Mappings of finite order and dimension theory

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§ 1. Introduction

Among several functions called dimension the following two definitions of dimension are surely the most important ones for non-separable spaces. One is the covering dimension due to H. Lebesgue defined as follows.

DEFINITION 1.1. Let R be a normal space. R has the *covering dimension* $\leq n^{(1)}$, $\dim R \leq n$, if the following condition is satisfied.

(1.1) For any finite open covering $\mathfrak{G}$ there exists an open covering $\mathfrak{H}$ with order $\mathfrak{H}^{(2)} \leq n+1$ such that $\mathfrak{H}$ refines $\mathfrak{G}$.³⁾ We call $\dim R = n$ if $\dim R \leq n$ is true and $\dim R \leq n-1$ is false. When $R = \phi$ = the empty set, we call $\dim \phi = -1$. When there is no such n as $\dim R \leq n$, we call $\dim R = \infty$.

Another is the large inductive dimension due to E. Čech defined as follows.

DEFINITION 1.2. Let R be a normal space. The empty set ϕ has the *large inductive dimension* -1 , $\text{Ind } \phi = -1$. R has the large inductive dimension $\leq n$, $\text{Ind } R \leq n$, if

(1.2) For any pair of a closed set F and an open set G with $F \subset G$ there exists an open set U with $F \subset U \subset G$ such that $\text{Ind } (\bar{U} - U) \leq n-1$. We call $\text{Ind } R = n$ if $\text{Ind } R \leq n$ is true and $\text{Ind } R \leq n-1$ is false. When there is no such n , we call $\text{Ind } R = \infty$.

Our notion "Ind" is the same as "ind dim" of K. Morita [16] or " d_2 " of W. Hurewicz and H. Wallman [8, Appendix].

1) Throughout this paper n denotes an integer with $n \geq -1$.

2) order $\mathfrak{H}$ denotes the order of $\mathfrak{H}$, i.e., the largest integer m such that there are m elements of $\mathfrak{H}$ which have a non-empty intersection. When there does not exist such m , order $\mathfrak{H} = \infty$.

3) Henceforth $\mathfrak{G} > \mathfrak{H}$ denotes the fact that $\mathfrak{H}$ refines $\mathfrak{G}$. When every element of $\mathfrak{H}$ is contained in some element of $\mathfrak{G}$, we say that $\mathfrak{H}$ refines $\mathfrak{G}$ or $\mathfrak{H}$ is a refinement of $\mathfrak{G}$.

It is to be noted that the condition (1.1) used in the definition of the covering dimension can be replaced with any one of the following conditions (cf. C. H. Dowker [3] or K. Morita [14; 15]).

(1.3) Any locally finite⁴⁾ open covering $\mathfrak{G}$ can be refined by an open covering $\mathfrak{H}$ of order $\leq n+1$.

(1.4) Any locally finite open covering $\mathfrak{G}=\{G_\alpha; \alpha \in A\}$ can be refined by an open covering $\mathfrak{H}=\{H_\alpha; \alpha \in A\}$ of order $\leq n+1$ such that $H_\alpha \subset G_\alpha$ for every $\alpha \in A$.

Although we were able to set up a definition of dimension for non-separable spaces, yet it was not possible to establish a theory of dimension for such general spaces (cf. W. Hurewicz and H. Wallman [8, Appendix]). Thus the dimension theory had been obliged to confine itself to only separable metric spaces. Recently the dimension theory for general metric spaces has been established by M. Katětov [10] and by K. Morita [16] independently with different proofs. They have shown the equivalence of the converging dimension and the large inductive dimension for general metric spaces and extended the fundamental theorems of dimension theory to such spaces.

The principal contribution of this paper is to furnish the third systematic development of dimension theory, other than Katětov and Morita's, for general metric spaces. The full use of closed mappings of finite order will simplify greatly our theory, where the order of mappings is defined as follows.

DEFINITION 1.3. Let f be a mapping of a topological space R into another topological space S and x be a point of S . The *order* of f at x , $\text{order}(x, f)$, is $|f^{-1}(x)|$ (=the cardinal number of the set $f^{-1}(x)$). The *order* of f , $\text{order } f$, is $\sup \{|f^{-1}(x)|; x \in S\}$. When $\text{order}(x, f)$ is finite at every point $x \in S$, we call that the order of f is *point-wise finite*.

A characterization of zero-dimensional metric spaces which is essentially the same as in Katětov [10] and in Morita [16] is given in §2 for the sake of completeness and of preparation for the next paragraph.

