On saltus-functions and interval-surfaces

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Introduction

As regards terminology and notation we shall follow the *Theory of the Integral* by Saks as a rule (see the list of references at the end). The mentioned treatise will be quoted very frequently in the sequel and we shall henceforward refer to it simply as Saks for short.

The space that will be basic for all our considerations is a fixed Euclidean space R^m of dimension $m \geq 1$. When we specialize our space (to be, say, the straight line R^1), we shall say so explicitly. Thus the various sets that will be considered in the sequel are subsets of R^m , unless another meaning is obvious from the context. We shall always understand by *intervals*, by themselves, closed nondegenerate intervals in R^m . It may be remarked that, contrary to the use in Saks, we thus do not regard the void set as an interval. Furthermore, the letters δ and ε will invariably stand for finite positive numbers.

We shall be chiefly concerned with additive interval-functions of bounded variation, defined for the intervals in R^m , and with additive set-functions, defined for the bounded Borel sets in R^m . In what follows we shall refer to them for brevity of wording simply as *additive interval-functions* and *additive set-functions*, even when we use such expressions as an arbitrary additive interval-function, and so on.

The present paper is divided into two chapters. In the first chapter we aim at obtaining a complete extension to R^m of the theory of *saltus-functions* developed in the one-dimensional case in Chapter III of Saks. The reasonings that are valid for R^1 are no more valid for higher dimensions. We are thus obliged to define the saltus-functions in a different manner than in Saks, and we shall deduce their fundamental properties by way of diverse arguments of auxiliary character. As a typical result of this chapter we may mention that of § 40 to the effect that if F is an additive interval-function and W is its absolute variation, then $W(F^*; X) = W^*(X)$ for every bounded Borel set X which is disjoint with the hyperplanes of discontinuity of F . This constitutes a natural extension to m dimensions of the theorem given in Saks on p. 99.

A few words of caution are necessary for our use, throughout the paper, of the notation F^* appearing above. The function F^* is defined on p. 64 of the Saks treatise for all sets of the space R^m . It is mostly convenient for our purpose, however, to consider this set-function only for the bounded Borel sets in R^m . To prevent ambiguity, we therefore agree to make the convention that the symbol F^* should mean, in our sense, the restriction of F^* in the Saks sense to the class of all bounded Borel sets and that, when there is need, a special notation F_* should be used to denote F^* in the Saks sense. Thus F_* is defined for all the sets in R^m , and is an outer Carathéodory measure if F is

nonnegative, while F^* is an additive set-function (of bounded variation).

On the basis of the first chapter we shall treat in the second chapter the theory of *area* of interval-surfaces and of set-surfaces. We understand by an *interval-surface* any finite sequence of additive interval-functions (defined for $\mathbf{R}^m$, needless to say), while we call *set-surface* any finite sequence of additive set-functions. Let us observe in passing that the term *sequence* will always mean a nonvoid countable (i.e. finite or countably infinite) one, so that we shall never consider a void sequence. We shall clarify in the following the motivation for our introducing these novel concepts.

Let us consider a curve φ in $\mathbf{R}^k$ defined on the real line, i.e. a mapping of $\mathbf{R}^1$ into $\mathbf{R}^k$, which we write in the form $\varphi=(\varphi_1, \dots, \varphi_k)$, where φ_i denotes the i th coordinate-function of φ . We can attach to each φ_i an additive interval-function F_i defined for every linear interval $I=[a, b]$ by setting $F_i(I)=\varphi_i(b)-\varphi_i(a)$. We have thus associated with the given curve φ a k -tuple of additive interval-functions.

Suppose conversely that we are given an arbitrary finite sequence $f=(F_1, \dots, F_k)$ of additive interval-functions, all defined for the real line $\mathbf{R}^1$. We may suggestively call f an *interval-curve*. It is obvious that there is an infinity of curves φ of the aforesaid description to which f corresponds in the sense described just now. But any two of these curves φ plainly differ only by a parallel translation, so that all of them have equal length on each linear interval I , which is defined as the *length* over I of the interval-curve f and denoted by $f^\nabla(I)$. In other words, the length $f^\nabla(I)$ is the supremum of the sum $|f(I_1)|+\dots+|f(I_n)|$, where $(I_1, \dots, I_n)$ is an arbitrary subdivision of I into intervals and where $|f(J)|$ denotes for every linear interval J the magnitude of the k -dimensional vector $f(J)$ with the i th component $F_i(J)$.

It was these circumstances that led us to introduce the concept of interval-surface. By the same token, it is easy to guess what we can reasonably mean by the *area* f^∇ of an interval-surface f over an interval, or more generally, a figure. Now it is only a matter of course that we should consider set-surfaces and their areas side by side with interval-surfaces and investigate the interrelations of the two kinds of surfaces. Tracing this line of thought, we arrive at a number of results, of which we shall content ourselves with quoting the following one (§ 57): Given any interval-surface $f=(F_1, \dots, F_k)$, let us attach to f a set-surface f^* defined by $f^*=(F_1^*, \dots, F_k^*)$. If f is continuous, that is, has continuous component-functions F_i , then $f^{*\nabla}(X)=f^\nabla(X)$ for every bounded Borel set X , where $f^{*\nabla}$ means $(f^*)^\nabla$, i.e. the area of the set-surface f^* , while f^∇ is short for $(f^\nabla)^*$.

We shall conclude the present research with a few results on the Steiner inequality which concerns the areas of parametric or non-parametric surfaces and is of certain importance to the study of the problem of Plateau.

The proofs of the propositions will often be omitted when they are comparatively easy or not indispensable.

Chapter I. Saltus-functions

1. Inside excess of a nonnegative additive interval-function. Given such a function U , we construct from U another interval-function U_0 , finite and non-

negative, whose value $U_0(I)$ for any interval I we define as the infimum of $U(I \ominus J)$, where J ranges over the intervals contained in the interior I° of I . Thus defined, $U_0(I)$ will be termed *inside excess* of U over I .

2. Subadditivity of the inside excess. *The inside excess U_0 satisfies the inequality*

$$U_0(I_1 \cup I_2) \leq U_0(I_1) + U_0(I_2)$$

for every pair I_1, I_2 of non-overlapping intervals whose join is also an interval.

More generally, a similar inequality holds when we consider a finite number, greater than two, of non-overlapping intervals $I_1, I_2, \dots, I_n$ whose join is likewise an interval.

PROOF. We may restrict ourselves to the first half of the assertion, the second half being proved quite analogously. Let us write for short $I = I_1 \cup I_2$. Given any ε we can, by definition of inside excess, find a pair of intervals J_1 and J_2 , contained respectively in I_1° and in I_2° and such that $U(I_i \ominus J_i) < U_0(I_i) + \varepsilon$ for $i=1, 2$. Now choose an interval J such that $J_1 \cup J_2 \subset J \subset I^\circ$. Then $U(J) \geq U(J_1) + U(J_2)$, and it follows that

$$\begin{aligned} U_0(I) &\leq U(I \ominus J) = U(I) - U(J) \leq U(I_1) + U(I_2) - U(J_1) - U(J_2) \\ &< U_0(I_1) + U_0(I_2) + 2\varepsilon. \end{aligned}$$

By making $\varepsilon \rightarrow 0$ here, we get at once $U_0(I) \leq U_0(I_1) + U_0(I_2)$.

3. Saltus-part of a nonnegative additive interval-function. In the preceding section we have associated with every nonnegative additive interval-function U its inside excess U_0 . With the help of U_0 we shall now attach to U still another interval-function M , which we shall call *saltus-part* of U . The value $M(I)$ of M for any given interval I we define namely as the supremum of the sum $U_0(I_1) + \dots + U_0(I_n)$, where $I_1, \dots, I_n$ form an arbitrary subdivision of I into intervals. Since $U_0(J) \leq U(J)$ for every interval J , we find that $M(I) \leq U(I)$, so that M is a finite nonnegative function.

4. Additivity of the saltus-part. *The saltus-part M of any nonnegative additive interval-function U is additive.*

PROOF. Let I_1 and I_2 be any pair of non-overlapping intervals whose join I is also an interval. We have to show that $M(I) = M(I_1) + M(I_2)$. Since evidently $M(I) \geq M(I_1) + M(I_2)$ by the definition of M , we need only prove the opposite inequality. Given an arbitrary ε , there is a subdivision $J_1, \dots, J_n$ of I into intervals such that $U_0(J_1) + \dots + U_0(J_n) > M(I) - \varepsilon$, and in view of subadditivity of U_0 we may suppose here that each J_i ($i=1, 2, \dots, n$) is situated either in I_1 or in I_2 . Then obviously $M(I_1) + M(I_2) \geq U_0(J_1) + \dots + U_0(J_n)$, and in consequence $M(I_1) + M(I_2) > M(I) - \varepsilon$, whence we derive at once $M(I_1) + M(I_2) \geq M(I)$, as desired.

5. *The inside excess U_0 of any nonnegative additive interval-function U coincides identically with the inside excess M_0 of the saltus-part M of U . Thus M is the saltus-part of its own.*

PROOF. On account of the inequality $U \geq M$ we have $U_0(I) \geq M_0(I)$ for every interval I , and hence it suffices to derive the opposite inequality $U_0(I) \leq M_0(I)$.

For this purpose, consider an arbitrary interval $J \subset I^\circ$ and an arbitrary subdivision of the figure $I \ominus J$ into intervals $I_1, \dots, I_n$. Then

$$M(I \ominus J) = \sum_{i=1}^n M(I_i) \geq \sum_{i=1}^n U_0(I_i). \quad (1)$$

Given any ε , it follows from the definition of U_0 that there exists a sequence $J_1, \dots, J_n$ of intervals such that $J_i \subset I_i^\circ$ for each i and such that

$$\sum_{i=1}^n U_0(I_i) + \varepsilon > \sum_{i=1}^n U(I_i \ominus J_i) = U(I \ominus J \ominus R), \quad (2)$$

where we write for short $R = J_1 \cup \dots \cup J_n$. Now take an interval K satisfying $I^\circ \supset K \supset J \cup R$. Then clearly $I \ominus K \subset I \ominus J \ominus R$, and so $U(I \ominus K) \leq U(I \ominus J \ominus R)$, which in combination with (1) and (2) gives $U_0(I) < M(I \ominus J) + \varepsilon$. Since ε is arbitrary, we get $U_0(I) \leq M(I \ominus J)$, and this implies that $U_0(I) \leq M_0(I)$, since J is arbitrary as long as $J \subset I^\circ$. The proof is thus complete.

6. If W is the sum of a pair U, V of nonnegative additive interval-functions and if we denote by U_0, V_0, W_0 the inside excesses, and by U_1, V_1, W_1 the saltus-parts, of U, V, W respectively, then we have $W_0 = U_0 + V_0$ and $W_1 = U_1 + V_1$ identically.

