8. On Blaschke's Theorem.

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1. W. Blaschke has proved the theorem⁽¹⁾:

If $f(x)$ is regular and bounded for $|x| < 1$ and $x_1, x_2, \dots, x_n, \dots$ be the zero points of $f(x)$ in $|x| < 1$, then $\sum_{n=1}^{\infty} (1 - |x_n|)$ is convergent.

I will in this paper modify this theorem and deduce some consequences from it.

Lemma⁽²⁾. When $x = 1 - \rho (\cos \varphi + i \sin \varphi)$, where $|\varphi| \leq \varphi_0 < \frac{\pi}{2}$ and $0 < \rho \leq \rho_0 < 2 \cos \varphi_0$, then there exists a constant A such that

$$(1) \quad \frac{1-x}{1-|x|} \leq A.$$

Considering only the roots $x_1, x_2, \dots, x_n, \dots$ of $f(x)$ which lie in the angular region G as defined in the above lemma, it follows from Blaschke's theorem that $\sum_{n=1}^{\infty} (1 - |x_n|)$ is convergent and considering (1) we see that

$\sum_{n=1}^{\infty} |1 - x_n|$ is also convergent. Hence we have

Theorem I. Let $x_1, x_2, \dots, x_n$ be the roots of $f(x)$ in the angular region G defined in the lemma, then $\sum_{n=1}^{\infty} |1 - x_n|$ is convergent.

2. Let $f(x)$ be regular and bounded in an angular domain G ;

G : $0 < \arg x < \theta$, $0 < |x| < 1$,

and let G_1 denote an angular domain contained in G such that

G_1 : $\varepsilon < \arg x < \theta - \varepsilon$, $0 < |x| < 1 - \varepsilon'$.

We map the domains G and G_1 conformally on the domains G' and G'_1 of the z -plane by a transformation;

$$(2) \quad z = x^{\frac{\pi}{\theta}}.$$

Then G' is a semicircle

G' : $0 < \arg z < \pi$, $0 < |z| < 1$,

G'_1 : $\varepsilon_1 < \arg z < \pi - \varepsilon_1$, $0 < |z| < 1 - \varepsilon'_1$.

⁽¹⁾ W. Blaschke, Leipz. Ber. 67 (1915), 194-200.

⁽²⁾ Cf. Knopp, Theorie und Anwendungen der unendlichen Reihen, 2. Aufl. 407.

Again we map G' and G_1' conformally on G'' and G_1'' on the w -plane, by a transformation,

$$(3) \quad w = \varphi(z),$$

so that G'' becomes a whole circle $|w|=1$ and to $z=0$ corresponds $w=1$. Since the boundaries of G' and G'' in the neighbourhood of $z=0$ and $w=1$ are analytic curves, the mapping function $w=\varphi(z)$ is regular at $z=0$ and since at the corresponding points $z=0$, $w=1$, the mapping is conformal, we see that $\varphi'(0) \neq 0, \neq \infty$. Hence if $|z|$ is very small, there exist two constants A and B such that

$$(4) \quad A \leq \frac{|z|}{|1-w|} \leq B.$$

Now suppose $f(x)$ has infinitely many zero points $x_1, x_2, \dots$ in G_1 . By the above mapping $f(x)$ becomes a function $F(w)$, which is regular and bounded in the circle $|w|=1$, hence by Theorem I, $\sum_{n=1}^{\infty} |1-w_n|$ is convergent, where $w_1, w_2, \dots, w_n, \dots$ are the corresponding zero points of $F(w)$ in the angular domain G_1'' .

From (4) we see that $\sum_{n=1}^{\infty} |z_n|$ is convergent and from (2), $\sum_{n=1}^{\infty} |x_n|^{\frac{\pi}{\theta}}$ is also convergent. Hence we have

Theorem II. Let $f(x)$ be regular and bounded in an angular domain G ;
 G ;
 $\alpha < \text{arc } x < \beta, \quad 0 < |x| < r,$
and $x_1, x_2, \dots, x_n, \dots$ be the zero points of $f(x)$ in an angular domain G_1 ;
 G_1 ;
 $\alpha + \varepsilon < \text{arc } x < \beta - \varepsilon, \quad 0 < |x| < r - \varepsilon,$
then

$$\sum_{n=1}^{\infty} |x_n|^{\frac{\pi}{\theta}} \text{ is convergent}$$

where $\theta = \beta - \alpha$; ε' and ε are any small positive quantities.

3. By the transformation $z = \frac{1}{x}$, we get as a special case of the above theorem, the following theorem on integral functions:

Theorem III. Let an integral function $f(x)$ be bounded in an angular domain G

$$G; \quad \alpha < \text{arc } x < \beta,$$

and the zero points of $f(x)$ in another angular domain G_1 ,

$$G_1; \quad \alpha + \varepsilon < \text{arc } x < \beta - \varepsilon$$

be denoted by $x_1, x_2, \dots, x_n, \dots$ then

$$\sum_{n=1}^{\infty} \frac{1}{|x_n|^{\frac{\pi}{\theta}}} \text{ is convergent}$$

where $\theta = \beta - \alpha$ and ε is any small quantity.