In §3 the fundamental six propositions which characterize n -dimensionality of metric spaces will be proved to be equivalent to each other (Theorem 3.2) with the aid of Theorem 3.1 which asserts that a closed continuous image S of a 0-dimensional metric space R under a mapping f such that $\text{order}(y, f)=k(<\infty)$ for any point $y \in S$ has dimension 0. Theorem 3.1 can be generalized for the case of n -dimension (Theorem 3.7)⁵⁾. Although the proof of Theorem 3.7 is more simple, with the aid of other results, than that of Theorem 3.1, rather complicated proof of Theorem 3.1 is self-contained and of some interest by itself. Two sorts of covering theorems will also be stated in Theorems 3.5 and 3.6. These theorems have been proved independently by Prof. Morita and are closely related with works of Morita [16, §9; 18] and of Katětov [11, Sect. 2].

§4 deals with an open mapping and dimension. It will be shown that an open continuous mapping whose order is point-wise finite cannot increase dimension for some class of non-metrizable spaces.

§5 gives a definition of the cut and a proof of the existence of it. Some

4) A collection $\mathfrak{K}$ of subsets of R is called locally finite if for every point $x \in R$ there is a neighborhood of x which meets only a finite number of elements of $\mathfrak{K}$.

5) Theorem 3.7 is proved by J. Suzuki [28] independently.

properties of it is also studied. The cut and Theorem 3.5 will be the main instrument to solve the problem dealt with in the next paragraph.

In § 6 we construct a metric space T with $\dim T = m$ for an arbitrary metric space R with $\dim R = n$, $0 < n (< \infty)$, and for an arbitrary integer m with $0 \leq m < n$ such that there exists a closed continuous mapping π_0 of T onto R with order $\pi_0 = n - m + 1$. This is an answer for W. Hurewicz's problem for general metric spaces.

The following is the well-known P. Alexandroff's approximation theorem (cf. Hurewicz and Wallman [8, p. 72]).

In order that a compact metric space R has dimension $\leq n$ it is necessary and sufficient that for every positive ε there is an ε -mapping of R into a finite polytope of dimension $\leq n$.

It is also well-known that this theorem can be expressed in the following form.

In order that a compact metric space R has dimension $\leq n$ it is necessary and sufficient that there exists a sequence of open coverings $\mathfrak{U}_i$, $i = 1, 2, \dots$, of R which satisfies the following conditions.

(1.5) order $\mathfrak{U}_i \leq n + 1$ for every i .

(1.6) mesh $\mathfrak{U}_i^{(6)} \rightarrow 0$ as $i \rightarrow \infty$.

Whether this proposition is valid for non-compact (even separable) metric spaces is a famous Alexandroff's problem. Recently this problem has been solved in the negative by K. Sitnikov [27] who has constructed an example of a metric space R with $\dim R = 2$ whose metric dimension⁷⁾ is 1. More recently M. Katětov [12] has proved that if R is a metric space, then $\text{met dim } R \leq \dim R \leq 2 \text{ met dim } R$. The reason why the existence of a sequence satisfying the above conditions characterizes n -dimensionality of a compact metric space is as follows: Let $\mathfrak{U}_i$, $i = 1, 2, \dots$, be such a sequence of a compact metric space R . Let $\mathfrak{F}_i$ be a finite subcovering of $\mathfrak{U}_i$. Then there exists a subsequence $\mathfrak{F}_{i'}$, $i = 1, 2, \dots$, such that $\mathfrak{F}_{i'} > \mathfrak{F}_{(i+1)'}^{(8)}$, for every i . Then the subsequence $\mathfrak{F}_{i'}$, $i = 1, 2, \dots$, satisfies the condition (B) of § 3, which proves $\dim R \leq n$ by Theorem 3.2. By these observations we know that the equivalence of (A) and (B) of Theorem 3.2 is a generalization of Alexandroff's theorem cited above. Besides ours characterizations of dimension by means of a sequence of coverings have already been given by J. Nagata [24, Theorem 3] and by C. H. Dowker and W. Hurewicz [6]. In § 7 we shall give weaker conditions imposed on a sequence which characterize dimension than those in (B) of § 3, J. Nagata [24, Theorem 3] and C. H. Dowker and W. Hurewicz [6] (Theorems 7.2 and 7.5).

In § 8 we observe the dimension of subordinate metric spaces (as for the definition see Definition 8.1) as an application of § 7.

The existence of a dimension-preserving completion of a metric space has been proved by Katětov [10, Theorem 3.10] and by Morita [16, Theorem 5.6] independently. In § 9 we shall execute to construct a dimension-preserving

6) The mesh of a collection $\mathfrak{U}$ of subsets of R is the largest of the diameters of the elements of $\mathfrak{U}$.

