intersection of D and some neighborhood of z_0 , then $w=f(z)$ takes every value, with one possible exception, belonging to each component Ω_n of Ω , infinitely often in any neighborhood of z_0 ;

(v) if each point of E belongs to a boundary component of D which consists of a non-degenerate continuum and if Ω is not empty, then $w=f(z)$ assumes every value with two possible exceptions, belonging to each component Ω_n of Ω , infinitely often in any neighborhood of z_0 .⁷⁾

Proof. First we note analytic properties of KQC mappings in Theorem 1 and the fact that Ahlfors' theory of covering surfaces [1] is also useful in the case of KPA functions. We can prove (i), (iv), (v) by simple recapitulation of the proofs of corresponding theorems in the case of analytic functions. To prove (ii), we follow the entirely similar argument used in the proof of Tsuji's theorem (see, Noshiro [5]; Theorem 4) and arrive at the stage to consider the case in which $w_0 \in C_{\Gamma-E}(f, z_0)$ is an asymptotic value of $w=f(z)$ along a path A in D terminating at z_0 . For a sufficiently small positive number ρ , we denote by A the component of the counter-image $f^{-1}(G)$ of $G: |w-w_0| < \rho$ which contains the last part of the path A . We may assume that the boundary γ of A consists of a compact subset e of E and at most countably many simple curves (boundary relative to D). The Riemann covering surface Φ_A , generated by $w=f(z)$ on A , of the basic surface G has Gross' property and so Iversen's property. Hence, by Stoilow's principle (see, Stoilow [2–5]), either Φ_A covers every point of G the same finite number of times except for a totally disconnected (relatively) closed set which consists of the asymptotic values of $f(z)$ at some points of e ; or the cluster set $C_A(f)$ (in the large) of $f(z)$ is $\bar{G}$.⁸⁾ However, since $z_0 \in \overline{(I-E)}$, $C_A(f, z_0)$ contains a continuum which connects w_0 to a point of $|w-w_0|=\rho$. Consequently, $C_A(f)=\bar{G}$ and Φ_A has an infinite area, by Gross' property. We shall show that $C_A(f, z_0)$ is also $\bar{G}$. For this purpose, it suffices to prove that $G-R_A(f, z_0)$ is of capacity zero. Otherwise, there would exist a positive number r_0 , such that there is no point of e on $|z-z_0|=r_0$, and a totally disconnected compact set e_w of positive capacity lying completely in G such that $w_0 \notin e_w$ and the set of values of $w=f(z)$ in the intersection A_{r_0} of A with the disc $|z-z_0| < r_0$ contains no points of e_w . The open set A_{r_0} consists of a finite number of connected domains. Hence, the Riemannian image Φ_{r_0} of A_{r_0} by $w=f(z)$ consists of the same number of open covering surfaces of the basic surface $G_1=G-e_w$. Introducing the conformal metric (5) in G_1 , we see that the area of Φ_{r_0} is infinite, since the part of Φ_{r_0} above a small disc $|w-w_0| < \rho_1$ has an infinite area, by the fact just stated above. Now, with the aid of the level curve $\Gamma_\lambda: \rho(z)=\text{const.}=\lambda$ ($0 < \lambda < \infty$) of an Evans-Selberg function $\zeta=\chi(z)=\rho(z)e^{i\psi(z)}$ (see Noshiro [2]) associated with the set e , we denote by $A(\lambda)$ the common part of A_{r_0} and the domain exterior to Γ_λ , by $A(\lambda)$ the area of the Riemannian image, by $w=f(z)$, of $A(\lambda)$ and by $L(\lambda)$ the total length of the image of the intersection of Γ_λ and A_{r_0} with respect to the metric (5). Then, Φ_{r_0} is regularly exhaustible in the sense of Ahlfors; i. e.

7) (i), (ii), (iii), (iv), (v) are extensions of results of Noshiro [1, 2], Tsuji [1], Hervé [1]. As to further extensions of (ii) and (iii), see Ohtsuka [2].