PROOF. Writing J for a generic interval contained in the interior of a given interval I , we find

$$W(I \ominus J) = U(I \ominus J) + V(I \ominus J) \geq U_0(I) + V_0(I),$$

and this implies that $W_0(I) \geq U_0(I) + V_0(I)$. On the other hand, given any ε we can choose in I° a pair of intervals J_1 and J_2 satisfying

$$U(I \ominus J_1) < U_0(I) + \varepsilon \quad \text{and} \quad V(I \ominus J_2) < V_0(I) + \varepsilon.$$

Let us take now in I° an arbitrary interval K containing both J_1 and J_2 . Then

$$\begin{aligned} W_0(I) &\leq W(I \ominus K) = U(I \ominus K) + V(I \ominus K) \leq U(I \ominus J_1) + V(I \ominus J_2) \\ &< U_0(I) + V_0(I) + 2\varepsilon, \end{aligned}$$

whence we obtain immediately $W_0(I) \leq U_0(I) + V_0(I)$. This, conjointly with the opposite inequality proved already, establishes $W_0(I) = U_0(I) + V_0(I)$.

Consider next any subdivision $I_1, \dots, I_n$ of I into intervals. Then, by what has been proved in the above,

$$\sum_{i=1}^n W_0(I_i) = \sum_{i=1}^n U_0(I_i) + \sum_{i=1}^n V_0(I_i) \leq U_1(I) + V_1(I),$$

and hence we get $W_1(I) \leq U_1(I) + V_1(I)$. To deduce the opposite inequality, let ε be arbitrarily given and take the above subdivision (I_i) so as to fulfill both

$$\sum_{i=1}^n U_0(I_i) > U_1(I) - \varepsilon \quad \text{and} \quad \sum_{i=1}^n V_0(I_i) > V_1(I) - \varepsilon.$$

This is clearly possible by § 2. It follows that

$$W_1(I) \geq \sum_{i=1}^n W_0(I_i) = \sum_{i=1}^n U_0(I_i) + \sum_{i=1}^n V_0(I_i) > U_1(I) + V_1(I) - 2\varepsilon,$$

whence we get at once $W_1(I) \geq U_1(I) + V_1(I)$, as desired.

7. In order that a nonnegative additive interval-function U be continuous on a

figure R , it is necessary and sufficient that the inside excess U_0 of U should vanish identically within R , i.e. for every interval contained in R .

PROOF. We may clearly suppose R nonvoid. Let I_0 be a fixed interval in R . If U is continuous on R , then given any ε there exists a δ such that $U(J) < 3^{-m}\varepsilon$ for every interval $J \subset R$ with $|J| < \delta$, where m is the dimension of the underlying space $\mathbf{R}^m$. Consider now an interval I_1 contained in the interior of I_0 and such that $|I_0| < |I_1| + \delta$. The figure $I_0 \ominus I_1$ can clearly be subdivided in a sequence of intervals $I_2, \dots, I_n$, where $n = 3^m$ and $|I_2| + \dots + |I_n| = |I_0| - |I_1| < \delta$. It follows that

$$U(I_0 \ominus I_1) = U(I_2) + \dots + U(I_n) < n3^{-m}\varepsilon = \varepsilon.$$

This implies that $U_0(I_0) = 0$ and thus completes the necessity proof.

Supposing now conversely that U_0 vanishes within R , we shall show the continuity of U on R by induction on m . The case $m=1$ being evident, we assume that $m > 1$ and further that the result holds for the space $\mathbf{R}^{m-1}$. Given any interval $I_0 \subset R$, let us express I_0 in the direct product form $I_0 = J_0 \times K_0$, where $J_0 \subset \mathbf{R}^1$ and $K_0 \subset \mathbf{R}^{m-1}$, and let us consider the pair V, W of interval-functions, defined respectively for the intervals J in $\mathbf{R}^1$ and K in $\mathbf{R}^{m-1}$ by $V(J) = U(J \times K_0)$ and by $W(K) = U(J_0 \times K)$. Both V and W are then plainly nonnegative additive interval-functions, whose inside excesses vanish respectively within J_0 and within K_0 . Consequently, by the case $m=1$ and the induction hypothesis, these two functions must be continuous on J_0 and on K_0 respectively, and hence, given any ε we can find a δ such that $V(J) < \varepsilon$ for every linear interval $J \subset J_0$ with $L_1(J) < \delta$ and that $W(K) < \varepsilon$ for every $(m-1)$ -dimensional interval $K \subset K_0$ with $L_{m-1}(K) < \delta$ (for the meaning of L_1 and L_{m-1} vide p. 59 of Saks). Now let I be any m -dimensional interval in I_0 with $L_m(I) < \delta^2$ and let us write $I = J \times K$, where $J \subset J_0$ and $K \subset K_0$. In view of the equation $L_m(I) = L_1(J) \cdot L_{m-1}(K)$ we then have the two alternatives: either $L_1(J) < \delta$ and hence $U(I) \leq U(J \times K_0) = V(J) < \varepsilon$, or else $L_{m-1}(K) < \delta$ and so $U(I) \leq U(J_0 \times K) = W(K) < \varepsilon$. We thus obtain $U(I) < \varepsilon$ in both cases, and this clearly implies that U is continuous on the interval I_0 . It follows immediately that U is continuous on the figure R , and the proof is complete.

8. Continuous part of a nonnegative additive interval-function. Given such a function U , let U_1 be the saltus-part of U and let us write $U_2 = U - U_1$. The interval-function U_2 thus defined, which is plainly nonnegative, is moreover additive on account of § 4, and it follows from § 5 and § 6 that the inside excess of U_2 vanishes identically. Consequently U_2 must be continuous by the preceding section. We are thus entitled to call U_2 *continuous part* of U .

9. Saltus-part and continuous part of an additive interval-function. Given such a function F (of bounded variation, of course, in conformity with the convention made in the introduction), let U, V, W be the three variations of F , respectively upper, lower, and absolute. It must be remarked that we use the term *lower variation* in a different sense than in the Saks treatise, inasmuch as we understand by V the upper variation (in the Saks sense) of the function $-F$. We thus have the well-known identities $F = U - V$ and $W = U + V$.

Let us write U_0, U_1, U_2 respectively for the inside excess, the saltus-part,

and the continuous part of U and let further the symbols V_i and W_i ($i=0, 1, 2$) have corresponding obvious meanings. It follows at once from § 6 that we have identically $W_i = U_i + V_i$ for $i=0, 1, 2$.

This being so, we define now for F the *inside excess* F_0 , the *saltus-part* F_1 , and the *continuous part* F_2 by $F_i = U_i - V_i$ ($i=0, 1, 2$), so that $F = F_1 + F_2$ identically. These definitions are legitimate since if F is nonnegative, then $V=0$ and hence $V_i=0$ for each i , so that we are led back to the old definitions. Thus defined, both F_1 and F_2 are clearly additive interval-functions.

10. CONTINUATION. *The functions U_i, V_i, W_i of the foregoing section are respectively the upper, lower, and absolute variations of F_i for both $i=1$ and $i=2$.*

PROOF. The following proof applies equally to both values of i . If R is an arbitrary figure in a given figure R_0 , then

$$F_i(R) = U_i(R) - V_i(R) \leq U_i(R) \leq U_i(R_0),$$

and hence we get $G_i(R_0) \leq U_i(R_0)$, where G_i denotes the upper variation of F_i .

We shall next deduce the opposite inequality $G_i(R_0) \geq U_i(R_0)$. Given an arbitrary ε there exists in R_0 , by definition of U , a figure R such that $U(R) - V(R) = F(R) > U(R_0) - \varepsilon$ and hence that $U(S) + V(R) < \varepsilon$, where we write for short $S = R_0 \ominus R$. Since $U_i(J) \leq U(J)$ and $V_i(J) \leq V(J)$ for any interval J , it follows that $U_i(S) + V_i(R) < \varepsilon$, or what comes to the same thing, that $F_i(R) = U_i(R) - V_i(R) > U_i(R_0) - \varepsilon$. But this implies that $G_i(R_0) > U_i(R_0) - \varepsilon$, whence we find $G_i(R_0) \geq U_i(R_0)$, as desired. We have thus established the identity $G_i = U_i$.

Let us now replace the function F by $-F$. Then U and V are replaced respectively by V and U , and so U_i and V_i respectively by V_i and U_i . Therefore F_i is changed into $V_i - U_i = -F_i$, and it follows from what has already been proved that V_i is the upper variation of $-F_i$, i.e. the lower variation of F_i . In view of the identity $W_i = U_i + V_i$ (see the preceding section) we infer further from the results obtained already that W_i must coincide with the absolute variation of F_i .

11. The following self-evident remark will sometimes be useful to us in the sequel. If two additive interval-functions F and G coincide identically within a figure R , that is, for every interval in R (and hence for every figure in R), then the inside excess, the saltus-part, and the continuous part of F , as well as the upper, lower, and absolute variations of F , coincide respectively with those of G identically within R .

12. Saltus-functions. Any additive interval-function which coincides with its own saltus-part will be called *saltus-function*. As we shall see below, we can also define a saltus-function as an interval-function that can be expressed as the saltus-part of some additive interval-function. We shall further term an additive interval-function F to be of *saltus type* on a figure R , iff (i.e. if and only if) F coincides with its own saltus-part identically within R . For example, every additive interval-function is of saltus type on the void figure.

13. *The saltus-part of every additive interval-function is a saltus-function. More generally, in order that an additive interval-function G be of saltus type on a figure R , it is necessary and sufficient that G should coincide within R with the saltus-part of some additive interval-function.*

PROOF. Let F be an arbitrary additive interval-function and let us use the same notation as in § 9, so that the identity $F_1 = U_1 - V_1$ is by § 10 the Jordan decomposition of the saltus-part F_1 of F . In view of § 5 this shows at once that the saltus-part of F_1 is identical with $U_1 - V_1 = F_1$. The first half of the assertion is thus established.

Suppose next that there exists for the function G an additive interval-function F such that $G = F_1$ identically within R . Then by the remark of § 11 the saltus-part G_1 of G coincides within R with the saltus-part of F_1 , that is, with F_1 itself by what has already been proved. In other words, $G_1 = G$ identically within R , and thus the condition of the assertion is sufficient. This completes the proof, since the necessity of the condition is evident.

14. The one-dimensional case. Suppose that the underlying Euclidean space R^m is the real line R^1 , and let F be any additive interval-function defined for R^1 . Then the saltus-part of F in our sense coincides with that of F in the sense of Saks.

15. Given a pair G, H of additive interval-functions, let G_0 and G_1 denote respectively the inside excess and the saltus-part of G and let H_0, H_1 be defined similarly for H . If $G_0 = H_0$ identically within a figure R , then we have also $G_1 = H_1$ within R identically.