7) When there is a sequence of open coverings $\mathfrak{U}_i$, $i = 1, 2, \dots$, of a metric space R satisfying (1.5) and (1.6), we call that R has the metric dimension $\leq n$, $\text{met dim } R \leq n$. When $\text{met dim } R \leq n$ is true and $\text{met dim } R \leq n - 1$ is false, we call that $\text{met dim } R = n$.

8) $\bar{\mathfrak{U}}$ denotes $\{\bar{U}; U \in \mathfrak{U}\}$, where $\mathfrak{U}$ is a collection of subsets of a topological space.

completion of a metric space with the aid of our characterization theorem in § 7 (Theorem 9.1). From our construction we can know that a completion of a metrizable space with respect to J. Nagata's metric (cf. Definition 9.3) is a dimension-preserving one (Theorem 9.4).

In § 10 our concern is devoted to the notion "length of a multiplicative covering" which is due to P. Alexandroff and A. Kolmogoroff [1] and has been recently applied to the characterization of dimension of metric spaces by J. Nagata [24, Theorem 4]. Our Theorem 10.2 gives a weaker condition than those in (B) of § 3.

Finally § 11 gives a characterization of the covering dimension of normal spaces by means of the length of coverings.

The content of this paper includes a development in detail of our brief notes [21; 22; 23].

I have been stimulated interest in dimension theory by Professor K. Morita who has been kind enough to give instructive advices and constant encouragement to me for many years. During the preparation of this paper he has given me many valuable suggestions. I express here my hearty thanks to Professor Morita.

§ 2. Zero-dimensional spaces

THEOREM 2.1. *A topological space R is a metric space with $\dim R \leq 0$ if and only if R is a dense subset of an inverse limiting space of a sequence of discrete spaces.*

PROOF. To prove the necessity let R be a metric space with $\dim R \leq 0$. Then for every positive integer i there exists an open covering $\mathfrak{U}_i$ with order $\mathfrak{U}_i \leq 1$ and mesh $\mathfrak{U}_i < 2^{-i}$. When $R \neq \phi$ i.e. $\dim R = 0$, we assume with no loss of generality that each element of $\mathfrak{U}_i$ is not empty. Let $\mathfrak{B}_i = \bigcap_{j \leq i} \mathfrak{U}_j$ ⁹⁾; then $\mathfrak{B}_i = \{V_\alpha; \alpha \in A_i\}$ is an open covering of R such that i) order $\mathfrak{B}_i \leq 1$, ii) mesh $\mathfrak{B}_i < 2^{-i}$, iii) $\mathfrak{B}_{i+1}$ refines $\mathfrak{B}_i$. Define $f_{i+1,i}: A_{i+1} \rightarrow A_i$ in such a way that $f_{i+1,i}(\alpha) = \beta$ yields $V_\alpha \subset V_\beta$. Since $\mathfrak{B}_{i+1}$ is a refinement of $\mathfrak{B}_i$, $f_{i+1,i}$ can be defined on A_{i+1} . Consider A_i as a topological space with the discrete topology. Let A be an inverse limiting space of a system $\{A_i, f_{i+1,i}\}$, $A = \lim \{A_i, f_{i+1,i}\}$. Let A_0 be a subspace of A consisting of points $(\alpha_1, \alpha_2, \dots) \in A$ such that $\bigcap_{i=1}^{\infty} V_{\alpha_i} \neq \phi$. When $\bigcap_{i=1}^{\infty} V_{\alpha_i} \neq \phi$, then it consists of a single point. Let φ be a mapping of A_0 into R such that $\varphi(a) = \bigcap_{i=1}^{\infty} V_{\pi_i(a)}$, where π_i is the projection of A_0 onto A_i .

Let x be an arbitrary point of R ; then for every i there exists one and only one element V_{α_i} of $\mathfrak{B}_i$ such that $x \in V_{\alpha_i}$. Since $V_{\alpha_{i+1}} \subset V_{\alpha_i}$ for every i , $a = (\alpha_1, \alpha_2, \dots)$ is a point of A_0 and $\varphi(a) = x$. Hence φ is onto.

Let a and b be points of A_0 with $a \neq b$; then there exists i such that $\pi_i(a) \neq \pi_i(b)$. Since $V_{\pi_i(a)} \cap V_{\pi_i(b)} = \phi$, we get $\varphi(a) \neq \varphi(b)$. Thus φ is one-to-one.