8) $\alpha \in C_A(f)$ if there is a sequence of points $\{z_n\}$ in A , which converges to the boundary γ of A , such that $\lim_{n \rightarrow \infty} f(z_n) = \alpha$.

$$\lim_{\lambda \rightarrow \infty} L(\lambda)/A(\lambda) = 0$$

(cf. Noshiro [2]). On the other hand, by the preceding lemma, $A(\lambda) \leq h(L(\lambda) + L_0)$, where L_0 denotes the total length of the image of the intersection of $|z - z_0| = r_0$ and Δ ; hence

$$\lim_{\lambda \rightarrow \infty} L(\lambda)/A(\lambda) \geq 1/h > 0.$$

Contradiction.

To prove (iii), we have only to note the fact that if Δ is a connected component of the counter-image of any circular disc $G: |w - w_0| < \rho$ on the w -plane such that the boundary of Δ consists of some simple curves (boundary relative to D) and a compact subset e of E , then the Riemannian image Φ_Δ of Δ covers every point of G except for a (relatively) closed set of capacity zero. Since we cannot use the theory of functions of class (U) in Seidel's sense in the case of KPA functions, we prove this fact in the following way. Assume, contrary to the assertion, that Φ_Δ does not cover any point of a totally disconnected compact set e_w of positive capacity in $G: |w - w_0| < \rho$. We now denote by $\Delta(\lambda)$ the part of Δ contained in the exterior of the level curve Γ_λ (in the proof of (ii)). We note that Φ_Δ is regularly exhaustible in Ahlfors' sense: $\lim_{\lambda \rightarrow \infty} L(\lambda)/A(\lambda) = 0$, even when Φ_Δ is of finite area with respect to the metric (5).⁹⁾ On the other hand, $A(\lambda) \leq h L(\lambda)$ in the present case for (almost) all λ ($\lambda_0 \leq \lambda < \infty$, λ_0 being a fixed positive number) by the preceding lemma. Contradiction.

REMARK. Using an idea of Ohtsuka [1], we give an alternative proof for (iii). Assume that Φ_Δ does not cover any point of a totally disconnected compact set e_w of positive capacity in $G: |w - w_0| < \rho$. We consider the function

$$(4) \quad \zeta(w) = \exp \left\{ -\frac{2\pi}{d}(w + i\tilde{w}) \right\}.$$

Then, the inverse function $w = w(\zeta)$ defines a single-valued univalent analytic function $w = \psi(\zeta)$ in the annulus $A_\varepsilon: e^{-2\pi/d} < |\zeta| < 1$ (cf. foot-note⁶⁾, p. 86) with radial slits whose arguments ϑ in the interval $(0, 2\pi)$ form a set of measure zero. Let G' be the image of A_ε in G by $w = \psi(\zeta)$ and consider the part $\Phi_{\Delta'}$ of Φ_Δ above G' . We denote by Δ' the counter-image of $\Phi_{\Delta'}$ by $w = f(z)$. We can now consider the KPA function $\zeta = \zeta(z) = \psi^{-1}(f(z))$ in the open set Δ' . The Riemannian image $\Psi_{\Delta'}$ of Δ' by $\zeta = \zeta(z)$ consists of at most countably many covering surfaces of the basic surface A_ε . We may assume, without loss of generality, that a quadrilateral $Q_\varepsilon: 1 - \varepsilon < |\zeta| < 1, \vartheta_1 < \arg \zeta < \vartheta_2$ ($\varepsilon > 0$) is contained in $\Psi_{\Delta'}$. Then, $\Psi_{\Delta'}$ contains a quadrilateral $Q: e^{-2\pi/d} < |\zeta| < 1, \vartheta_1 < \arg \zeta < \vartheta_2$ with radial slits whose arguments form a set of measure zero by an extension of Gross' star theorem.¹⁰⁾ Consequently, Q contains the radial segment: $\zeta = |\zeta|e^{i\vartheta}, e^{-2\pi/d} < |\zeta| < 1$ for almost every ϑ in the interval $(\vartheta_1, \vartheta_2)$. On the other hand, the image of the segment by the KQC mapping $z = z(\zeta)$ of Q is an asymptotic path in Δ' starting from a point of the relative boundary of Δ and terminating at some point of e of capacity zero. This is a contradiction.

4. Beurling and Ahlfors [1] have proved that for each $K > 1$, there exists a

9) As to a related result, see Noshiro [4].

10) Cf. Kaplan [1].