PROOF. Write U, V respectively for the upper and lower variations of G , let U_0 and V_0 denote the inside excesses of U, V respectively, and further let U_1, V_1 be the saltus-parts of U, V respectively. Denoting by X, Y respectively the upper and lower variations of H , we define X_0, Y_0, X_1, Y_1 correspondingly. Given any interval $I \subset R$ and an arbitrary ε we can find by § 2 a subdivision $I_1, \dots, I_n$ of I into intervals satisfying $U_1(I) < U_0(I_1) + \dots + U_0(I_n) + \varepsilon$ as well as similar inequalities for V_1, X_1, Y_1 . Now by definition $G_i = U_i - V_i$ and $H_i = X_i - Y_i$ for both $i=0$ and $i=1$, and in consequence

$$\left| G_1(I) - \sum_{j=1}^n G_0(I_j) \right| < \varepsilon \quad \text{and} \quad \left| H_1(I) - \sum_{j=1}^n H_0(I_j) \right| < \varepsilon.$$

If, therefore, G_0 coincides with H_0 within R , we find at once that $|G_1(I) - H_1(I)| < 2\varepsilon$, and hence that $G_1(I) = H_1(I)$ since ε is arbitrary. This proves the assertion.

16. The inside excess F_0 , as well as the saltus-part F_1 , of any additive interval-function F which is continuous on a figure R , vanishes identically within R . F thus coincides within R with the continuous part of its own.

PROOF. Let us denote by U and V respectively the upper and lower variations of F , and by U_0 and V_0 respectively the inside excesses of U and V . Given any interval $I \subset R$ and an arbitrary ε , we can choose in the interior of I a pair of intervals I_1 and I_2 in such a manner that

$$U(I \ominus I_1) < U_0(I) + \varepsilon \quad \text{and} \quad V(I \ominus I_2) < V_0(I) + \varepsilon.$$

By continuity of F on R , there exists in the interior of I an interval $J \supset I_1 \cup I_2$ such that $|F(I \ominus J)| < \varepsilon$. We then find at once

$$U_0(I) \leq U(I \ominus J) < U_0(I) + \varepsilon \quad \text{and} \quad V_0(I) \leq V(I \ominus J) < V_0(I) + \varepsilon,$$

whence, in view of $F(I) = U(I) - V(I)$ and $F_0(I) = U_0(I) - V_0(I)$, it follows that

$$F_0(I) < F(I \ominus J) + \varepsilon < 2\varepsilon \quad \text{and} \quad F_0(I) > F(I \ominus J) - \varepsilon > -2\varepsilon.$$

Consequently $F_0(I) = 0$, since ε is arbitrary. Thus F_0 vanishes within R , and hence, by the foregoing section, F_1 must also vanish within R . This completes the proof.

17. As an application of the results obtained hitherto, we can easily prove the following known proposition, which is essentially the same as the theorem given in Saks at the end of p. 63.

The upper, lower, and absolute variations of any additive interval-function F which is continuous on a figure R , are also continuous on R .

18. If F and G are a pair of additive interval-functions, we have identically $H_0 = F_0 + G_0$, where we put $H = F + G$ and F_0, G_0, H_0 denote the inside excesses of the functions F, G, H respectively.

19. Given any pair of additive interval-functions F and G , suppose that F is continuous on a figure R and that G is of saltus type on R . Then G coincides within R with the saltus-part of the sum $H = F + G$, while F therefore coincides within R with the continuous part of H .

PROOF. We have $H_0 = F_0 + G_0$ by the preceding section, where $F_0 = 0$ identically within R in virtue of § 16. Hence the saltus-part of H and that of G must coincide within R , in accordance with § 15. But G coincides by hypothesis with its own saltus-part within R . Hence the result.

20. **Mutually singular sequences of additive interval-functions.** Let $f = (F_1, \dots, F_k)$ be a k -tuple of additive interval-functions (all defined for R^m). We shall term f *mutually singular on a figure R* , iff (i.e. if and only if) for each ε there exists a subdivision $R_1, R_2, \dots, R_k$ of R into figures such that the absolute variation W_i of F_i fulfils $W_i(R \ominus R_i) < \varepsilon$ for every $i = 1, 2, \dots, k$. In particular, f is mutually singular on every figure in the special case $k = 1$; indeed we may take $R_1 = R$ for any given ε .

21. We shall simply say that a finite sequence of additive interval-functions is *mutually singular*, iff it is so on every figure. An interesting example of this property is furnished by the pair of the upper variation U and the lower variation V of any additive interval-function F .

22. Given a pair of additive interval-functions F and G , suppose that $|F(I)| \leq G(I)$ for every interval I . Then $W(R) \leq G(R)$ for every figure R , where W stands for the absolute variation of F .

23. Given any two additive interval-functions F and G , suppose that F is continuous on a figure R and that G is of saltus type on R . Then F and G form on R a mutually singular pair.

PROOF. By § 17 the absolute variation of F is continuous on R , and it follows from §§ 10–12 that the absolute variation of G is of saltus type on R . Hence we may suppose from the first that both F and G are nonnegative. Given an arbitrary ε we can, by hypothesis and by definition of saltus-part (see § 3) of G , choose a subdivision $I_1, \dots, I_n$ of R into intervals such that $G_0(I_1) + \dots + G_0(I_n) > G(R) - \varepsilon$, where G_0 denotes the inside excess of G . Taking in the interior

of each I_i an arbitrary interval J_i , we therefore find that

$$\sum_{i=1}^n G(I_i \ominus J_i) \geq \sum_{i=1}^n G_0(I_i) > \sum_{i=1}^n G(I_i) - \varepsilon,$$

which implies that $G(S) < \varepsilon$, where we write for short $S = J_1 \cup \dots \cup J_n$. By continuity of F on R we may suppose here that $F(I_i \ominus J_i) < n^{-1}\varepsilon$ for every i and hence that $F(R \ominus S) < \varepsilon$. This completes the proof.

24. *Given a figure R and a pair of finite sequences $(F_1, \dots, F_p)$ and $(G_1, \dots, G_q)$ of additive interval-functions, suppose that the pair (F_i, G_j) is mutually singular on R for every combination of $i=1, 2, \dots, p$ and $j=1, 2, \dots, q$. Then the sums $F = F_1 + \dots + F_p$ and $G = G_1 + \dots + G_q$ form on R a mutually singular pair.*

PROOF. By replacing, if necessary, each F_i and each G_j with their respective absolute variations and by taking account of § 22, we may suppose that every F_i as well as every G_j is nonnegative. According to the definition of mutual singularity we can, given any ε , subdivide R into a pair of figures A_{ij} and B_{ij} , for every choice of the indexes i and j , in such a manner that $F_i(A_{ij}) < \varepsilon$ and $G_j(B_{ij}) < \varepsilon$. Writing $A_i = A_{i1} \cup \dots \cup A_{iq}$ and $B_i = R \ominus A_i$ for each i , we find at once that, on the one hand, $F_i(A_i) \leq F_i(A_{i1}) + \dots + F_i(A_{iq}) < q\varepsilon$ and that, on the other hand, $G_j(B_i) \leq G_j(B_{ij}) < \varepsilon$ ($j=1, 2, \dots, q$). Putting $B = B_1 \cup \dots \cup B_p$ and $A = R \ominus B$, we therefore find that $G_j(B) \leq G_j(B_1) + \dots + G_j(B_p) < p\varepsilon$ for every j and that $F_i(A) \leq F_i(A_i) < q\varepsilon$ for every i . Consequently we get $F(A) < pq\varepsilon$ and $G(B) < pq\varepsilon$, completing the proof.

25. *In order that a sequence $h = (H_1, \dots, H_k)$ of additive interval-functions be mutually singular on a figure R , it is necessary and sufficient that both $f = (F_1, \dots, F_k)$ and $g = (G_1, \dots, G_k)$ should be so on the same figure R , where F_i denotes the continuous part, and G_i the saltus-part, of H_i for each $i=1, 2, \dots, k$.*

PROOF. Let us denote for each i by U_i, V_i, W_i the absolute variations of F_i, G_i, H_i respectively. Then U_i and V_i are by § 10 respectively the continuous part and the saltus-part of W_i , so that V_i is a saltus-function in virtue of § 13. Further, mutual singularity on R of f [g , or h] is evidently equivalent to the same property of the sequence (U_i) [(V_i) , or (W_i)]. We may therefore suppose from the first that F_i, G_i, H_i are all nonnegative for each i . The necessity of the condition is then manifest.

To establish the sufficiency of the condition, let ε be arbitrarily given. In view of § 23 we infer from the result of the preceding section that the figure R can be subdivided in a pair of figures A and B subject to the requirements $F_i(B) < \varepsilon$ and $G_i(A) < \varepsilon$ for every i . Suppose now that both f and g are mutually singular on R . Then there exist two subdivisions $(A_1, \dots, A_k)$ and $(B_1, \dots, B_k)$ of R into figures, respectively fulfilling $F_i(R \ominus A_i) < \varepsilon$ and $G_i(R \ominus B_i) < \varepsilon$ for every i . If we write $C_i = (A \odot A_i) \cup (B \odot B_i)$, then the figures $C_1, \dots, C_k$ clearly form a subdivision of R and further $R \ominus C_i = (A \ominus A_i) \cup (B \ominus B_i)$. Consequently

$$F_i(R \ominus C_i) = F_i(A \ominus A_i) + F_i(B \ominus B_i) \leq F_i(R \ominus A_i) + F_i(B) < 2\varepsilon,$$

and similarly $G_i(R \ominus C_i) < 2\varepsilon$. We thus get finally

$$H_i(R \ominus C_i) = F_i(R \ominus C_i) + G_i(R \ominus C_i) < 4\varepsilon,$$

which shows that the sequence h is mutually singular on R .

26. Given a pair F, G of additive interval-functions, we shall say that F is *singular on a figure R relatively to G* (or *with respect to G*), iff F and G form on R a mutually singular pair. When this is the case for an arbitrary R , we shall usually omit the reference to the figure R and simply call F *singular relatively to G* .

In order that an additive interval-function F be of saltus type on a figure R , it is necessary and sufficient that F be singular on R relatively to every nonnegative continuous additive interval-function.

PROOF. We need only prove the sufficiency part of the assertion, the necessity part being included in the proposition of § 23. Supposing that F satisfies the stated condition, let us denote by U the upper variation of F , so that U also fulfils the condition of the assertion. In particular, U is singular on R relatively to the continuous part K of U . Given any figure $Q \subset R$ and an arbitrary ε , we can therefore subdivide Q into a pair Q_1, Q_2 of figures such that $K(Q_1) < \varepsilon$ and $U(Q_2) < \varepsilon$. The latter of these inequalities clearly implies that $K(Q_2) < \varepsilon$. Consequently we get $K(Q) = K(Q_1) + K(Q_2) < 2\varepsilon$, whence we see that K must vanish for every figure in R and that U therefore coincides within R identically with its own saltus-part. By symmetry we find further that the lower variation V of F is within R the saltus-part of V itself. It follows finally in view of the definition of saltus-part (see § 9) that the function F must be identical within R with the saltus-part of its own and so is of saltus type on R . This establishes the assertion.

27. Given any finite sequence $H_1, \dots, H_k$ of additive interval-functions, let us denote by F_i and G_i respectively the continuous part and the saltus-part of H_i for each $i=1, 2, \dots, k$. Then the sums $F=F_1+\dots+F_k$ and $G=G_1+\dots+G_k$ are respectively the continuous part and the saltus-part of the sum $H=H_1+\dots+H_k$.