Let a be an arbitrary point of A_0 ; then $\varphi(\pi_i^{-1}(\pi_i(a))) = V_{\pi_i(a)}$ for every i , which proves the continuity of φ and φ^{-1} . Therefore φ is a homeomorphism of A_0 onto R . Since $\pi_i(A_0) = A_i$ for every i , A_0 is a dense subset of A .

To prove the sufficiency let R be a dense subset of an inverse limiting space $\lim \{A_i, f_{i+1,i}\}$ of discrete spaces A_i and $\mathfrak{G} = \{G_1, \dots, G_m\}$ an arbitrary

9) $\bigcap_{j \leq i} \mathfrak{U}_j$ denotes the aggregate of sets of type $\bigcap_{j \leq i} \{U_j; U_j \in \mathfrak{U}_j\} \neq \phi$.

finite open covering of R . Let π_i be the projection of R onto A_i . Let $B_{ij} = \{\alpha; \alpha \in A_i - \bigcup\{f_{ik}^{-1}(B_{kl}); l=1, \dots, m, k=1, \dots, i-1\}, \pi_i^{-1}(\alpha) \subset G_j \text{ and } \pi_i^{-1}(\alpha) \not\subset G_k \text{ for } k=1, \dots, j-1\}^{(10)}$. Then it can easily be seen that $\{U_{ij} = \bigcup\{\pi_i(\alpha); \alpha \in B_{ij}\}; j=1, \dots, m, i=1, 2, \dots\}$ is a covering of R consisting of mutually disjoint open sets. Moreover $U_j = \bigcup_{i=1}^{\infty} U_{ij}$ is included in G_j for $j=1, \dots, m$. Hence $\mathfrak{U} = \{U_j; j=1, \dots, m\}$ is an open covering of R with order $\mathfrak{U} \leq 1$ which refines $\mathfrak{G}$. Therefore $\dim R \leq 0$ and the theorem is completely proved.

§ 3. Fundamental relations in dimension theory for metric spaces

Throughout this paragraph R is a metric space. The following propositions which are fundamental in dimension theory for metric spaces are proved to be equivalent in such a way as $(A) \rightarrow (B) \rightarrow (C) \rightarrow (D) \rightarrow (E) \rightarrow (A)$.

(A) $\dim R \leq n$.

(B) There exists a sequence of locally finite coverings $\mathfrak{F}_i = \{F_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, of R such that

(3.1) $\mathfrak{F}_{i+1}$ refines $\mathfrak{F}_i$ for every i ,

(3.2) order $\mathfrak{F}_i \leq n+1$ for every i ,

(3.3) for any neighborhood U of any point x of R there exists i with $\text{Star}(x, \mathfrak{F}_i)^{(11)} \subset U$.

(C) There exist a metric space A_0 with $\dim A_0 \leq 0$ and a closed continuous mapping f of A_0 onto R with order $f \leq n+1$.

(D) R is the sum of $n+1$ subspaces R_i , $i=1, \dots, n+1$, with $\dim R_i \leq 0$.

(E) $\text{Ind } R \leq n$.

(A) *implies* (B). Let $\mathfrak{U}_1 = \{U_\alpha; \alpha \in A_1\}$ be a locally finite open covering of R of order $\leq n+1$ such that the diameter of each $U_\alpha < 1$. Since a locally finite open covering is shrinkable⁽¹²⁾, there exist a closed covering $\mathfrak{F}_1 = \{F_\alpha; \alpha \in A_1\}$ and an open covering $\mathfrak{V}_1 = \{V_\alpha; \alpha \in A_1\}$ such that $U_\alpha \supset F_\alpha \supset V_\alpha$ for every $\alpha \in A_1$. Let $\mathfrak{U}_2 = \{U_\alpha; \alpha \in A_2\}$ be a locally finite open covering of order $\leq n+1$ such that the diameter of each $U_\alpha (\alpha \in A_2) < 1/2$ and $\mathfrak{U}_2$ refines $\mathfrak{V}_1$. Let $\mathfrak{F}_2 = \{F_\alpha; \alpha \in A_2\}$ and $\mathfrak{V}_2 = \{V_\alpha; \alpha \in A_2\}$ be respectively a closed covering and an open covering of R such that $U_\alpha \supset F_\alpha \supset V_\alpha$ for every $\alpha \in A_2$. Repeating this procedure we get a sequence of closed coverings $\mathfrak{F}_i = \{F_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, which satisfies the conditions (3.1), (3.2) and (3.3).

