On equilibrium potential of m -dimensional sphere

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1. Introduction.

The equilibrium potential of the order α of m -dimensional sphere S^* whose center is at the origin of the co-ordinates and which is with radius R , was given by M. Riesz [4] as follows:

$$(I) \quad u(P) = \pi^{-(m/2+1)} \Gamma\left(\frac{m}{2}\right) \sin \frac{\alpha\pi}{2} \int_{S^*} (R^2 - r_Q^2)^{-\alpha/2} r_{PQ}^{\alpha-m} dQ,$$

where $r_Q = \overline{OQ}$, $r_{PQ} = \overline{PQ}$ and dQ denotes the volume element.

The function $u(P)$ has the property that $u(P)=1$ for $P \in S^*$ and $u(P) < 1$ for $P \notin S^*$.

The proof of (I) in the case $m=1, 2, 3$ is given by G. Pólya and G. Szegő [3]. M. Riesz used the formula (I) without proof. The author will give a concrete demonstration of (I) by using the method which was used by G. Pólya and G. Szegő. It is a remarkable fact that the proof of (I) when m is an even number is different from that of (I) when m is an odd number.

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2. Lemmas.

LEMMA 1. Let $v=1-2a \cos \theta + a^2$ ($0 \leq a < 1$) and let n and p be both positive integers, then

$$(1) \quad I_n^{(p)} = \int_0^\pi \frac{\cos p\theta}{v^n} d\theta = \frac{\pi a^p}{(1-a^2)^{2n-1}} \frac{(p+1)(p+2)\cdots(p+n-1)}{(n-1)!} \\ \times \left[1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{(n-p-1)(n-p-2)\cdots(n-p-k)}{(p+1)(p+2)\cdots(p+k)} a^{2k} \right].$$

In a special case when $p=0$ we have

$$(2) \quad I_n^{(0)} = \int_0^\pi \frac{d\theta}{v^n} = \frac{\pi}{(1-a^2)^{2n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} a^{2k}.$$

Concerning the proof of this lemma we can find it, for example, in [1] or [2].

LEMMA 2. Let it be $z = \rho e^{i\varphi}$, $\zeta = r e^{i\theta}$ and $\zeta' = r' e^{i\theta'}$, then by the transformation

$$(3) \quad \zeta = \frac{z + \zeta'}{1 + \bar{z}\zeta'}$$

we have

1) He gave an elegant proof of (I).

$$(4) \quad 4r^2\rho^2 \sin^2\Theta = \frac{4r'^2\rho^2(1-\rho^2)^2 \sin^2\Theta'}{|1+\bar{z}\bar{\zeta}'|^4},$$

where $\Theta = \theta - \varphi$ and $\Theta' = \theta' - \varphi$.

Proof. As $|z - \zeta|^2 = r^2 - 2r\rho \cos \Theta + \rho^2$, we have

$$(5) \quad \begin{aligned} 4r^2\rho^2 \sin^2\Theta &= 4r^2\rho^2 - (r^2 + \rho^2 - |z - \zeta|^2)^2 \\ &= -\{-(1-\rho^2) + (1-r^2)\}^2 + 2\{(1+\rho^2) - (1-r^2)\} \cdot |z - \zeta|^2 - |z - \zeta|^4. \end{aligned}$$

From (3) we have

$$1 - r^2 = \frac{(1-\rho^2)(1-r'^2)}{|1+\bar{z}\bar{\zeta}'|^2}, \quad |z - \zeta| = \frac{(1-\rho^2)r'}{|1+\bar{z}\bar{\zeta}'|}.$$

Therefore (5) becomes

$$(6) \quad \begin{aligned} 4r^2\rho^2 \sin^2\Theta &= -(1-\rho^2)^2 \left(1 - \frac{1-r'^2}{|1+\bar{z}\bar{\zeta}'|^2}\right)^2 \\ &\quad + 2\left\{1 + \rho^2 - \frac{(1-\rho^2)(1-r'^2)}{|1+\bar{z}\bar{\zeta}'|^2}\right\} \frac{(1-\rho^2)^2 r'^2}{|1+\bar{z}\bar{\zeta}'|^2} - \frac{(1-\rho^2)^4 r'^4}{|1+\bar{z}\bar{\zeta}'|^4} \\ &= (1-\rho^2)^2 \left[-1 + 2\frac{1+r'^2\rho^2}{|1+\bar{z}\bar{\zeta}'|^4} - \frac{(1-r'^2\rho^2)^2}{|1+\bar{z}\bar{\zeta}'|^4}\right]. \end{aligned}$$