PROOF. The sum F is continuous, and we have $H=F+G$ identically since $H_i=F_i+G_i$ for every i . Hence, in view of § 19, we need only show that G is a saltus-function. This may be easily seen by means of the criterion of the foregoing section. In point of fact, let us consider an arbitrary continuous additive interval-function K . Then K is singular relatively to every G_i by § 23, and hence, in virtue of § 24, K and G form a mutually singular pair. This completes the proof.

28. If a k -tuple $(F_1, \dots, F_k)$ of additive interval-functions is mutually singular on a figure Q , and if we denote by F_0 the sum $F_1+\dots+F_k$ and further by U_ν, V_ν, W_ν respectively the upper, lower, and absolute variations of F_ν for each $\nu=0, 1, \dots, k$, then we have, for every figure R in Q ,

$$U_0(R) = \sum_{i=1}^k U_i(R), \quad V_0(R) = \sum_{i=1}^k V_i(R), \quad W_0(R) = \sum_{i=1}^k W_i(R).$$

PROOF. We may clearly assume $k \geq 2$. The inequality $U_0(R) \leq U_1(R) + \dots + U_k(R)$ is obvious since for every figure S in R we have

$$F_0(S) = F_1(S) + \dots + F_k(S) \leq U_1(R) + \dots + U_k(R).$$

To obtain the opposite inequality, let ε be arbitrarily given and let us choose, as we can, a subdivision $R_1, \dots, R_k$ of R into figures such that $W_i(R \ominus R_i) < \varepsilon$

for each $i=1, \dots, k$. We find that, for any k -tuple $S_1, \dots, S_k$ of figures in R ,

$$\sum_{i=1}^k F_i(S_i) = \sum_{i=1}^k F_i(R_i \odot S_i) + \sum_{i=1}^k F_i(S_i \ominus R_i), \quad (1)$$

where the last sum fulfils

$$\sum_{i=1}^k F_i(S_i \ominus R_i) \leq \sum_{i=1}^k W_i(R \ominus R_i) < k\varepsilon. \quad (2)$$

Now write for short $T_i = R_i \odot S_i$, so that $T_1, \dots, T_k$ form a non-overlapping sequence of figures in R . Then $T_i \subset R_i \subset R \ominus R_j$ for any distinct pair i, j of positive indexes $\leq k$, and hence

$$F_i(T_i) = F_0(T_i) - \sum_{j \neq i} F_j(T_i) \leq F_0(T_i) + \sum_{j \neq i} W_j(R \ominus R_j) < F_0(T_i) + (k-1)\varepsilon.$$

It therefore follows with the help of (1) and (2) that

$$\sum_{i=1}^k F_i(S_i) < F_0\left(\bigcup_{i=1}^k T_i\right) + (k^2 - k)\varepsilon + k\varepsilon \leq U_0(R) + k^2\varepsilon.$$

Let us now subject for each $i=1, 2, \dots, k$ the arbitrary figure S_i to the condition $F_i(S_i) > U_i(R) - \varepsilon$. Then

$$\sum_{i=1}^k U_i(R) < \sum_{i=1}^k F_i(S_i) + k\varepsilon < U_0(R) + 2k^2\varepsilon,$$

and this implies the required inequality $U_0(R) \geq U_1(R) + \dots + U_k(R)$, since ε is arbitrary.

We show further analogously that $V_0(R) = V_1(R) + \dots + V_k(R)$, and it follows by addition that $W_0(R) = W_1(R) + \dots + W_k(R)$.

29. In order that two nonnegative additive interval-functions U and V be respectively the upper and lower variations of the function $F = U - V$, it is necessary and sufficient that U and V should form a mutually singular pair.

30. From now on we shall consider not only additive interval-functions as heretofore, but also additive set-functions. The latter will invariably be regarded, in accordance with the convention made in the introduction, as defined for the bounded Borel sets contained in the underlying space $\mathbf{R}^m$, which will be kept fixed throughout. As in the case of additive interval-functions, we shall understand by the lower variation of an additive set-function Φ , the upper variation of $-\Phi$ (in the Saks sense).

(i) Given any additive set-function Φ and any bounded Borel set X , there exist for every ε a bounded open set $G \supset X$ and a closed set $F \subset X$, such that we have respectively

$$W(\Phi; G - X) < \varepsilon \quad \text{and} \quad W(\Phi; X - F) < \varepsilon,$$

where the functional W denotes the absolute variation (see p. 10 of Saks).

(ii) Given a pair of additive set-functions Φ and Ψ , suppose that $\Phi(D) \geq \Psi(D)$ for every bounded open set D . Then $\Phi(X) \geq \Psi(X)$ for every bounded Borel set X . In particular, therefore, Φ and Ψ coincide identically provided they coincide for every bounded open set.

PROOF. *re (i):* Let Φ_1 and Φ_2 denote respectively the upper and lower variations of Φ . Given an arbitrary ε , it follows from the second theorem on

p. 71 of the Saks treatise and the theorem on p. 69 of the same book that there exist for each $i=1, 2$ a bounded open set $G_i \supset X$ and a closed set $F_i \subset X$ such that $\Phi_i(G_i - X) < 2^{-1}\varepsilon$ and $\Phi_i(X - F_i) < 2^{-1}\varepsilon$. Writing $G = G_1 \cap G_2$ and $F = F_1 \cup F_2$, we therefore find that $\Phi_i(G - X) < 2^{-1}\varepsilon$ and $\Phi_i(X - F) < 2^{-1}\varepsilon$ for both values of i . It follows immediately that, as required,

$$W(\Phi; G - X) = \sum_{i=1}^2 \Phi_i(G - X) < \varepsilon \quad \text{and} \quad W(\Phi; X - F) = \sum_{i=1}^2 \Phi_i(X - F) < \varepsilon.$$

re (ii): Without loss of generality we may suppose that Ψ vanishes identically. Given any bounded Borel set X , part (i) ensures for every ε the existence of a bounded open set $G \supset X$ fulfilling $|W(\Phi; G - X)| \leq W(\Phi; G - X) < \varepsilon$, so that we find $\Phi(X) = \Phi(G) - \Phi(G - X) > -\varepsilon$. This plainly yields $\Phi(X) \geq 0$, Q.E.D.

31. (i) *Given an additive interval-function F and a nonvoid bounded open set D , express D as the limit of an ascending sequence (R_n) of figures $R_1, R_2, \dots$ such that $R_n \subset R_{n+1}^\circ$ for every $n=1, 2, \dots$. Then $F(R_n) \rightarrow F^*(D)$ as $n \rightarrow \infty$ (for the notation F^* cf. the introduction).*

(ii) *If we write F for the sum of a finite sequence $F_1, \dots, F_k$ of additive interval-functions, then $F^* = F_1^* + \dots + F_k^*$ identically.*

(iii) *If F is a continuous additive interval-function, we have $F^*(R) = F(R)$ for every figure R .*

PROOF. *re (i):* Denoting by U and V respectively the upper and lower variations of F , we find by the theorem given in Saks at the top of p. 68 that, for every n ,

$$U^*(R_n) \leq U^*(R_{n+1}^\circ) \leq U(R_{n+1}) \leq U^*(R_{n+1}).$$

Since evidently $U^*(D) = \lim U^*(R_n)$, it follows at once that $U^*(D) = \lim U(R_n)$. By symmetry we get also $V^*(D) = \lim V(R_n)$. We thus have

$$F^*(D) = U^*(D) - V^*(D) = \lim [U(R_n) - V(R_n)] = \lim F(R_n).$$

re (ii): We deduce from part (i) proved already that, for every nonvoid bounded open set D ,

$$\sum_{i=1}^k F_i^*(D) = \sum_{i=1}^k \lim_n F_i(R_n) = \lim_n \sum_{i=1}^k F_i(R_n) = \lim F(R_n) = F^*(D),$$

and this implies, in virtue of § 30(ii), the required identity.

re (iii): Writing U and V for the upper and lower variations of F , we get $U^*(R) = U(R)$ and $V^*(R) = V(R)$ by the first theorem on p. 68 of Saks. Hence $F^*(R) = U^*(R) - V^*(R) = U(R) - V(R) = F(R)$.

32. *Given any additive interval-function F , let W and Ω denote respectively the absolute variations of F and F^* . Then we have $\Omega(X) \leq W^*(X)$ for every bounded Borel set X .*

PROOF. Let us denote by U and V respectively the upper and lower variations of F and by Φ and Ψ respectively the upper and lower variations of F^* . Since $W = U + V$ identically, we find by § 31(ii) the identity $W^* = U^* + V^*$. Now by definition, $\Phi(X)$ is the supremum of $F^*(Y)$ for the Borel subsets Y of X . In view of the identity $F^* = U^* - V^*$, it follows that $\Phi(X) \leq \sup U^*(Y) = U^*(X)$.

We obtain similarly $\Psi(X) \leq V^*(X)$. Thus $\Omega(X) = \Phi(X) + \Psi(X) \leq U^*(X) + V^*(X) = W^*(X)$, as required.

33. Consider an arbitrary k -tuple of additive set-functions, $\varphi = (\Phi_1, \dots, \Phi_k)$. We shall term φ *mutually singular on a bounded Borel set X* , iff there exists a decomposition (disjoint, of course) of X into k Borel subsets $X_1, \dots, X_k$ such that for every $i=1, 2, \dots, k$ the function Φ_i vanishes identically within the set $X - X_i$, i.e. that $\Phi_i(A) = 0$ for every i and every Borel set $A \subset X - X_i$. When this is the case for every X , we shall omit the reference to X and say simply that φ is *mutually singular*.

Given two additive set-functions Φ and Ψ , we shall say that Φ is *singular* [on a bounded Borel set X] *relatively to Ψ* , when and only when the pair Φ, Ψ is mutually singular [on the set X].

34. A typical example of a mutually singular pair is furnished by the upper and lower variations of any additive set-function. This is an immediate consequence of the Hahn decomposition theorem given on p. 32 of Saks.

35. (i) Given two sequences $(\Phi_1, \dots, \Phi_p)$ and $(\Psi_1, \dots, \Psi_q)$ of additive set-functions and given a bounded Borel set X , suppose that the pair (Φ_i, Ψ_j) is mutually singular on X for every combination of $i=1, 2, \dots, p$ and $j=1, 2, \dots, q$. Then the sums $\Phi = \Phi_1 + \dots + \Phi_p$ and $\Psi = \Psi_1 + \dots + \Psi_q$ form on X a mutually singular pair.