(B) *implies* (C). Let $\mathfrak{F}_i = \{F_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, be locally finite coverings of R which satisfy the conditions (3.1), (3.2) and (3.3). For every i let us define a single-valued mapping $f_{i+1, i}$ of A_{i+1} into A_i in such a way that $f_{i+1, i}(\alpha) = \beta$ yields $\bar{F}_\alpha \subset F_\beta$. For i and j with $i > j$ let $f_{ij} = f_{j+1, j} \cdots f_{i-1, i-2} f_{i, i-1}$. Let A be an inverse limiting space of a system $\{A_i, f_{ij}\}$, $A = \lim\{A_i, f_{ij}\}$ where A_i are topological spaces with discrete topology. Let A_0 be a subspace of A consisting of points $(\alpha_1, \alpha_2, \dots) \in A$ such that $\bigcap_{i=1}^{\infty} F_{\alpha_i} \neq \emptyset$. When $\bigcap_{i=1}^{\infty} F_{\alpha_i} \neq \emptyset$, then it consists of a single point by (3.3). Let f be a mapping of A_0 into R such that $f(a) = \bigcap_{i=1}^{\infty} F_{\pi_i(a)}$, where π_i is the projection of A_0 into A_i .

10) $f_{ik} = f_{k+1, k} \cdots f_{i-1, i-2} f_{i, i-1}$.

11) $\text{Star}(x, \mathfrak{F}_i) = \bigcup\{F_\alpha; x \in F_\alpha \in \mathfrak{F}_i\}$.

12) An open covering $\{U_\alpha; \alpha \in A\}$ is called *shrinkable* if there exists a close covering $\{F_\alpha; \alpha \in A\}$ such that $U_\alpha \supset F_\alpha$ for every $\alpha \in A$. A point-finite open covering of a normal space is shrinkable (cf. [13]).

Let x be an arbitrary point of R . Since $\mathfrak{F}_i$ is locally finite, $B_i = \{\alpha; x \in F_\alpha \in \mathfrak{F}_i\}$ ($\subset A_i$) is a non-empty finite set of indices. Since $f_{i+1,i}(B_{i+1}) \subset B_i$ for every i , $\{B_i, f_{i+1,i}|B_{i+1}\}$ forms an inverse system. Consider $B = \lim\{B_i, f_{i+1,i}|B_{i+1}\}$ and then evidently $B \subset A$. Since an inverse limiting space of a sequence of non-empty compact spaces is not empty, $B \neq \emptyset$. Let $(\alpha_1, \alpha_2, \dots)$ be a point of B ; then $\bigcap_{i=1}^{\infty} F_{\alpha_i} = x$ and hence $f(a) = x$, which proves that f is onto.

Let a be an arbitrary point of A_0 and U an arbitrary neighborhood of $f(a)$; then there exists i with $\text{Star}(f(a), \mathfrak{F}_i) \subset U$ by (3.3). Since $f(a) \in F_{\pi_i(a)}$ and $f(\pi_i^{-1}(\pi_i(a))) \subset F_{\pi_i(a)}$, we get $f(\pi_i^{-1}(\pi_i(a))) \subset U$, which proves the continuity of f .

To prove $\text{order } f \leq n+1$ assume that $\text{order } f > n+1$ and then there exists a point x of R with $\text{order}(x, f) > n+1$. Let $a_i, i=1, \dots, n+2$, be mutually different points of $f^{-1}(x)$; then there exists j such that $\pi_j(a_i), i=1, \dots, n+2$, are mutually different indices of A_j . Since $x = f(a_i) \in f(\pi_j^{-1}(\pi_j(a_i))) \subset F_{\pi_j(a_i)}$ for $i=1, \dots, n+2$, $\text{order } \mathfrak{F}_j > n+1$ which contradicts to (3.2). Thus we get $\text{order } f \leq n+1$.