When $f(x)$ is an integral function of order ρ , we know by Hadamard's theorem⁽³⁾ that $\sum_{n=1}^{\infty} \frac{1}{|x_n|^{\rho+\varepsilon}}$ is convergent. Hence if $\frac{\pi}{\theta} > \rho$, the above theorem asserts nothing more than Hadamard's theorem. But by Phragmén's theorem⁽⁴⁾ an integral function of order ρ can not be bounded in an angle whose opening is greater than $\pi\left(2 - \frac{1}{\rho}\right)$, hence $\theta \leq \pi\left(2 - \frac{1}{\rho}\right)$ or $\frac{\rho}{2\rho-1} \leq \frac{\pi}{\theta}$; and from this it does not necessarily follow $\frac{\pi}{\theta} > \rho$.

4. We will apply the above theorem to prove an extension of Vitali's theorem⁽⁵⁾. If $f_n(x)$ are regular and uniformly bounded in a domain G and converge at points which have a limiting point in the domain, then $f_n(x)$ converge at every point of the domain and the convergence is uniform in any closed domain contained in G .

W. Blaschke⁽⁶⁾ has extended this theorem when G is a circle in the following way.

Blaschke's extension of Vitali's theorem: If $f_n(x)$ are regular and uniformly bounded in a circle $|x| < 1$ and converge at such points $x_1, x_2, \dots, x_n, \dots$ that $\sum_{n=1}^{\infty} (1 - |x_n|)$ diverges, then $f_n(x)$ converge at every point in the unit circle and the convergence is uniform in a circle $|x| \leq r$, where r is any positive quantity smaller than 1.

We will prove the following

Theorem IV. (*Extension of Vitali's theorem*). Let $f_n(x)$ be regular and uniformly bounded in a domain G and $P(x=\xi)$ be such a point on the boundary of G , that an angular domain APB is entirely contained in G , and let A_1PB_1 be another angular domain contained in APB having the same vertex with it. If $f_n(x)$ converge at such points $x_1, x_2, \dots, x_n, \dots$ of A_1PB_1 that

(³) Borel, Leçons sur les fonctions entières. p. 75.

(⁴) Phragmén, Acta Mathematica, 28(1904), 351-358.

(⁵) Cf. Julia, Leçons sur les fonctions uniformes à point singulier essentiel isolé p. 55.

(⁶) Blaschke. l. c.

$\sum_{n=1}^{\infty} |x_n - \xi|^{\frac{\pi}{\theta}}$ is divergent, where $\theta = \widehat{APB}$, then $f_n(x)$ converge at every point of G and the convergence is uniform in any closed domain contained in G .

Proof. We consider only the angular domain APB . In this domain $f_n(x)$ are regular and uniformly bounded. If $f_n(x)$ do not converge at a point x_0 , say, in this domain, then we can find two sub-sequences $f_{n_i}(x)$ and $f_{m_i}(x)$ so that $\lim_{i \rightarrow \infty} f_{n_i}(x_0) = a$ and $\lim_{i \rightarrow \infty} f_{m_i}(x_0) = b$, where $a \neq b$. From $f_{n_i}(x)$, $f_{m_i}(x)$ we select by Montel's theorem sub-sequences $f_{n_{i'}}(x)$ and $f_{m_{i'}}(x)$ so that $f_{n_{i'}}(x)$, $f_{m_{i'}}(x)$ converge uniformly in any closed domain contained in APB . We denote the limiting functions, which are evidently regular in APB , by $\varphi(x)$, $\psi(x)$ so that

$$\lim_{i \rightarrow \infty} f_{n_{i'}}(x) = \varphi(x), \quad \lim_{i \rightarrow \infty} f_{m_{i'}}(x) = \psi(x).$$

Since $\varphi(x_0) = a$, $\psi(x_0) = b$, we have $\varphi(x) \neq \psi(x)$.

Consider a function $F(x) = \varphi(x) - \psi(x)$, then $F(x_0) = 0$ and $F(x)$ is bounded in APB . Hence by theorem III, $\sum_{n=1}^{\infty} |x_n - \xi|^{\frac{\pi}{\theta}}$ is convergent, which con-

tradicts the hypothesis that $\sum_{n=1}^{\infty} |x_n - \xi|^{\frac{\pi}{\theta}}$ is divergent, therefore $f_n(x)$ must converge at every point in APB . By Vitali's theorem $f_n(x)$ converge at every point in G and the convergence is uniform in any closed domain contained in G . Our theorem is thus proved.

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