KQC mapping of $|z| < 1$ onto $|\zeta| < 1$ such that a certain set E_z of linear measure zero on $\Gamma_z: |z|=1$ is mapped onto a set E_ζ of linear measure 2π on $\Gamma_\zeta: |\zeta|=1$.¹¹⁾ We now construct an example, applying the result of Beurling and Ahlfors. Let $\zeta = T(z)$ be a KQC mapping ($K > 1$) of $|z| < 1$ onto $|\zeta| < 1$ such that a certain compact set E_z of measure zero on $\Gamma_z: |z|=1$ is mapped onto a compact set E_ζ of positive measure on $\Gamma_\zeta: |\zeta|=1$. We may assume that E_z and E_ζ are totally disconnected perfect sets. First, we construct the harmonic measure $u(\zeta)$ of E_ζ with respect to the unit disc $|\zeta| < 1$. Then, $u(\zeta)$ is continuous and equal to 0 on the (relatively open) set $\Gamma_\zeta - E_\zeta$. Further, the angular limit $u(e^{i\theta})$ is 1 at almost every point $e^{i\theta} \in E_\zeta$. We consider the function $w = \varphi(\zeta) = 1/\exp[u(\zeta) + iv(\zeta)]$ in $|\zeta| < 1$, where $v(\zeta)$ is the conjugate of $u(\zeta)$. Then, by the monodromy theorem, $w = \varphi(\zeta)$ is single-valued, analytic and bounded: $e^{-1} < |\varphi(\zeta)| < 1$ in $|\zeta| < 1$. Evidently, $w = \varphi(\zeta)$ is continuous and $|\varphi(\zeta)| = 1$ on the (open) set $\Gamma_\zeta - E_\zeta$. Without loss of generality, we may assume that every point of E_ζ is a singularity of $w = \varphi(\zeta)$. By Fatou's theorem, for almost every $e^{i\theta} \in E_\zeta$ the angular limit $\varphi(e^{i\theta})$ exists and has modulus e^{-1} . It is evident that the set E_ζ^1 of all the points $e^{i\theta} \in E_\zeta$, for which $|\varphi(e^{i\theta})| = e^{-1}$, is everywhere dense on E_ζ . We now consider the composed function $w = f(z) \equiv \varphi[T(z)]$ in the disc $|z| < 1$. Then, $w = f(z)$ is a KPA function which satisfies: (i) $f(z)$ is continuous and $|f(z)| = 1$ on $\Gamma_z - E_z$; (ii) $f(z)$ is discontinuous at every point $e^{i\theta}$ of E_z , since the counter-image $T^{-1}(E_\zeta^1) = E_z^1$ is everywhere dense on E_z ; (iii) for every $e^{i\theta} \in E_z^1$, the angular limit $f(e^{i\theta})$ is of modulus e^{-1} (cf. Mori [1]; Theorem 6).¹²⁾

REMARK. The function $w = f(z) \equiv \varphi[T(z)]$ is regarded as a K -pseudo-analytic function of class (U) in Seidel's sense, but the set of values of $w = f(z)$ does not cover any point of the disc $|w| < 1/e$. Accordingly, we cannot extend the theory of functions of class (U) to the case of pseudo-analytic functions without any additional condition.¹³⁾ Almost all results on cluster sets which are related to a compact set of capacity zero can be extended to the case of KPA functions. On the contrary, it is impossible to extend the results based on the theory of functions of class (U) , the theory of meromorphic functions of bounded type and Lusin-Privaloff's theorem [1] to the case of KPA functions without any additional condition.¹⁴⁾ However, it is easy to see that we can extend the theory of normal meromorphic functions introduced by Lehto-Virtanen [1] to the case of KPA functions by Mori's theorems [1] on normal families of KPA functions and on the angular limits of KQC mappings between the unit discs.

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11) By Mori [1], we see that the set E_z is of positive capacity.

12) By the reflection principle, $w = f(z)$ can be extended to be a KPA function in the complement D_z of E_z with respect to the z -plane and the Riemannian image of D_z by the extended KPA function $w = f(z)$ is a covering surface Φ_w of the annulus $e^{-1} < |w| < e$. We note that $w = f(z)$ is discontinuous at every point $e^{i\theta}$ of E_z . This example is of some interest in contrast with L. Myrberg's example [1].

13) Cf. Storvick [1], Lohwater [1, 3].

14) This fact has been also pointed out by Jenkins [2].

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