To show the closedness of f let F be an arbitrary closed set of A_0 . Then $f(F) = \bigcup \{f(F \cap \pi_1^{-1}(\alpha)); \alpha \in \pi_1(A_0)\}$. Since $f(\pi_1^{-1}(\alpha)) \subset F_\alpha$ for any $\alpha \in \pi_1(A_0)$ and $\mathfrak{F}_1 = \{F_\alpha; \alpha \in A_1\}$ is locally finite, we get $\overline{f(F)} = \bigcup \{\overline{f(F \cap \pi_1^{-1}(\alpha))}; \alpha \in \pi_1(A_0)\}$. Let x_0 be a given point of $\overline{f(F)}$ and then there exists $\alpha_1 \in \pi_1(A_0)$ such that $x_0 \in \overline{f(F \cap \pi_1^{-1}(\alpha_1))}$. By the inductive process we can find an index $\alpha_i \in \pi_i(A_0)$ for $i=1, 2, \dots$, such that $x_0 \in \overline{f(F \cap \pi_1^{-1}(\alpha_1) \cap \dots \cap \pi_i^{-1}(\alpha_i))}$. Since $\pi_i^{-1}(\alpha_i) \cap \pi_{i+1}^{-1}(\alpha_{i+1}) \neq \emptyset$ for any i , it holds that $f_{i+1,i}(\alpha_{i+1}) = \alpha_i$ for any i . Therefore $a = (\alpha_1, \alpha_2, \dots)$ is a point of A . Since $f(\pi_{i+1}^{-1}(\alpha_{i+1})) \subset F_{\alpha_{i+1}}$ and $\overline{F_{\alpha_{i+1}}} \subset F_{\alpha_i}$ for any i ,

$$x_0 \in \bigcap_{i=1}^{\infty} \overline{f(F \cap \pi_1^{-1}(\alpha_1) \cap \dots \cap \pi_i^{-1}(\alpha_i))} \subset \bigcap_{i=1}^{\infty} \overline{f(\pi_i^{-1}(\alpha_i))} \subset \bigcap_{i=1}^{\infty} \overline{F_{\alpha_i}} \subset \bigcap_{i=2}^{\infty} \overline{F_{\alpha_i}} \subset \bigcap_{i=1}^{\infty} F_{\alpha_i}.$$

Since $\bigcap_{i=1}^{\infty} F_{\alpha_i}$ is either the empty set or a single point, we get $x_0 = \bigcap_{i=1}^{\infty} F_{\alpha_i}$. Thus we can know that a is a point of A_0 and $f(a) = x_0$. On the other hand $\{\pi_i^{-1}(\alpha_i); i=1, 2, \dots\}$ is a complete system of neighborhoods of a and $F \cap \pi_i^{-1}(\alpha_i) \neq \emptyset$ for any i . Therefore $a \in \overline{F} = F$ and $x_0 = f(a) \in f(F)$, which shows the closedness of $f(F)$. Thus the closedness of f is proved. Since A_0 is evidently metrizable and $\dim A_0 \leq 0$ by Theorem 2.1, the implication is completely proved.

To prove the next implication we need the following theorem¹³⁾.

THEOREM 3.1. *Let R and S be metric spaces and f a closed continuous mapping of R onto S . If $f^{-1}(y)$ consists of exactly $k(<\infty)$ points for every point $y \in S$ and $\dim R \leq 0$, then we have $\dim S \leq 0$.*

PROOF. By Theorem 2.1 we can assume that R is a dense subset of $\varprojlim R_i$ obtained from $\{R_i, f_{ji}: R_j \rightarrow R_i (j > i)\}$ with discrete spaces $R_i = \{p_{i\alpha}; \alpha \in A_i\}$. We can assume that points of R_i are linearly-ordered such that for any $p_{i\alpha}, p_{i\beta}$ with $f_{ij}(p_{i\alpha}) \neq f_{ij}(p_{i\beta}), i > j$, it holds that $p_{i\alpha} > p_{i\beta}$ if and only if $f_{ij}(p_{i\alpha}) > f_{ij}(p_{i\beta})$. We introduce into points $(p_{1\alpha_1}, p_{2\alpha_2}, \dots)$ of $\varprojlim R_i$ the lexicographic order with respect to the one of R_i just defined. Let $x_1(y), \dots, x_k(y) \in R$ be the inverse image of $y \in S$ with $x_1(y) < \dots < x_k(y)$ and then R is decomposed into mutually disjoint subsets $T_i = \{x_i(y); y \in S\}, i=1, \dots, k$.

We shall show that every T_i is an F_σ . To do so it suffices to prove T_1 is an F_σ since the rest case is proved similarly. Let $r(y), y \in S$, be the smallest

13) This theorem will be generalized in Theorem 3.7.