On the other hand, we have

$$(7) \quad \begin{aligned} 4r'^2\rho^2 \sin^2\Theta' &= 4r'^2\rho^2 - (1+r'^2\rho^2)^2 + 2(1+r'^2\rho^2)|1+\bar{z}\bar{\zeta}'|^2 - |1+\bar{z}\bar{\zeta}'|^4 \\ &= -(1-r'^2\rho^2)^2 + 2(1+r'^2\rho^2)|1+\bar{z}\bar{\zeta}'|^2 - |1+\bar{z}\bar{\zeta}'|^4. \end{aligned}$$

From (6) and (7) we get easily

$$4r^2\rho^2 \sin^2\Theta = \frac{4r'^2\rho^2(1-\rho^2)^2 \sin^2\Theta'}{|1+\bar{z}\bar{\zeta}'|^4}.$$

LEMMA 3. Let m, k be integers and let $m \geq 3$. Then we have

$$(8) \quad \begin{aligned} \sum_{p=0}^{m-2} \sum_{q=0}^{m-2-p} (-1)^q \binom{m-2}{p} \binom{m-2-p}{q} (\alpha - 2m + 2 + 4p + 2q)(\alpha - 2m + 2 + 4p + 2q - 1) \cdots \\ \times (\alpha - 2m + 2 + 4p + 2q - (2m - 4) + 1 - k) \\ = \begin{cases} 2^{2m-4} (2m-4)! & \text{for } k=0, \\ 0 & \text{for } 1 \leq k \leq 2m-4. \end{cases} \end{aligned}$$

Proof. We consider

$$(9) \quad \begin{aligned} G(x) &= \sum_{p=0}^{m-2} \sum_{q=0}^{m-2-p} (-1)^q 2^q \binom{m-2}{p} \binom{m-2-p}{q} (1+x)^{\alpha-2m+2+4p+2q} \\ &= (1+x)^{\alpha-2m+2} [1 - 2(1+x)^2 + (1+x)^4]^{m-2} \\ &= 2^{2m-4} (1+x)^{\alpha-2m+2} \left(1 + \frac{x}{2}\right)^{2m-4} x^{2m-4}. \end{aligned}$$

When $k=0$, the left hand member of (8) is equal to $G^{(2m-4)}(0)$ and from (9) we obtain

$$(10) \quad G^{(2m-4)}(0) = 2^{2m-4} (2m-4)!.$$

In the same way as before, when $k \geq 1$, the left hand side member of (8) is equal to $G^{(2m-4-k)}(0)$ and this vanishes evidently.

LEMMA 4. Let m be an integer such that $m \geq 3$, then we have

$$(11) \quad \sum_{h=0}^{m-2} (-1)^h \frac{\binom{m-2}{h}}{2m-3-2h} = (-1)^{m-2} \frac{2^{m-2}(m-2)!}{(2m-3)(2m-5)\cdots 3 \cdot 1}.$$

Proof. If we put

$$(12) \quad H(x) = \sum_{h=0}^{m-2} (-1)^h \binom{m-2}{h} x^{2(m-2-h)} = (-1)^{m-2} (1-x^2)^{m-2},$$

then we have

$$(13) \quad \sum_{h=0}^{m-2} (-1)^h \binom{m-2}{h} \Big/ (2m-3-2h) = \int_0^1 H(x) dx.$$

And the last integral becomes

$$(14) \quad \begin{aligned} \int_0^1 H(x) dx &= \int_0^1 (-1)^{m-2} (1-x^2)^{m-2} dx = (-1)^{m-2} \frac{1}{2} B\left(\frac{1}{2}, m-1\right) \\ &= (-1)^{m-2} \frac{2^{m-2}(m-2)!}{(2m-3)(2m-5)\cdots 3 \cdot 1}. \end{aligned}$$

From (13) and (14) the equality (11) holds.