(ii) If a k -tuple $(\Theta_1, \dots, \Theta_k)$ of additive set-functions is mutually singular on a bounded Borel set X and if we denote by Θ_0 the sum $\Theta_1 + \dots + \Theta_k$ and further by Φ_i, Ψ_i, Ω_i respectively the upper, lower, and absolute variations of Θ_i for each $i=0, 1, \dots, k$, then we have within X the identities

$$\Phi_0 = \Phi_1 + \dots + \Phi_k, \quad \Psi_0 = \Psi_1 + \dots + \Psi_k, \quad \Omega_0 = \Omega_1 + \dots + \Omega_k.$$

(iii) In order that two nonnegative additive set-functions Φ and Ψ be respectively the upper and lower variations of the difference $\Phi - \Psi$, it is necessary and sufficient that Φ and Ψ should form a mutually singular pair.

36. If a pair F, G of additive interval-functions, of which we assume one at least, say F , to be continuous, is mutually singular, then so is the pair F^*, G^* of the corresponding additive set-functions.

PROOF. Mutual singularity of F and G evidently implies the same property of the absolute variations U, V of the respective functions F, G . Consequently, given any interval J and any ε , there is a figure $R \subset J$ such that $U(R) < \varepsilon$ and $V(J \ominus R) < \varepsilon$. Then, noting that U is by §17 continuous together with F , we see easily, by successive use of §32 and the first theorem on p. 68 of Saks, that $W(F^*; J^\circ \cap R) \leq U^*(J^\circ \cap R) = U(R) < \varepsilon$ as well as $W(G^*; J^\circ - R) \leq V^*(J^\circ - R) \leq V(J \ominus R) < \varepsilon$. These results in combination ensure the mutual singularity of the pair $W(F^*), W(G^*)$ on account of the first theorem on p. 32 of Saks. Hence F^* and G^* form a mutually singular pair.

37. Given a continuous additive interval-function F , let us denote by U, V, W respectively the upper, lower, and absolute variations of F . Then U^*, V^*, W^* coincide respectively with the upper, lower, and absolute variations of F^* .

PROOF. Let Φ, Ψ, Ω denote respectively the upper, lower, and absolute variations of F^* . Since U and V form a mutually singular pair as we remarked

in § 21 and since they are both continuous in conformity with § 17, we infer by the preceding section that U^* and V^* together form a mutually singular pair. In view of the identity $F^*=U^*-V^*$ it therefore follows from § 35(iii) that $\Phi=U^*$ and $\Psi=V^*$, and hence that $\Omega=\Phi+\Psi=U^*+V^*$. But $U^*+V^*=W^*$ by § 31(ii), and this completes the proof.

38. THEOREM. *Given any additive interval-function F , with W for its absolute variation, let us denote by F_1 the saltus-part of F and by W_1 the absolute variation of F_1 . Then, for every bounded Borel set X ,*

$$W^*(X)-W(F^*; X)=W_1^*(X)-W(F_1^*; X).$$

PROOF. Let us denote by F_2 the continuous part of F and by W_2 the absolute variation of F_2 . Then $F=F_1+F_2$, and we have moreover $W=W_1+W_2$ since W_1 and W_2 are by § 10 respectively the saltus-part and the continuous part of W . It follows at once from § 31(ii) that $F^*=F_1^*+F_2^*$ and $W^*=W_1^*+W_2^*$ identically. Since F_1^* and F_2^* , as well as W_1^* and W_2^* , form a mutually singular pair on account of § 23 and § 36, we find readily by § 35(ii) that $W(F^*; X)=W(F_1^*; X)+W(F_2^*; X)$ for every X . But the preceding section gives $W_2^*(X)=W(F_2^*; X)$, and we therefore obtain, as desired,

$$\begin{aligned} W^*(X)-W(F^*; X) &= W_1^*(X)+W_2^*(X)-W(F_1^*; X)-W(F_2^*; X) \\ &= W_1^*(X)-W(F_1^*; X). \end{aligned}$$

39. *Given a nonnegative saltus-function F (defined for $\mathbf{R}^m$ as heretofore), let M be the join of the hyperplanes of discontinuity of F . Then $F_*(\mathbf{R}^m-M)=0$, where we use the notation F_* in the sense explained in the introduction.*

PROOF. Let I be any fixed interval. Given an arbitrary ε , we can choose a subdivision $I_1, \dots, I_n$ of I into intervals such that $F(I) < F_0(I_1) + \dots + F_0(I_n) + \varepsilon$, where F_0 stands for the inside excess of F ; indeed, the saltus-function F coincides by definition with its own saltus-part. Now take in the interior I_i° of each I_i a variable interval J_i . Then $F_*(J_i^\circ) \leq F(J_i)$ for each i by definition of F_* (see p. 64 of Saks), and we find further that $F_*(I_i^\circ)$ is the supremum of $F_*(J_i^\circ)$. Moreover, $F_0(I_i) \leq F(I_i \ominus J_i) = F(I_i) - F(J_i)$ for every i , and so we obtain

$$\sum_{i=1}^n F(I_i) = F(I) < \sum_{i=1}^n F(I_i) - \sum_{i=1}^n F(J_i) + \varepsilon,$$

that is, $F(J_1) + \dots + F(J_n) < \varepsilon$. Consequently $F_*(J_1^\circ) + \dots + F_*(J_n^\circ) < \varepsilon$. Writing for short $N = I_1^\circ \cup \dots \cup I_n^\circ$, we therefore find that $F_*(N) = F_*(I_1^\circ) + \dots + F_*(I_n^\circ) \leq \varepsilon$.

On the other hand it is easily seen that $I^\circ - M \subset N \cup S$, where S is the join of a certain finite sequence of hyperplanes of continuity of F . But then $F_*(S) = 0$ since, as is well known, F_* vanishes for every hyperplane of continuity of F (cf. the lemma on p. 67 of Saks). We thus have

$$F_*(I^\circ - M) \leq F_*(N) + F_*(S) = F_*(N) \leq \varepsilon,$$

and this requires that $F_*(I^\circ - M) = 0$ since ε is arbitrary. Now $F_*(\mathbf{R}^m - M)$ is evidently the supremum of $F_*(I^\circ - M)$ with respect to the interval I and hence must vanish, as asserted.

40. THEOREM. *If F is an additive interval-function and W is its absolute variation, then $W(F^*; X) = W^*(X)$ for every bounded Borel set X which is disjoint*

with the hyperplanes of discontinuity of F .

PROOF. Applying the result of § 38 to F , we see at once that it suffices to verify the equality $W_1^*(X) = W(F_1^*; X)$, where F_1 denotes the saltus-part of F , while W_1 is the absolute variation of F_1 . According to the second theorem on p. 63 of Saks, the discontinuity hyperplanes for W_1 must coincide with those for F_1 . On the other hand, F_1 and F evidently have the same hyperplanes of discontinuity since $F - F_1$, being the continuous part of F , is continuous. It thus follows from the foregoing section that $W_1^*(X) = 0$. But then we also have $W(F_1^*; X) = 0$ in virtue of § 32, and this completes the proof.

Chapter II. Interval-surfaces

41. Area of an interval-surface. As we have already mentioned in the introduction, we understand by an *interval-surface* any finite sequence of additive interval-functions, say $f = (F_1, \dots, F_k)$, where the components F_i are all defined for the given space R^m . However, we shall use this term chiefly in connection with the concept of area of f that we are going to introduce presently. In the special case $k=1$ we may and do identify f with F_1 .

We call f *continuous* iff all its components F_i are continuous. Further, by the *sum* $f+g$ of a pair $f = (F_1, \dots, F_k)$ and $g = (G_1, \dots, G_k)$ of interval-surfaces, both having k components, we mean the interval-surface $(F_1 + G_1, \dots, F_k + G_k)$.

We shall call *area* of f on a figure R and denote by $f^\nabla(R)$, the supremum of the sum $|f(R_1)| + \dots + |f(R_n)|$, where $(R_1, \dots, R_n)$ denotes an arbitrary subdivision of R into figures and $|f(R_j)|$ means for each $j=1, 2, \dots, n$ the magnitude of the k -dimensional vector $f(R_j)$ with the i th component $F_i(R_j)$ ($i=1, 2, \dots, k$).

42. The area f^∇ of any interval-surface $f = (F_1, \dots, F_k)$ is a nonnegative additive interval-function when we restrict the domain of definition of f^∇ to intervals, and we have, for every figure R and every $i=1, 2, \dots, k$,

$$W_i(R) \leq f^\nabla(R) \leq W_1(R) + \dots + W_k(R),$$

where W_i stands for the absolute variation of F_i .

In particular, therefore, the area F^∇ of an additive interval-function F , i.e. of the interval-surface consisting only of F , coincides with the absolute variation of F .

43. In the same notation as in the preceding section, the area f^∇ coincides identically with the area w^∇ of the interval-surface $w = (W_1, \dots, W_k)$.

PROOF. We need only show that $f^\nabla(R) \geq w^\nabla(R)$ for every figure R , the opposite inequality being obvious. Given any ε there exists, by definition of area, a subdivision $R_1, \dots, R_n$ of R into figures such that $|w(R_1)| + \dots + |w(R_n)| > w^\nabla(R) - \varepsilon$. Now denote by U_i and V_i respectively the upper and lower variations of F_i for each i . On account of § 21 we can, for any δ and for each pair of $i=1, 2, \dots, k$ and $j=1, 2, \dots, n$, subdivide R_j into a pair of figures A_{ij} and B_{ij} such that $U_i(B_{ij}) < \delta$ and $V_i(A_{ij}) < \delta$.

This being so, let us write $c=2^k$ for short and let us keep the index j fixed for the moment. Then R_j can easily be subdivided, by what has been said in the above, into a sequence $R_j^{(1)}, \dots, R_j^{(c)}$ of figures such that for each pair of $i=1, 2, \dots, k$ and $p=1, 2, \dots, c$ we have either $U_i(S) < \delta$ or $V_i(S) < \delta$, where we

write for brevity $S=R_j^{(p)}$. We thus find that

$$W_i(S)=U_i(S)+V_i(S)<|U_i(S)-V_i(S)|+2\delta=|F_i(S)|+2\delta.$$

It follows at once, by means of the triangular inequality, that

$$|w(S)|=\left[\sum_{i=1}^k W_i^2(S)\right]^{\frac{1}{2}}<\left[\sum_{i=1}^k F_i^2(S)\right]^{\frac{1}{2}}+2k\delta=|f(S)|+2k\delta,$$

which, on being summed over j and p , gives

$$\sum_{j=1}^n \sum_{p=1}^c |w(S)| < \sum_{j=1}^n \sum_{p=1}^c |f(S)| + 2nck\delta \leq f^\nabla(R) + 2nck\delta.$$

Recalling the definition of the subdivision (R_j) , we therefore find

$$w^\nabla(R) < \sum_{j=1}^n |w(R_j)| + \varepsilon \leq \sum_{j=1}^n \sum_{p=1}^c |w(S)| + \varepsilon < f^\nabla(R) + \varepsilon + 2nck\delta,$$

and making here $\delta \rightarrow 0$ in the first place and then $\varepsilon \rightarrow 0$, we get finally $w^\nabla(R) \leq f^\nabla(R)$. This completes the proof.