integer such that $\pi_r(x_1(y)), \dots, \pi_r(x_k(y))$ are mutually different points of R_r , where $\pi_r: R \rightarrow R_r$ is the projection. Let $S_t = \{y; y \in S, r(y) \leq t\}$, $t=1, 2, \dots$, and $T_{jt} = T_j \cap f^{-1}(S_t)$; then evidently i) $S = \bigcup_{t=1}^{\infty} S_t$, ii) $T_1 = \bigcup_{t=1}^{\infty} T_{1t}$, iii) $T_{1t} \subset T_{1,t+1}$. The family $\{f(V(p_{t\alpha})); V(p_{t\alpha}) = \{x; x \in R, \pi_t(x) = p_{t\alpha}\}, \alpha \in A_t\}$ is a closed covering of S such that the sum of any subfamily is also closed. Let y be an arbitrary point of S_t and then it is not hard to see that $W_y = S - \bigcup \{f(V(p_{t\alpha})); \alpha \in A_t, y \notin f(V(p_{t\alpha}))\}$ is an open set of S which contains y and that $z \in S_t \cap W_y$ implies $\pi_t(x_j(y)) = \pi_t(x_j(z))$ for $j=1, \dots, k$. Therefore the open set $G_{ty} = \bigcup_{j=2}^k (f^{-1}(W) \cap V(\pi_t(x_j(y))))$ is unable to meet T_{1t} . Thus $F_t = R - \bigcup \{G_{ty}; y \in S_t\}$ is a closed set with $F_t \supset T_{1t}$ and $F_t \cap (\bigcup_{j=2}^k T_{jt}) = \emptyset$. Since $H_j = \bigcap_{t=j}^{\infty} F_t$ is a closed set with $H_j \supset T_{1j}$ and $H_j \cap (\bigcup_{i=2}^k T_i) = \emptyset$, $T_1 = \bigcup_{j=1}^{\infty} H_j$ and T_1 is an F_σ . Since $f|_{H_j}$ is a homeomorphism, $\dim f(H_j) \leq 0$. Moreover $f(H_j)$ is closed in S and $S = \bigcup_{j=1}^{\infty} f(H_j)$ and hence $\dim S \leq 0$ by the sum theorem, which completes the proof.

(C) implies (D). Let A_0 be a metric space with $\dim A_0 \leq 0$ and f a closed continuous mapping of A_0 onto R of order $\leq n+1$. Let R_i be the aggregate of points $x \in R$ such that $f^{-1}(x)$ consists of exactly i points. Then by Theorem 3.1 $\dim R_i \leq 0$. Thus R is the sum of $n+1$ subspaces R_i , $i=1, \dots, n+1$, with $\dim R_i \leq 0$.

(D) implies (E). We prove this by the induction on n . When $\dim R \leq 0$, it is almost evident that $\text{Ind } R \leq 0$.

Let R be the sum of $n+1$ subspaces R_i , $i=1, \dots, n+1$, with $\dim R_i \leq 0$ and assume that $\text{Ind } \bigcup_{i=2}^{n+1} R_i \leq n-1$. Let F and G be respectively a closed set and an open set of R with $F \subset G$. Let H_1 and H_2 be open sets such that $F \subset H_1 \subset \bar{H}_1 \subset H_2 \subset \bar{H}_2 \subset G$. Since $\text{Ind } R_1 \leq 0$, there exists an open and closed subset U_1 of R_1 with $R_1 \cap \bar{H}_1 \subset U_1 \subset R_1 \cap H_2$. Let $P_1 = F \cup U_1$ and $P_2 = (R - G) \cup (R_1 - U_1)$ and then we get $(\bar{P}_1 \cap P_2) \cup (P_1 \cap \bar{P}_2) = \emptyset$. Hence there exists an open set U of R such that $P_1 \subset U \subset \bar{U} \subset R - P_2$. Then it can easily be seen that $(\bar{U} - U) \cap R_1 = \emptyset$ and $F \subset U \subset G$. Since $\text{Ind } \bigcup_{i=2}^{n+1} R_i \leq n-1$ and $\bar{U} - U$ is a closed subset of $\bigcup_{i=2}^{n+1} R_i$, we get $\text{Ind } (\bar{U} - U) \leq n-1$, which proves $\text{Ind } R \leq n$.

(E) implies (A). This is well-known and has been verified for the first time by N. Vedenisoff [30]. For the proof see for instance [6], [16], [17], [20] or [30].

THEOREM 3.2.¹⁴⁾ *Let R be a metric space. Then the propositions (A), (B), (C), (D), (E) are all equivalent.*

THEOREM 3.3. *Let R and S be metric spaces with $\dim R \leq 0$ and f a closed continuous mapping of R onto S of finite order. Then $\dim S + 1 \leq$ the number of integers i with $\{y; |f^{-1}(y)| = i\} \neq \emptyset$.*

This is almost evident by Theorem 3.1 and the implication (D) $\rightarrow$ (A).