3. The case when the dimension is even.

Let S_m be the unit sphere in the m -dimensional space Ω_m . Then we define two integrals as follows:

$$(II) \quad I_{2m} \equiv \int_{S_{2m}} (1-r_Q^2)^{-\alpha/2} r_{PQ}^{\alpha-2m} dQ$$

and

$$(III) \quad I_{2m-1} \equiv \int_{S_{2m-1}} (1-r_Q^2)^{-\alpha/2} r_{PQ}^{\alpha-(2m-1)} dQ.$$

Now let $r_Q = r$; $r_{PQ} = u = (r^2 - 2r\rho \cos \theta + \rho^2)^{1/2}$ ($0 \leq \rho < 1$), where θ denotes the angle between the radius vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Further, by Bertrand's method²⁾, Jacobian of the transformation from rectangular co-ordinates $(x_1, x_2, x_3, \dots, x_m)$ to polar co-ordinates $(r, \theta, \theta_1, \dots, \theta_{m-2})$ is

$$J = \frac{\partial(x_1, x_2, x_3, \dots, x_m)}{\partial(r, \theta, \theta_1, \dots, \theta_{m-2})} = (-1)^{m-1} r^{m-1} \sin^{m-2} \theta \sin^{m-3} \theta_1 \cdots \sin \theta_{m-3}.$$

Then, in general, we have

$$(1) \quad \begin{aligned} I_n &= \int_{S_n} (1-r_Q^2)^{-\alpha/2} r_{PQ}^{\alpha-n} dQ \\ &= 4 \cdot 2^{n-3} \int_0^1 \int_0^\pi \int_0^{\pi/2} \cdots \int_0^{\pi/2} (1-r^2)^{-\alpha/2} (r^2 - 2r\rho \cos \theta + \rho^2)^{(\alpha-n)/2} \\ &\quad \times r^{n-1} \sin^{n-2} \theta \sin^{n-3} \theta_1 \cdots \sin \theta_{n-3} dr d\theta d\theta_1 \cdots d\theta_{n-2} \\ &= 2^{n-1} \frac{\Gamma\left(\frac{1}{2}\right)^2}{2\Gamma(1)} \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{2}\right)} \cdots \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n-1}{2}\right)} \end{aligned}$$

2) See, for example, [1] pp. 31-33.

$$\begin{aligned}
& \times \int_0^1 \int_0^\pi (1-r^2)^{-\alpha/2} u^{\alpha-n} r^{n-1} \sin^{n-2} \theta \, dr d\theta \\
& = 2 \frac{\Gamma\left(\frac{1}{2}\right)^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 \int_0^\pi (1-r^2)^{-\alpha/2} u^{\alpha-n} r^{n-1} \sin^{n-2} \theta \, dr d\theta.
\end{aligned}$$

Computation of (II). Putting $2m$ for n in (1), we have

$$(2) \quad I_{2m} = \frac{\Gamma\left(\frac{1}{2}\right)^{2m-1}}{\Gamma\left(\frac{2m-1}{2}\right)} \int_0^1 \int_{-\pi}^\pi (1-r^2)^{-\alpha/2} u^{\alpha-2m} r^{2m-1} \sin^{2m-2} \theta \, dr d\theta.$$

We put

$$(3) \quad I_{2m} = \frac{\Gamma\left(\frac{1}{2}\right)^{2m-1}}{\Gamma\left(\frac{2m-1}{2}\right)} I = \frac{(2\pi)^{m-1}}{(2m-3)(2m-5)\cdots 3 \cdot 1} I,$$

where

$$(4) \quad I = \int_0^1 \int_{-\pi}^\pi (1-r^2)^{-\alpha/2} u^{\alpha-2m} r^{2m-1} \sin^{2m-2} \theta \, dr d\theta.$$