44. *Given two arbitrary interval-surfaces $f=(F_1, \dots, F_p)$ and $g=(G_1, \dots, G_q)$, let us write $h=(F_1, \dots, F_p, G_1, \dots, G_q)$ and $s=(f^\nabla, g^\nabla)$. Then $h^\nabla(R)=s^\nabla(R)$ for every figure R .*

A similar result holds as well if we are given a finite number, greater than two, of interval-surfaces.

PROOF. We restrict ourselves for simplicity to the first half of the assertion, the second half being proved in almost the same way. Consider an arbitrary subdivision $R_1, \dots, R_k$ of R into figures. Then

$$\begin{aligned} \sum_{i=1}^k |h(R_i)| &= \sum_{i=1}^k \sqrt{|f(R_i)|^2 + |g(R_i)|^2} \leq \sum_{i=1}^k \sqrt{[f^\nabla(R_i)]^2 + [g^\nabla(R_i)]^2} \\ &= \sum_{i=1}^k |s(R_i)| \leq s^\nabla(R), \end{aligned}$$

and this leads at once to $h^\nabla(R) \leq s^\nabla(R)$.

On the other hand, given any figure Q and any ε , we can subdivide Q into a finite number of figures $Q_1, \dots, Q_n$ such that $|f(Q_1)| + \dots + |f(Q_n)| > f^\nabla(Q) - \varepsilon$, and we may clearly suppose that in addition $|g(Q_1)| + \dots + |g(Q_n)| > g^\nabla(Q) - \varepsilon$. We then find, on account of the triangular inequality, that

$$\begin{aligned} |s(Q)| &= \sqrt{[f^\nabla(Q)]^2 + [g^\nabla(Q)]^2} < \sum_{j=1}^n \sqrt{|f(Q_j)|^2 + |g(Q_j)|^2} + 2\varepsilon \\ &= \sum_{j=1}^n |h(Q_j)| + 2\varepsilon \leq h^\nabla(Q) + 2\varepsilon, \end{aligned}$$

whence we get readily $|s(Q)| \leq h^\nabla(Q)$.

Returning to the subdivision $R_1, \dots, R_k$ considered above, we therefore obtain the relation

$$\sum_{i=1}^k |s(R_i)| \leq \sum_{i=1}^k h^\nabla(R_i) = h^\nabla(R),$$

whence we deduce immediately $s^\nabla(R) \leq h^\nabla(R)$. This, combined with the opposite inequality established already, completes the proof.

45. Suppose given an interval-surface $f=(F_1, \dots, F_k)$, with W_i for the absolute variation of F_i ($i=1, 2, \dots, k$). In order that the sequence f be mutually singular on a figure R , it is necessary and sufficient that we should have

$$f^\nabla(R) = W_1(R) + \dots + W_k(R).$$

PROOF. In view of §43 we may assume in the sequel that the functions F_i are all nonnegative, so that $W_i=F_i$ identically for every i .

1) Necessity. If f is mutually singular on R , then we can, given an arbitrary ε , subdivide R into a sequence $R_1, \dots, R_k$ of figures such that $F_i(R \ominus R_i) < \varepsilon$ for every i . It follows that

$$f^\nabla(R) \geq \sum_{i=1}^k |f(R_i)| \geq \sum_{i=1}^k F_i(R_i) = \sum_{i=1}^k \left[F_i(R) - \sum_{j=1}^{k'} F_i(R_j) \right] > \sum_{i=1}^k F_i(R) - k^2 \varepsilon,$$

where $\sum'$ means that the summation is extended over $j \neq i$ (so that $\sum'$ reduces to zero when $k=1$). By making $\varepsilon \rightarrow 0$, we get $f^\nabla(R) \geq F_1(R) + \dots + F_k(R)$, and this combined with the opposite inequality, which holds by virtue of §42, proves the necessity of the condition.

2) Sufficiency. The condition being trivially sufficient when $k=1$, we shall begin with the case $k=2$. Assuming the equality $f^\nabla(R) = F_1(R) + F_2(R)$, let us choose, as we may, a subdivision $R_1, \dots, R_n$ of R into figures such that $|f(R_1)| + \dots + |f(R_n)| > f^\nabla(R) - \varepsilon$, where ε is any given positive number. Let us denote further by S_1 the (possibly void) join of those of the figures R_j ($j=1, 2, \dots, n$) which satisfy $F_1(R_j) \geq F_2(R_j)$, and by S_2 the join of the remaining figures R_j , so that S_1 and S_2 together form a subdivision of R . In view of the easily proved relation $b \leq 2(a+b-\sqrt{a^2+b^2})$, which is valid for every pair of real numbers a, b fulfilling $a \geq b \geq 0$, we see at once that

$$\begin{aligned} 2^{-1} F_2(S_1) &\leq \sum_{j=1}^n [F_1(R_j) + F_2(R_j) - |f(R_j)|] \\ &= F_1(R) + F_2(R) - \sum_{j=1}^n |f(R_j)| = f^\nabla(R) - \sum_{j=1}^n |f(R_j)| < \varepsilon, \end{aligned}$$

and by symmetry that $F_1(S_2) < 2\varepsilon$. Thus F_1 and F_2 are found to form on R a mutually singular pair, and the sufficiency proof is complete for the case $k=2$.

This being so, we pass on now to the general case by induction. Suppose that $k > 2$ and that, further, the sufficiency of the condition is established for every sequence of less than k additive interval-functions. Writing $g=(F_2, \dots, F_k)$ and $h=(F_1, g^\nabla)$, we find by §44 that $f^\nabla(Q) = h^\nabla(Q)$ for every figure Q . Assume now that $f^\nabla(R) = F_1(R) + \dots + F_k(R)$. Then, on account of the inequalities $g^\nabla(R) \leq F_2(R) + \dots + F_k(R)$ and $h^\nabla(R) \leq F_1(R) + g^\nabla(R)$, both of which follow from §42, we infer immediately that $g^\nabla(R) = F_2(R) + \dots + F_k(R)$ and $h^\nabla(R) = F_1(R) + g^\nabla(R)$. But these together imply by the induction hypothesis that each of the sequences g and h is mutually singular on R . Thus there exists for any ε a subdivision T_1, T of R into figures such that $F_1(R \ominus T_1) = F_1(T) < \varepsilon$ and $g^\nabla(T_1) < \varepsilon$. Moreover, g is mutually singular on T too, since $T \subset R$, and so we can subdivide T into a sequence $T_2, \dots, T_k$ of figures such that $F_i(T \ominus T_i) < \varepsilon$ for every $i=2, \dots, k$. Consequently, for the same values of i , $F_i(R \ominus T_i) = F_i(T_1) + F_i(T \ominus T_i) < 2\varepsilon$, for we have $F_i(T_1) \leq g^\nabla(T_1)$ by virtue of §42. We thus find the sequence f mutually singular on R , completing the proof of the theorem.

46. Suppose given a pair of interval-surfaces $f=(F_1, \dots, F_p)$ and $g=(G_1, \dots, G_q)$. In order that the areas f^∇ and g^∇ should form on a figure R a mutually singular pair, it is necessary and sufficient that the pair F_i, G_j be mutually singular on R for every choice of $i=1, 2, \dots, p$ and $j=1, 2, \dots, q$.

47. Given a pair of interval-surfaces f and g with the same number of components, let us write $h=f+g$. If f^∇ and g^∇ form on a figure Q a mutually singular pair, then we have $h^\nabla=f^\nabla+g^\nabla$ identically within Q .

PROOF. For any figure $R \subset Q$ and any ε there exists by hypothesis a subdivision of R into a pair of figures A and B such that $f^\nabla(B)+g^\nabla(A)<\varepsilon$. On the other hand we can clearly take a subdivision $R_1, \dots, R_n$ of R into figures, in such a way that

$$\sum_{i=1}^n |f(R_i)| > f^\nabla(R) - \varepsilon \quad \text{and} \quad \sum_{i=1}^n |g(R_i)| > g^\nabla(R) - \varepsilon. \quad (1)$$

For this subdivision (R_i) we find at once that

$$\begin{aligned} |h(A \odot R_i)| &= |f(R_i) - f(B \odot R_i) + g(A \odot R_i)| \geq |f(R_i)| - |f(B \odot R_i)| - |g(A \odot R_i)| \\ &\geq |f(R_i)| - f^\nabla(B \odot R_i) - g^\nabla(A \odot R_i), \end{aligned}$$

and it follows by symmetry that

$$|h(B \odot R_i)| \geq |g(R_i)| - f^\nabla(B \odot R_i) - g^\nabla(A \odot R_i).$$

On combining these results and on taking into account (1) as well as $f^\nabla(B)+g^\nabla(A)<\varepsilon$, we are immediately led to

$$\begin{aligned} h^\nabla(R) &\geq \sum_{i=1}^n |h(A \odot R_i)| + \sum_{i=1}^n |h(B \odot R_i)| \\ &\geq \sum_{i=1}^n |f(R_i)| + \sum_{i=1}^n |g(R_i)| - 2f^\nabla(B) - 2g^\nabla(A) > f^\nabla(R) + g^\nabla(R) - 4\varepsilon, \end{aligned}$$

whence we obtain, by making $\varepsilon \rightarrow 0$, $h^\nabla(R) \geq f^\nabla(R) + g^\nabla(R)$.

On the other hand, we have for any subdivision $S_1, \dots, S_n$ of R ,

$$\sum_{i=1}^n |h(S_i)| = \sum_{i=1}^n |f(S_i) + g(S_i)| \leq \sum_{i=1}^n |f(S_i)| + \sum_{i=1}^n |g(S_i)| \leq f^\nabla(R) + g^\nabla(R),$$

whence it follows readily that $h^\nabla(R) \leq f^\nabla(R) + g^\nabla(R)$. This, in conjunction with the opposite inequality established just now, proves the theorem.

48. The area f^∇ of an interval-surface $f=(F_1, \dots, F_k)$ is continuous [or of saltus type] on a figure R if, and only if, the functions $F_1, \dots, F_k$ are all continuous [or of saltus type] on R .

REMARK. When this is the case, we shall say that f is continuous [of saltus type] on R .

PROOF. 1) Continuity on R of f^∇ plainly implies the same property for every F_i since by §42 we have $W_i(Q) \leq f^\nabla(Q)$ for every figure Q , where W_i denotes the absolute variation of F_i for $i=1, 2, \dots, k$. Suppose conversely every F_i continuous on R . Then every W_i is continuous on R by §17 and hence so is the sum $W=W_1+\dots+W_k$. But $f^\nabla(Q) \leq W(Q)$ for every figure Q in virtue of §42. It follows at once that f^∇ is continuous on R .

2) Let U be an arbitrary additive interval-function which is both non-negative and continuous. The inequalities $W_i(Q) \leq f^\nabla(Q)$ ($i=1, 2, \dots, k$), quoted above, then imply that if f^∇ is singular on R relatively to U , so is F_i for every i . On the other hand, §46 shows that f^∇ is singular on R relatively to U whenever every F_i is so too. It follows readily from §26 that f^∇ is of saltus type on R when and only when every F_i is of saltus type on R .

