5. On the Distribution of the Zero Points of Sections of a Power Series, III.

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A Remark on Okada's Theorem.

1. Prof. Okada⁽¹⁾ has proved the theorem :

Let r_n be the maximum modulus of the roots of the n -th section,

$$f_n(x) = a_0 + a_1x + \dots + a_nx^n$$

of a power series $f(x) = a_0 + a_1x + \dots + a_nx^n + \dots$,

then the necessary and sufficient condition that $f(x)$ is an integral function is that

$$\lim_{n \rightarrow \infty} r_n = \infty.$$

In this paper I will investigate the case when $f(x)$ is an integral function of order ρ and prove the following theorem which is conjectured by Prof. G. Szegő.

Theorem I. The necessary and sufficient condition that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an integral function of order ρ is that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log n}{\log r_n} = \rho,$$

where r_n is the maximum modulus of the roots of the n -th section $f_n(x) = \sum_{m=0}^n a_m x^m$.

Proof. Let $f(x)$ be an integral function of order ρ , then for

$$|x| \leq r, \quad |f(x)| < e^{\rho' r} \quad (2), \quad \text{where } \rho' \text{ is any quantity greater than } \rho.$$

From this follows

$$(1) \quad |a_n| \leq \left(\frac{e\rho'}{n} \right)^{\frac{n}{\rho'}} \quad (3)$$

Let the roots of the equation,

$$a_0 + a_1x + \dots + a_nx^n = 0,$$

(1) The science reports of the Tôhoku Imperial University, **11** (1922) 43-50. Cf. the simple proof of Prof. Kakeya this journal, **1** 131-133.

(2) We assume that this inequality holds from $r=0$, but it is easily seen that the final result is true, when we only assume that the inequality holds for $r \geq r_0$.

(3) E. Borel. Leçons sur les fonctions entières. p. 63.

be denoted by

$$x_1^{(n)}, \dots, x_n^{(n)}.$$

If we suppose, without loss of generality, $a_0=1$ in the following discussion, then we have

$$(2) \quad |x_1^{(n)} \dots x_n^{(n)}| = \frac{1}{|a_n|} \geq \left(\frac{n}{e\rho'}\right)^{\frac{n}{\rho'}}.$$

On the other hand

$$(3) \quad |x_1^{(n)} \dots x_n^{(n)}| \leq r_n^n.$$

Hence

$$(4) \quad r_n^n \geq \left(\frac{n}{e\rho'}\right)^{\frac{n}{\rho'}}.$$

From this follows at once

$$\overline{\lim}_{n=\infty} \frac{\log n}{\log r_n} \leq \rho'.$$

Since ρ' is any quantity greater than ρ , we have

$$(5) \quad \overline{\lim}_{n=\infty} \frac{\log n}{\log r_n} \leq \rho.$$

We will prove that $\overline{\lim}_{n=\infty} \frac{\log n}{\log r_n} = \rho_1 < \rho$ is inadmissible.

In fact, from $\overline{\lim}_{n=\infty} \frac{\log n}{\log r_n} = \rho_1$, follows

$$(6) \quad n^{\frac{1}{\rho'}} < r_n \quad \text{for } n \geq n_0,$$

where

$$\rho_1 < \rho' < \rho^{(4)}.$$

Let x_n be the root of $f_n(x)=0$, which has the maximum modulus r_n , then

$$1 + a_1 x_1 = 0$$

$$1 + a_1 x_2 + a_2 x_2^2 = 0$$

$$\dots\dots\dots$$

$$1 + a_1 x_n + \dots\dots\dots + a_n x_n^n = 0$$

$$\dots\dots\dots$$

Hence

$$|a_n| r_n^n \leq |a_{n-1}| r_n^{n-1} + \dots\dots\dots + |a_1| r_n + 1.$$

We choose M so that for $k=1, 2, \dots, n_0-1$,

$$(7) \quad M > 1 \quad |a_k| < M \frac{2^k}{(k!)^{\frac{1}{\rho'}}},$$

then from (6) and (7) we have

(4) When $\rho=0$, evidently $\overline{\lim}_{n=\infty} \frac{\log n}{\log r_n} = 0$.

$$\begin{aligned}
|a_{n_0}| &\leq \frac{|a_{n_0-1}|}{r_{n_0}} + \frac{|a_{n_0-2}|}{r_{n_0}^2} + \dots + \frac{|a_1|}{r_{n_0}^{n_0-1}} + \frac{1}{r_{n_0}^{n_0}} \\
&< M \left[\frac{2^{n_0-1}}{[(n_0-1)!]^{\frac{1}{\rho'}} n_0^{\frac{1}{\rho'}}} + \frac{2^{n_0-2}}{[(n_0-2)!]^{\frac{1}{\rho'}} n_0^{\frac{2}{\rho'}}} + \dots + \frac{1}{n_0^{\frac{n_0}{\rho'}}} \right] \\
&< \frac{M}{(n_0!)^{\frac{1}{\rho'}}} (2^{n_0-1} + \dots + 2 + 1) < \frac{2^{n_0} M}{(n_0!)^{\frac{1}{\rho'}}}.
\end{aligned}$$

Hence in general we have

$$(8) \quad |a_n| < \frac{2^n M}{(n!)^{\frac{1}{\rho'}}}, \quad (5),$$

so that the order of $f(x)$ is at most $\rho'(^6)$, contradictory to the hypothesis that $f(x)$ is of order ρ .

Consequently we have

$$(9) \quad \overline{\lim}_{n \rightarrow \infty} \frac{\log n}{\log r_n} = \rho.$$

Conversely, if $\overline{\lim}_{n \rightarrow \infty} \frac{\log n}{\log r_n} = \rho$, then, as we have seen, $f(x)$ is of order $\rho' \leq \rho$, so that, as we have proved, $\overline{\lim}_{n \rightarrow \infty} \frac{\log n}{\log r_n} = \rho'$.

Hence $\rho' = \rho$, or $f(x)$ is of order ρ , and our theorem is proved.

In my previous paper⁽⁷⁾ I have proved that, if in a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |a_n| = \Omega\left(\frac{1}{n^{kn+r}}\right) \text{ where } k > 0 \text{ and } r \text{ is any real number,}$$

then $r_n = \Omega(n^k)$.

From this follows at once $\lim_{n \rightarrow \infty} \frac{\log n}{\log r_n} = \frac{1}{k}$.

On the other hand it is easily seen that $\frac{1}{k}$ is the order of $f(x)$.

2. Since $\overline{\lim}_{n \rightarrow \infty} \frac{\log n}{\log r_n}$ is the convergence-exponent of the sequence $r_n(^8)$ we have the following theorem which is equivalent to Theorem I.

Theorem II. *The necessary and sufficient condition that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is*

(⁵) We find the inequality (8) in Okada's original paper. I.e. 49. We have assumed a_n not to be zero, but when a_n is zero, the inequality (8) evidently holds.

(⁶) Cf. Valiron: Lectures on the general theory of integral functions, p. 40.

(⁷) This journal, 1, 133.

(⁸) Cf. G. Pólya und G. Szegő. Aufgaben und Lehrsätze aus der Analysis, I, p. 20.

an integral function of order ρ is that $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho+\varepsilon}}$ is convergent and $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho-\varepsilon}}$ is divergent, for any positive quantity ε , where r_n is the maximum modulus of the roots of the n -th section.

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