The integral I can be written in the form

$$(5) \quad I = \frac{1}{(2\rho)^{2m-2}} \int_0^1 \int_{-\pi}^\pi (1-r^2)^{-\alpha/2} u^{\alpha-2m} \{-(r^2-\rho^2)^2 + 2(r^2+\rho^2)u^2 - u^4\}^{m-1} r \, dr d\theta.$$

Putting $z = \rho e^{i\varphi}$, $\zeta = r e^{i\theta}$, we have

$$(6) \quad r^2 - 2r\rho \cos \theta + \rho^2 = u^2 = |z - \zeta|^2.$$

Then we consider the following linear transformation which maps the unit circle of the ζ -plane to the unit circle of the ζ' -plane:

$$(7) \quad \zeta = \frac{z + \zeta'}{1 + \bar{z}\zeta'}.$$

From (7) we obtain

$$(8) \quad 1 - r^2 = \frac{(1 - |z|^2)(1 - |\zeta'|^2)}{|1 + \bar{z}\zeta'|^2}, \quad |z - \zeta| = \frac{(1 - |z|^2)|\zeta'|}{|1 + \bar{z}\zeta'|}, \quad \frac{d\zeta}{d\zeta'} = -\frac{1 - |z|^2}{|1 + \bar{z}\zeta'|^2}.$$

If we apply the transformation (7) to $4r^2\rho^2 \sin^2(\theta - \varphi)$, then by Lemma 2, there follows

$$(9) \quad 4r^2\rho^2 \sin^2(\theta - \varphi) = \frac{4|\zeta'|^2\rho^2(1 - \rho^2)^2 \sin^2(\theta' - \varphi)}{|1 + \bar{z}\zeta'|^4}.$$

Here if we rewrite $|\zeta'| = r$ and $\arg \zeta' = \theta$, then the right hand member of (9) becomes

$$(10) \quad \frac{4r^2\rho^2(1 - \rho^2)^2 \sin^2(\theta - \varphi)}{|1 + \bar{z}\zeta'|^4}.$$

Since θ varies from 0 to 2π , and we can take the initial line arbitrarily, the integral I can be written as follows:

$$I = \frac{1}{(2\rho)^{2m-2}} \int_0^1 \int_{-\pi}^\pi \frac{(1 - \rho^2)^{-\alpha/2} (1 - r^2)^{-\alpha/2}}{|1 + \bar{z}\zeta'|^{-\alpha}} \frac{(1 - \rho^2)^{\alpha-2m} r^{\alpha-2m}}{|1 + \bar{z}\zeta'|^{\alpha-2m}} \, dr d\theta.$$

$$(11) \quad \times \left\{ \frac{4r^2 \rho^2 (1-\rho^2)^2 \sin^2 \theta}{|1+\bar{z}\zeta'|^4} \right\}^{m-1} \frac{(1-\rho^2)^2}{|1+\bar{z}\zeta'|^4} r dr d\theta$$

$$= 2(1-\rho^2)^{\alpha/2} \int_0^1 \int_0^\pi \frac{(1-r^2)^{-\alpha/2} r^{\alpha-1}}{|1+\bar{z}\zeta'|^{2m}} \sin^{2m-2} \theta dr d\theta.$$

By proper choice of θ , we can put

$$(12) \quad J = \int_0^\pi \frac{\sin^{2m-2} \theta}{|1+\bar{z}\zeta'|^{2m}} d\theta = \int_0^\pi \frac{\sin^{2m-2} \theta}{(1-2r\rho \cos \theta + r^2 \rho^2)^m} d\theta = \int_0^\pi \frac{\sin^{2m-2} \theta}{v^m} d\theta$$

where $v = u^2$.