49. Given any interval-surface $h=(H_1, \dots, H_k)$, let us denote by F_i and G_i respectively the continuous part and the saltus-part of H_i for $i=1, 2, \dots, k$ and let us write $f=(F_1, \dots, F_k)$ and $g=(G_1, \dots, G_k)$. Then the areas f^∇ and g^∇ are respectively the continuous part and the saltus-part of the area h^∇ . In particular, continuity of f^∇ is equivalent to continuity of f .

PROOF. It follows at once from the preceding section that f^∇ is continuous and that g^∇ is a saltus-function. Now, in virtue of §23, every F_i and every G_j form together a mutually singular pair, and so it follows from §46 that the pair f^∇, g^∇ must also be mutually singular. We then deduce from §47 that the identity $h^\nabla = f^\nabla + g^\nabla$ is valid, whence the assertion follows immediately on account of §19.

REMARK. We shall call f and g respectively the *continuous part* and the *saltus-part* of the interval-surface h .

50. Area-measure of a set-surface. By a *set-surface* we shall understand any finite sequence, say $\varphi=(\Phi_1, \dots, \Phi_k)$, of additive set-functions, as we have already mentioned in the introduction. We may identify φ with Φ_1 in the special case $k=1$. The term set-surface will be chiefly used in connection with the concept of *area-measure* which we define as follows. We shall call area-measure of φ over a bounded Borel set X and denote by $\varphi^\nabla(X)$, the supremum of the sum $|\varphi(X_1)| + \dots + |\varphi(X_n)|$ for the decompositions $(X_1, \dots, X_n)$ of X into Borel subsets. Further, by the *sum* $\varphi + \psi$ of a pair of set-surfaces $\varphi=(\Phi_1, \dots, \Phi_k)$ and $\psi=(\Psi_1, \dots, \Psi_k)$ with the same number of components we shall mean the set-surface $(\Phi_1 + \Psi_1, \dots, \Phi_k + \Psi_k)$.

51. (i) The area-measure φ^∇ of any set-surface $\varphi=(\Phi_1, \dots, \Phi_k)$ is a non-negative additive set-function and we have, for every bounded Borel set X ,

$$\Omega_i(X) \leq \varphi^\nabla(X) \leq \Omega_1(X) + \dots + \Omega_k(X),$$

where Ω_i stands for the absolute variation of Φ_i ($i=1, 2, \dots, k$).

In particular, therefore, the area-measure Φ^∇ of an additive set-function Φ , i.e. of the set-surface consisting only of Φ , coincides identically with the absolute variation of Φ .

(ii) In the same notation as in the above, the area-measure φ^∇ coincides identically with the area-measure ω^∇ of the set-surface $\omega=(\Omega_1, \dots, \Omega_k)$.

(iii) Given any two set-surfaces $\varphi=(\Phi_1, \dots, \Phi_p)$ and $\psi=(\Psi_1, \dots, \Psi_q)$, let us write $\theta=(\Phi_1, \dots, \Phi_p, \Psi_1, \dots, \Psi_q)$ and $\omega=(\varphi^\nabla, \psi^\nabla)$. Then we have the identity $\theta^\nabla = \omega^\nabla$.

(iv) Suppose given a set-surface $\varphi=(\Phi_1, \dots, \Phi_k)$, with Ω_i for the absolute variation of Φ_i ($i=1, 2, \dots, k$). In order that the sequence φ be mutually singular on a bounded Borel set X , it is necessary and sufficient that we should have

$$\varphi^\nabla(X) = \mathcal{Q}_1(X) + \cdots + \mathcal{Q}_k(X) .$$

(v) Suppose given a pair of set-surfaces $\varphi = (\Phi_1, \dots, \Phi_p)$ and $\psi = (\Psi_1, \dots, \Psi_q)$. In order that the area-measures φ^∇ and ψ^∇ should form on a bounded Borel set X a mutually singular pair, it is necessary and sufficient that the pair Φ_i, Ψ_j be mutually singular on X for every choice of $i=1, 2, \dots, p$ and $j=1, 2, \dots, q$.

(vi) Given a pair of set-surfaces φ and ψ with the same number of components, let us write $\omega = \varphi + \psi$. If φ^∇ and ψ^∇ form on a bounded Borel set X a mutually singular pair, then we have $\omega^\nabla = \varphi^\nabla + \psi^\nabla$ identically within X .

52. In order that a sequence $\varphi = (\Phi_1, \dots, \Phi_k)$ of additive set-functions be mutually singular on a bounded Borel set X , it is necessary and sufficient that, given an arbitrary ε , we can decompose X into a (disjoint) sequence of Borel sets $X_1, \dots, X_k$ such that

$$W(\Phi_i; X - X_i) < \varepsilon \quad (i=1, 2, \dots, k) . \quad (1)$$

PROOF. The condition is clearly necessary. To ascertain it also sufficient, we may by § 51(ii) suppose every Φ_i nonnegative, and we shall carry out the proof by induction on the number k . The case $k=1$ being trivial and the case $k=2$ being given in Saks at the beginning of p. 32, let us assume that $k \geq 3$ and that the condition of the proposition is sufficient for any sequence of less than k additive set-functions.

Supposing now that the condition is fulfilled by the sequence $\varphi = (\Phi_1, \dots, \Phi_k)$, let us write $\psi = (\Phi_2, \dots, \Phi_k)$. We shall begin by observing that the sequence ψ is mutually singular on every Borel subset Y of X . In point of fact, the inequalities (1) give us $\Phi_i(Y - X_i \cap Y) \leq \Phi_i(X - X_i) < \varepsilon$ for every $i=1, 2, \dots, k$ on account of $Y - X_i \cap Y = Y - X_i \subset X - X_i$, and consequently, by the induction hypothesis, ψ must be mutually singular on Y , since $X_1 \cap Y, \dots, X_n \cap Y$ plainly form a decomposition of Y into Borel subsets.

This being so, we shall next deduce that Φ_1 and ψ form on X a mutually singular pair. Indeed, in view of § 51(i), it follows from (1) that

$$\varphi^\nabla(X_1) \leq \sum_{i=2}^k \Phi_i(X_1) \leq \sum_{i=2}^k \Phi_i(X - X_i) < k\varepsilon ,$$

which in conjunction with $\Phi_1^\nabla(X - X_1) = \Phi_1(X - X_1) < \varepsilon$ ensures the mentioned property of the pair Φ_1, ψ in virtue of the case $k=2$ of our proposition.

We may thus take a Borel subset A of X so as to fulfil $\Phi_1(A) = \psi^\nabla(X - A) = 0$. Now ψ is mutually singular on A by what has already been proved, and this enables us to decompose A into a sequence of Borel subsets $A_2, \dots, A_k$ such that we have $\Phi_i(A - A_i) = 0$ for $i=2, 3, \dots, k$. For the same values of i we then get $\Phi_i(X - A_i) = \Phi_i(X - A) + \Phi_i(A - A_i) = 0$, since $\Phi_i(X - A)$, which does not exceed $\psi^\nabla(X - A) = 0$, must vanish. Consequently we see, on writing $A_1 = X - A$, that the decomposition $A_1, \dots, A_k$ of X exhibits the mutual singularity on X of the sequence φ . This completes the proof.

53. Given a set-surface $\varphi = (\Phi_1, \dots, \Phi_k)$, we define on the class of all bounded open sets G in $\mathbf{R}^m$ a finite nonnegative set-function $\tilde{\varphi}$, whose value for the void set is zero by definition and whose value for a nonvoid G we determine as the supremum of the sum $|\varphi(I_1)| + \cdots + |\varphi(I_n)|$, where $(I_1, \dots, I_n)$ represents an arbitrary finite disjoint sequence of intervals in G .

We then have $|\varphi(G)| \leq \tilde{\varphi}(G)$ for every bounded open set G .

PROOF. A hyperplane H in R^m (orthogonal to one of the axes, in accordance with a convention made in Saks on p. 57) will for the nonce be called to be *admissible*, iff every component Φ_i of φ vanishes identically within H . Obviously there is at most a countable infinity of non-admissible hyperplanes. Assuming G nonvoid as we may, we can therefore express G , by the second theorem on p. 58 of Saks, as the join of a disjoint sequence of half-open cubes whose faces lie each on an admissible hyperplane. Given an arbitrary ε , we can choose a finite sequence of these half-open cubes, say $K_1, \dots, K_n$, in such a manner that $|\varphi(K_1)| + \dots + |\varphi(K_n)| > |\varphi(G)| - \varepsilon$. Now there plainly exists in each K_i an interval I_i which fulfils $|\varphi(I_i)| > |\varphi(K_i)| - n^{-1}\varepsilon$, so that we find directly $|\varphi(I_1)| + \dots + |\varphi(I_n)| > |\varphi(G)| - 2\varepsilon$. The intervals I_i being disjoint by construction, it follows at once that $\tilde{\varphi}(G) > |\varphi(G)| - 2\varepsilon$, which completes the proof since ε is arbitrary.

54. Let $\varphi = (\Phi_1, \dots, \Phi_k)$ be any set-surface. For every ε and for every non-void bounded open set G , the area-measure $\varphi^\nabla(G)$ is equal to the supremum of the sum $|\varphi(I_1)| + \dots + |\varphi(I_n)|$, where $(I_1, \dots, I_n)$ denotes an arbitrary finite disjoint sequence of intervals, contained in G and having diameters less than ε . We thus have a fortiori the identity $\varphi^\nabla(G) = \tilde{\varphi}(G)$.

PROOF. Let us denote for the moment by $\varphi_0(G)$ the supremum mentioned in the proposition. We need only show that $\varphi^\nabla(G) \leq \varphi_0(G)$, the opposite inequality being obvious. For this purpose, let δ be arbitrarily given and let us choose a decomposition $X_1, \dots, X_n$ of G into nonvoid Borel sets with diameters $< 3^{-1}\varepsilon$, such that $|\varphi(X_1)| + \dots + |\varphi(X_n)| > \varphi^\nabla(G) - \delta$. According to § 30(i), there is in each X_j a nonvoid closed set Q_j such that $|\Phi_i(X_j - Q_j)| < (kn)^{-1}\delta$ for every $i=1, 2, \dots, k$. Then $|\Phi_i(X_j)| < |\Phi_i(Q_j)| + (kn)^{-1}\delta$ and hence we get $|\varphi(X_j)| < |\varphi(Q_j)| + n^{-1}\delta$. It thus follows that $|\varphi(Q_1)| + \dots + |\varphi(Q_n)| > \varphi^\nabla(G) - 2\delta$.