Since the formula

$$(13) \quad \sin^{2m-2} \theta = (-1)^{m-1} \frac{1}{2^{2m-3}} \\ \times \left\{ \sum_{k=1}^{m-1} (-1)^{k-1} \binom{2m-2}{k-1} \cos(2m-2k)\theta + (-1)^{m-1} \frac{1}{2} \binom{2m-2}{m-1} \right\}$$

is well-known, (12) becomes

$$J = (-1)^{m-1} \frac{1}{2^{2m-3}} \\ \times \left[\sum_{k=1}^{m-1} (-1)^{k-1} \binom{2m-2}{k-1} \int_0^\pi \frac{\cos(2m-2k)\theta}{v^m} d\theta + (-1)^{m-1} \frac{1}{2} \binom{2m-2}{m-1} \int_0^\pi \frac{d\theta}{v^m} \right].$$

Since $a = r\rho < 1$, by Lemma 1, J can be written in the form

$$(14) \quad J = (-1)^{m-1} \frac{1}{2^{2m-3}} A,$$

where

$$(15) \quad A = \left[\sum_{k=1}^{m-1} (-1)^{k-1} \binom{2m-2}{k-1} \frac{(2m-2k+1)(2m-2k+2)\cdots(2m-2k+m-1)}{(m-1)!} a^{2m-2k} \right. \\ \times \left\{ \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(-1-m+2k)(-2-m+2k)\cdots(-q-m+2k)}{(1+2m-2k)(2+2m-2k)\cdots(q+2m-2k)} a^{2q} \right\} \\ \left. + (-1)^{m-1} \frac{1}{2} \binom{2m-2}{m-1} \sum_{q=0}^{m-1} \binom{m-1}{q}^2 a^{2q} \right] \frac{\pi}{(1-a^2)^{2m-1}}.$$

Now let m be a positive integer and p be an integer, and we use the notation ${}_m C_p$ defined as follows:

$${}_m C_p = \begin{cases} \binom{m}{p} & \text{for } 0 \leq p \leq m, \\ 0 & \text{for } p < 0 \text{ or } p \geq m+1. \end{cases}$$

Then, we can prove directly the following identity which is the sum of the coefficients of a^{2p} in (15):

$${}_{m-1} C_{p^2} - \frac{2}{\binom{2m-2}{m-1}} \left[\binom{2m-2}{m-2} \binom{m+1}{2} {}_{m-1} C_{p-1} \frac{(m-3)(m-4)\cdots(m-p+1)}{3 \cdot 4 \cdots (p-1)} \right. \\ \left. - \binom{2m-2}{m-3} \binom{m+3}{4} {}_{m-1} C_{p-2} \frac{(m-5)(m-6)\cdots(m-p-2)}{5 \cdot 6 \cdots (p+2)} + \cdots \right. \\ \left. + (-1)^{m-2} \binom{2m-2}{0} \binom{3m-3}{2m-2} {}_{m-1} C_{2p-2m+2} \frac{(-m+1)(-m)\cdots(m-2p)}{(2m-1)2m\cdots 2p} \right]$$

$$=(-1)^p \binom{2m-2}{p},$$

so that A becomes

$$(16) \quad \begin{aligned} A &= \frac{\pi}{(1-a^2)^{2m-1}} (-1)^{m-1} \frac{1}{2} \binom{2m-2}{m-1} \sum_{p=0}^{2m-2} (-1)^p \binom{2m-2}{p} a^{2p} \\ &= (-1)^{m-1} \frac{1}{2} \binom{2m-2}{m-1} \frac{\pi}{1-a^2}. \end{aligned}$$

From (14) and (16) we have

$$(17) \quad J = \frac{\binom{2m-2}{m-1}}{2^{2m-2}} \frac{\pi}{1-a^2} = \frac{\binom{2m-2}{m-1} \pi}{2^{2m-2} (1-r^2 \rho^2)}.$$

Hence, from (11), (12) and (17) we have

$$(18) \quad I = \frac{\binom{2m-2}{m-1} \pi}{2^{2m-3}} (1-\rho^2)^{\alpha/2} \int_0^1 \frac{(1-r^2)^{-\alpha/2} r^{\alpha-1}}{1-r^2 \rho^2} dr.$$

Now, changing the integration variable r to t by

$$\frac{1-r^2}{1-r^2 \rho^2} = t,$$

I becomes

$$(19) \quad \begin{aligned} I &= \frac{\binom{2m-2}{m-1} \pi}{2^{2m-2}} \int_0^1 t^{-\alpha/2} (1-t)^{\alpha/2-1} dt \\ &= \frac{\binom{2m-2}{m-1} \pi}{2^{2m-2}} \frac{\Gamma\left(1-\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2} + \frac{\alpha}{2}\right)} = \frac{\binom{2m-2}{m-1} \pi^2}{2^{2m-2} \sin \frac{\alpha\pi}{2}}. \end{aligned}$$

From (3) and (19) we finally obtain

$$(20) \quad I_{2m} = \frac{\pi^{(2m-1)/2}}{\Gamma\left(\frac{2m-1}{2}\right)} \frac{\pi^2}{2^{2m-2}} \frac{\binom{2m-2}{m-1}}{\sin \frac{\alpha\pi}{2}} = \frac{\pi^{m+1}}{(m-1)! \sin \frac{\alpha\pi}{2}}.$$

4. The case when the dimension is odd.

Now, we prove the equality

$$(III) \quad I_{2m-1} = \int_{S_{2m-2}} (1-r_Q^2)^{-\alpha/2} r_{PQ}^{\alpha-(2m-1)} dQ = \frac{\pi^{m+1/2}}{\Gamma\left(\frac{2m-1}{2}\right) \sin \frac{\alpha\pi}{2}}.$$

Let the notations be the same as in § 3. Putting $2m-1$ for n in (1) of § 3, we have

$$(1) \quad \begin{aligned} I_{2m-1} &= 2 \frac{\Gamma\left(\frac{1}{2}\right)^{2m-2}}{\Gamma(m-1)} \int_0^1 \int_0^\pi (1-r^2)^{-\alpha/2} (r^2 - 2r\rho \cos \theta + \rho^2)^{(\alpha-(2m-1))/2} r^{2m-2} \sin^{2m-3} \theta dr d\theta. \end{aligned}$$

Using the notation $u=(r^2-2r\rho\cos\theta+\rho^2)^{1/2}$, we can write (1) as follows:

$$(2) \quad I_{2m-1} = \frac{1}{2^{2m-5}} \frac{\Gamma\left(\frac{1}{2}\right)^{2m-2}}{\Gamma(m-1)} \frac{1}{\rho^{2m-3}} \\ \times \int_0^1 \int_{|r-\rho|}^{r+\rho} (1-r^2)^{-\alpha/2} r u^{\alpha-(2m-2)} \{(2r\rho)^2 - (r^2 + \rho^2 - u^2)^2\}^{m-2} du dr.$$

Then we put

$$(3) \quad I_{2m-1} = (-1)^{m-2} \frac{\Gamma\left(\frac{1}{2}\right)^{2m-2}}{2^{2m-5} \Gamma(m-1)} \frac{1}{\rho^{2m-3}} \\ \times \int_0^1 (1-r^2)^{-\alpha/2} r [A(u; r^2 - \rho^2, r^2 + \rho^2)]_{u=|r-\rho|}^{u=r+\rho} dr.$$

The symbol A is defined as follows. First we consider

$$(4) \quad \{(r^2 - \rho^2)^2 - 2(r^2 + \rho^2)u^2 + u^4\}^{m-2} \\ = \sum_{p=0}^{m-2} \sum_{q=0}^{m-2-p} (-1)^q 2^q \binom{m-2}{p} \binom{m-2-p}{q} (r^2 - \rho^2)^{2(m-2-p-q)} (r^2 + \rho^2)^q u^{4p+2q}.$$

If we multiply $u^{\alpha-(2m-2)}$ by the right hand of (4) and integrating with respect to u then the resulting expression contains $u; r, \rho$ which will be expressed by $A(u; r^2 - \rho^2, r^2 + \rho^2)$, that is, we have defined

$$(5) \quad A(u; r^2 - \rho^2, r^2 + \rho^2) \\ = \sum_{p=0}^{m-2} \sum_{q=0}^{m-2-p} (-1)^q 2^q \binom{m-2}{p} \binom{m-2-p}{q} (r^2 - \rho^2)^{2(m-2-p-q)} (r^2 + \rho^2)^q \\ \times \frac{u^{\alpha-2m+3+4p+2q}}{\alpha-2m+3+4p+2q}.$$