Noting that the sets Q_j have diameters less than $3^{-1}\varepsilon$, let us now enclose each Q_j in an open set $G_j \subset G$ with diameter $< \varepsilon$, in such a manner that $|\Phi_i(G_j - Q_j)| < (kn)^{-1}\delta$ for every $i=1, 2, \dots, k$. This is possible in virtue of § 30(i). Here we may clearly suppose in addition that the sets $G_1, \dots, G_n$ are disjoint. We deduce that $|\Phi_i(Q_j)| < |\Phi_i(G_j)| + (kn)^{-1}\delta$ and hence that $|\varphi(Q_j)| < |\varphi(G_j)| + n^{-1}\delta$, whence we obtain $|\varphi(G_1)| + \dots + |\varphi(G_n)| > \varphi^\nabla(G) - 3\delta$. Consequently we find by means of the preceding section that

$$\sum_{j=1}^n \tilde{\varphi}(G_j) \geq \sum_{j=1}^n |\varphi(G_j)| > \varphi^\nabla(G) - 3\delta.$$

On the other hand it follows from the definition of $\tilde{\varphi}$ that there exists a disjoint sequence of intervals $I_1, \dots, I_p$, contained each in one of the open sets G_j and further such that

$$|\varphi(I_1)| + \dots + |\varphi(I_p)| > \tilde{\varphi}(G_1) + \dots + \tilde{\varphi}(G_n) - \delta.$$

Since these intervals clearly have diameters smaller than ε , we find immediately

$$\varphi_0(G) \geq |\varphi(I_1)| + \dots + |\varphi(I_p)| > \varphi^\nabla(G) - 4\delta,$$

whence we get $\varphi_0(G) \geq \varphi^\nabla(G)$ by making $\delta \rightarrow 0$. The proof is thus complete.

55. Two additive set-functions Φ and Ψ coincide identically provided they coincide for every interval.

PROOF. Without loss of generality we may suppose that $\mathcal{V}$ vanishes identically. Consider the area-measure $\mathcal{O}^\nabla$, which, by the bye, is identical with the absolute variation of $\mathcal{O}$ as we have seen in §51(i). Then, by the preceding section, $\mathcal{O}^\nabla$ vanishes for every bounded open set. It therefore follows from §30(ii) that $\mathcal{O}^\nabla$, and hence also $\mathcal{O}$, must vanish identically. This completes the proof.

REMARK. This result includes the proposition III.1.47 of Radó [2].

56. THEOREM. *Given any interval-surface $f=(F_1, \dots, F_k)$, let us write $f^*=(F_1^*, \dots, F_k^*)$. Then $f^{\nabla*}(X) \geq f^*(X)$ for every bounded Borel set X , where $f^{\nabla*}$ and f^* mean respectively the set-functions $(f^\nabla)^*$ and $(f^*)^\nabla$.*

PROOF. In view of §30(ii) it is enough to show that $f^{\nabla*}(D) \geq f^*(D)$ for every nonvoid bounded open set D . This reduces on account of §54 to ascertaining that $f^{\nabla*}(D) \geq |f^*(I_1)| + \dots + |f^*(I_n)|$ for every disjoint sequence $I_1, \dots, I_n$ of intervals in D . Let us choose in D another disjoint sequence $K_1, \dots, K_n$ of intervals such that I_j is contained in the interior of K_j for each $j=1, 2, \dots, n$. Noting that $f^{\nabla*}(K_j) \geq f^\nabla(K_j)$ for each j in virtue of the first theorem on p. 68 of Saks, we have

$$f^{\nabla*}(D) \geq \sum_{j=1}^n f^{\nabla*}(K_j) \geq \sum_{j=1}^n f^\nabla(K_j).$$

It thus suffices to deduce the inequality $f^\nabla(K) \geq |f^*(I)|$ for every pair I, K of intervals such that I lies in the interior of K .

For this purpose, denote by W_i the absolute variation of F_i for each $i=1, 2, \dots, k$ and write $w=(W_1, \dots, W_k)$, so that $f^\nabla(K)=w^\nabla(K)$ by §43. But we have evidently $w^\nabla(K) \geq |w(K)|$, and the theorem quoted from Saks just now shows that $|w(K)| \geq |w^*(I)|$. Consequently $f^\nabla(K) \geq |w^*(I)|$. Now $W_i^*(I) \geq W(F_i^*; I) \geq |F_i^*(I)|$ for every i by §32, whence $|w^*(I)| \geq |f^*(I)|$. We thus get $f^\nabla(K) \geq |f^*(I)|$, completing the proof.

57. THEOREM. *We have the identity $f^{\nabla*}=f^*$ for every continuous interval-surface f .*

PROOF. In view of the foregoing section we need only prove that $f^{\nabla*}(X) \geq f^*(X)$ for every bounded Borel set X . For this purpose it is enough, on account of §30(ii), to show that $f^{\nabla*}(D) \geq f^*(D)$ for every nonvoid bounded open set D . Given any ε we find by §31(i) that there is a figure R in D fulfilling $f^\nabla(R) > f^{\nabla*}(D) - \varepsilon$. By definition of f^∇ , we can further choose a subdivision $I_1, \dots, I_n$ of R into intervals such that $|f(I_1)| + \dots + |f(I_n)| > f^\nabla(R) - \varepsilon$. Now, since f is continuous, we have $f^*(I)=f(I)$ for every interval I by §31(iii), so that there exists in the interior of each I_i ($i=1, 2, \dots, n$) an interval J_i subject to the condition $|f^*(J_i)| > |f(I_i)| - n^{-1}\varepsilon$. Thus

$$f^{\nabla*}(D) \geq \sum_{i=1}^n |f^*(J_i)| > \sum_{i=1}^n |f(I_i)| - \varepsilon > f^\nabla(R) - 3\varepsilon,$$

whence we get at once $f^{\nabla*}(D) \geq f^*(D)$ by making $\varepsilon \rightarrow 0$.

58. THEOREM. *Given any interval-surface $h=(H_1, \dots, H_k)$, let g denote the saltus-part of h (cf. the remark at the end of §49). Then, for every bounded Borel set X ,*

$$h^{\nabla*}(X) - h^{*\nabla}(X) = g^{\nabla*}(X) - g^{*\nabla}(X).$$

PROOF. Writing f for the continuous part of h , we find by § 49 that f^{∇} and g^{∇} are respectively the continuous part and the saltus-part of h^{∇} . Hence the pair f^{∇}, g^{∇} is mutually singular on account of § 23, and so we have $h^{\nabla*} = f^{\nabla*} + g^{\nabla*}$ identically by virtue of § 31(ii). For the same reason we also have the identity $h^{*} = f^{*} + g^{*}$. On the other hand, the components of f^{*} are singular relatively to every component of g^{*} on account of § 36 and § 23, and hence it follows from § 51(v) that $f^{*\nabla}$ and $g^{*\nabla}$ form a mutually singular pair. Therefore, by § 51(vi), we have $h^{*\nabla} = f^{*\nabla} + g^{*\nabla}$ identically. It follows on account of the preceding section that, as required,

$$h^{\nabla*} - h^{*\nabla} = (f^{\nabla*} + g^{\nabla*}) - (f^{*\nabla} + g^{*\nabla}) = g^{\nabla*} - g^{*\nabla}.$$

59. THEOREM. *If $f = (F_1, \dots, F_k)$ is an interval-surface, we have $f^{\nabla*}(X) = f^{*\nabla}(X)$ for any bounded Borel set X which is disjoint for each $i = 1, 2, \dots, k$ with the hyperplanes H of discontinuity of F_i .*

PROOF. Denoting by g the saltus-part of f , we find on account of the preceding section and § 56 that it is enough to verify $g^{\nabla*}(X) = 0$. Now the interval-functions f^{∇} and g^{∇} have the same hyperplanes of discontinuity, since g^{∇} is the saltus-part of f^{∇} in virtue of § 49. On the other hand, we easily see with the help of § 42 that every discontinuity hyperplane of f^{∇} must coincide for some index i with an H considered in the theorem (cf. the second theorem on p. 63 of Saks). The set X is therefore disjoint with every hyperplane of discontinuity of g^{∇} , and hence the desired relation $g^{\nabla*}(X) = 0$ is an immediate consequence of § 39.

60. *A sequence $f = (F_1, \dots, F_k)$ of nonnegative additive interval-functions is mutually singular whenever the sequence $f^{*} = (F_1^{*}, \dots, F_k^{*})$ of the associated set-functions is mutually singular.*

PROOF. It suffices to ascertain that f is mutually singular on every interval I at whose boundary the oscillation of the area f^{∇} vanishes; in fact, every figure can be enclosed in such an interval since there is at most a countable infinity of hyperplanes of discontinuity for each additive interval-function (*vide* the first theorem on p. 63 of Saks). It follows from the theorem at the top of p. 68 of Saks that $f^{\nabla}(I) = f^{\nabla*}(I)$ and that $F_i(I) \leq F_i^{*}(I)$ for every $i = 1, 2, \dots, k$. We therefore find by using successively §§ 42, 51(iv), 56 that

$$f^{\nabla}(I) \leq \sum_{i=1}^k F_i(I) \leq \sum_{i=1}^k F_i^{*}(I) = f^{*\nabla}(I) \leq f^{\nabla*}(I) = f^{\nabla}(I),$$

which obviously requires that $f^{\nabla}(I) = F_1(I) + \dots + F_k(I)$. Therefore f is mutually singular on I by § 45. This completes the proof.

61. We shall now prove a result which extends the second theorem on p. 71 of Saks to the case of an arbitrary additive set-function.

Given any additive set-function Θ , there is always an additive interval-function F such that $\Theta = F^{}$ identically.*

PROOF. Let us denote by Φ and Ψ respectively the upper and lower variations of Θ , so that Φ and Ψ form a mutually singular pair as we have seen in § 34. Now, by the theorem quoted from Saks just now, there exists a pair U ,

V of nonnegative additive interval-functions such that $\phi = U^*$ and $\psi = V^*$ identically. Consequently U and V must form a mutually singular pair, on account of the foregoing section. It then follows from § 29 that U and V are respectively the upper and lower variations of the interval-function $F = U - V$, and therefore that $F^* = U^* - V^* = \phi - \psi = \theta$ identically, Q.E.D.

62. The inequality of Steiner. For the origin and the significance of this we refer the reader to I.17 of Radó [1] and III.4.6 of Radó [2].

(i) *Given a pair φ, ψ of set-surfaces with the same number of components, write $\omega = \varphi + \psi$. Then $\omega^\nabla(X) \leq \varphi^\nabla(X) + \psi^\nabla(X)$ for every bounded Borel set X .*

(ii) *Similarly, if we denote by h the sum of any pair f, g of interval-surfaces with the same number of components, we have $h^\nabla(R) \leq f^\nabla(R) + g^\nabla(R)$ for every figure R .*

63. Case of equality in the Steiner inequality. *Given any pair h_1, h_2 of interval-surfaces with the same number of components, let us write $h_3 = h_1 + h_2$ and let f_i and g_i denote respectively the continuous part and the saltus-part of h_i for each $i = 1, 2, 3$. In order that $h_3^\nabla(R) = h_1^\nabla(R) + h_2^\nabla(R)$ should hold for a given figure R , it is necessary and sufficient that we should have $f_3^\nabla(R) = f_1^\nabla(R) + f_2^\nabla(R)$ and $g_3^\nabla(R) = g_1^\nabla(R) + g_2^\nabla(R)$ simultaneously.*

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