Now, since

$$(6) \quad \int_0^1 (1-r^2)^{-\alpha/2} r A(r+\rho; r^2 - \rho^2, r^2 + \rho^2) dr \\ = - \int_{-1}^0 (1-r^2)^{-\alpha/2} r A(|r-\rho|; r^2 - \rho^2, r^2 + \rho^2) dr$$

we can write

$$(7) \quad \int_0^1 (1-r^2)^{-\alpha/2} r \{A(r+\rho; r^2 - \rho^2, r^2 + \rho^2) - A(|r-\rho|; r^2 - \rho^2, r^2 + \rho^2)\} dr \\ = - \int_{-1}^1 (1-r^2)^{-\alpha/2} r A(|r-\rho|; r^2 - \rho^2, r^2 + \rho^2) dr.$$

To evaluate the integral of the right side of (7), we put

$$(8) \quad f(z) = (1-z^2)^{-\alpha/2} z A(z-\rho; z^2 - \rho^2, z^2 + \rho^2),$$

where z is a complex variable.

The function $f(z)$ has the singularities only at $z=-1$, $z=\rho$, and $z=1$ for $0 < \alpha < 2$. We consider two integrating paths. The path C denotes the contour which encloses the points $z=-1$, $z=\rho$, and $z=1$ in its interior. On the other hand, the path K denotes the circle whose center is at $z=0$ and which has radius R (sufficiently large). In both cases the paths C and K are sensed positively. In the region

between the contours C and K , $f(z)$ is single-valued and regular, so that we have

$$(9) \quad \int_C f(z) dz = \int_K f(z) dz.$$

Also we get

$$(10) \quad \int_C f(z) dz = (1 - e^{-\alpha\pi i}) \int_{-1}^1 f(z) dz.$$

In addition, when $|z|$ is large, we can write

$$(11) \quad f(z) = e^{-\alpha\pi i/2} z^{2m-4} \left(1 - \frac{1}{z^2}\right)^{-\alpha/2} A\left(1 - \frac{\rho}{z}; 1 - \frac{\rho^2}{z^2}, 1 + \frac{\rho^2}{z^2}\right).$$

We denote the coefficient of $1/z$ in the expansion of (11), i. e., the coefficient of $(1/z)^{2m-3}$ in the expansion of

$$(12) \quad e^{-\alpha\pi i/2} \left(1 - \frac{1}{z^2}\right)^{-\alpha/2} A\left(1 - \frac{\rho}{z}; 1 - \frac{\rho^2}{z^2}, 1 + \frac{\rho^2}{z^2}\right)$$

by

$$(13) \quad (-1)^{m-2} 2^{2m-4} e^{-\alpha\pi i/2} B \rho^{2m-3}.$$

Then we have

$$(14) \quad \lim_{R \rightarrow \infty} \int_K f(z) dz = 2\pi i (-1)^{m-2} 2^{2m-4} e^{-\alpha\pi i/2} B \rho^{2m-3}.$$

Therefore, from (9), (10) and (14),

$$(15) \quad (1 - e^{-\alpha\pi i}) \int_{-1}^1 f(z) dz = \pi i (-1)^{m-2} 2^{2m-3} e^{-\alpha\pi i/2} B \rho^{2m-3}.$$

Hence, from (3), (7) and (15) we obtain

$$(16) \quad I_{2m-1} = (-1)^{m-2} \frac{\Gamma\left(\frac{1}{2}\right)^{2m-2}}{2^{2m-5} \Gamma(m-1)} \frac{1}{\rho^{2m-3}} \left[-(-1)^{m-2} \frac{2^{2m-3} e^{-\alpha\pi i/2} \pi B \rho^{2m-3} i}{1 - e^{-\alpha\pi i}} \right] \\ = - \frac{\pi^m}{(m-2)!} \frac{2B}{\sin \frac{\alpha\pi}{2}}.$$

Since it will be shown later that

$$(17) \quad B = - \frac{2^{m-2}(m-2)!}{(2m-3)(2m-5)\cdots 3 \cdot 1},$$

(16) becomes finally

$$(18) \quad I_{2m-1} = - \frac{\pi^m}{(m-2)!} \frac{2}{\sin \frac{\alpha\pi}{2}} \left[- \frac{2^{m-2}(m-2)!}{(2m-3)(2m-5)\cdots 3 \cdot 1} \right] = \frac{\pi^{m+1/2}}{\Gamma\left(\frac{2m-1}{2}\right)} = \frac{1}{\sin \frac{\alpha\pi}{2}},$$

which proves our assertion (III).

We are now in position to compute the value of B . We put

$$(19) \quad F(t) = (1-t^2)^{-\alpha/2} A(1-\rho t; 1-\rho^2 t^2, 1+\rho^2 t^2) \\ = (1-t^2)^{-\alpha/2} \sum_{p=0}^{m-3} \sum_{q=0}^{m-2-p} (-1)^q 2^q \binom{m-2}{p} \binom{m-2-p}{q} (1-\rho^2 t^2)^{2(m-2-p-q)} (1+\rho^2 t^2)^q \\ \times \frac{(1-\rho t)^{\alpha-2m+3+4p+2q}}{\alpha-2m+3+4p+2q}.$$

From the definitions (5) and (13), the coefficient of t^{2m-3} in the expansion of (19) is exactly given by

$$(20) \quad (-1)^{m-2} 2^{2m-4} \rho^{2m-3} B.$$

In order to compute the value (20) explicitly, we put the expansions of the terms in (19) as follows:

$$\begin{aligned} (1-t^2)^{-\alpha/2} &= a_0 + a_2 t^2 + a_4 t^4 + \cdots & (a_0=1), \\ (1-\rho^2 t^2)^{2(m-2-p-q)} (1+\rho^2 t^2)^q &= b_0 + b_2 \rho^2 t^2 + b_4 \rho^4 t^4 + \cdots + b_{2s} \rho^{2s} t^{2s} \\ & & (b_0=1, s=2m-4-2p-q), \\ \frac{(1-\rho t)^{\alpha-2m+3+4p+2q}}{\alpha-2m+3+4p+2q} &= c_0 + c_1 \rho t + c_2 \rho^2 t^2 + \cdots & (c_1=-1). \end{aligned}$$

Then the desired value (20) is given by the sum of terms $a_{2\lambda} b_{2\mu} c_\nu \rho^{2\mu+\nu}$ where the indices λ, μ and ν run over all non-negative integers with $2\lambda+2\mu+\nu=2m-3$. From Lemma 3, the sums in p, q of the terms $a_{2\lambda} b_{2\mu} c_\nu$ vanish except for $\lambda=0, \nu=2m-3$. The sums of the terms $a_{2\lambda} b_{2\mu} c_\nu$ (μ : fixed) reduces to a similar summation in Lemma 3 with $m-2$ replaced by $m-2-2\mu$. Hence the sums in p, q of terms $a_{2\lambda} b_{2\mu} c_\nu$ vanish except for $\lambda=0, 2\mu+\nu=2m-3$ which gives the value

$$2^{2m-4} (-1)^\mu \frac{\binom{m-2}{\mu} (2m-4-2\mu)!}{(2m-3-2\mu)!} \rho^{2m-3}$$

for a fixed μ ($\nu=2m-2\mu-3$), by Lemma 3. Hence the value (20) is equal to

$$(21) \quad 2^{2m-4} \sum_{\mu=0}^{m-2} (-1)^\mu \frac{\binom{m-2}{\mu}}{2m-3-2\mu} \rho^{2m-3}.$$

Using Lemma 4, it becomes

$$(22) \quad (-1)^{m-2} 2^{2m-4} \frac{2^{m-2} (m-2)!}{(2m-3)(2m-5)\cdots 3 \cdot 1} \rho^{2m-3}.$$

Since (20) equals (22), we have finally the equality (17):

$$B = - \frac{2^{m-2} (m-2)!}{(2m-3)(2m-5)\cdots 3 \cdot 1}.$$

REMARK. The integrals I_{2m} and I_{2m-1} are improper. But they converge uniformly in ρ ($0 \leq \rho \leq 1$). Therefore the above proof of (II) and (III) remains valid even when $\rho \rightarrow 1$. So that (I) holds for any ρ ($0 \leq \rho \leq 1$).

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