THEOREM 3.4. *Let R be a metric space with $\dim R = n$. Then for any open covering $\mathcal{U}$ there exists a locally finite closed covering $\mathfrak{F}$ such that*

(3.4) $\mathfrak{F}$ refines $\mathcal{U}$,

(3.5) $\text{order } \mathfrak{F} \leq n+1$,

(3.6) *for any i with $1 \leq i \leq n+1$ there exists a point x of R with $\text{order}(x, \mathfrak{F})^{15)} = i$.*

14) (A) $\Rightarrow$ (D) $\Rightarrow$ (E) have been proved by Katětov [10] and Morita [16] independently. (E) $\Rightarrow$ (C) has been proved by Morita [18]. (B) is a weaker condition which characterizes dimension than that of C.H. Dowker and W. Hurewicz [6].

15) $\text{order}(x, \mathfrak{F})$ is the number of elements of $\mathfrak{F}$ which contain x .

PROOF. Let A_0 be a dense subset of $\lim \{A_i (i=1, 2, \dots), f_{ij}: A_i \rightarrow A_j (i > j)\}$, where $A_i, i=1, 2, \dots$, are topological spaces with discrete topology, and f a closed continuous mapping of A_0 onto R with order $f \leq n+1$ such that $\{f(\pi_1^{-1}(\alpha)); \alpha \in A_1\}$ refines $\mathfrak{U}$, where $\pi_i, i=1, 2, \dots$, is the projection of A_0 onto A_i , and $\{f(\pi_i^{-1}(\alpha)); \alpha \in A_i\}$ is locally finite for every i . The existence of such A_0 and f is essentially proved in the proofs of (A) $\rightarrow$ (B) $\rightarrow$ (C) of Theorem 3.2. By Theorem 3.3 there exists, for any i with $1 \leq i \leq n+1$, a point y_i of R with order $(y_i, f) = i$. Let j be an integer such that the points of $\bigcup \{f^{-1}(y_i); i=1, \dots, n+1\}$ have mutually different representatives in A_j . Let $\mathfrak{F} = \{f(\pi_j^{-1}(\alpha)); \alpha \in A_j\}$ and then it can easily be seen that order $(y_i, \mathfrak{F}) = i$ for $i=1, \dots, n+1$, and that $\mathfrak{F} < \{f(\pi_i^{-1}(\alpha)); \alpha \in A_i\} < \mathfrak{U}$. Since f is closed, $\mathfrak{F}$ is a locally finite closed covering of R . Thus $\mathfrak{F}$ satisfies all conditions required, and the theorem is proved.

THEOREM 3.5.¹⁶⁾ *Let R be a metric space with $\dim R \leq n$ and $\mathfrak{U}$ be an arbitrary open covering of R . Then there exists a sequence of locally finite closed coverings $\mathfrak{H}_i = \{H_\alpha; \alpha \in A_i\}, i=1, 2, \dots$, and a sequence of mappings $f_{i+1,i}: A_{i+1} \rightarrow A_i, i=1, 2, \dots$, which satisfy the following conditions.*

(3.7) $\mathfrak{H}_1$ refines $\mathfrak{U}$.

(3.8) $\text{mesh } \mathfrak{H}_i \rightarrow 0$ as $i \rightarrow \infty$.

(3.9) order $\mathfrak{H}_i \leq n+1$ for every i .

(3.10) For any i and any $\alpha \in A_i, H_\alpha = \bigcup \{H_\beta; \beta \in f_{i+1,i}^{-1}(\alpha)\}$.

(3.11) For any i any $s \leq n+2, \dim \bigcap_{j=1}^s \{H_{\alpha(j)}; \alpha(1), \dots, \alpha(s) \text{ are mutually different indices of } A_i\} \leq n-s+1$.

Moreover if a sequence of locally finite closed coverings $\mathfrak{H}_i, i=1, 2, \dots$, of R satisfies (3.8), (3.9), (3.10), then it satisfies (3.11).

PROOF. The existence of a sequence satisfying (3.7), (3.8), (3.9), (3.10) is essentially proved in the proof of the above theorem, if we set $H_\alpha = f(\pi_i^{-1}(\alpha))$ for $\alpha \in A_i$.

Let $\mathfrak{H}_i, i=1, 2, \dots$, be a sequence of locally finite closed coverings satisfying conditions (3.8), (3.9), (3.10). We show that it satisfies (3.11). Let $\alpha(j), j=1, \dots, s (\leq n+2)$, are mutually different indices of A_i and put $H = \bigcap_{j=1}^s H_{\alpha(j)}$. When $H = \phi, \dim H = -1 \leq n-s+1$. Let us consider the case $H \neq \phi$. Consider a sequence of locally finite closed coverings $H \cap \mathfrak{H}_j, j=i, i+1, \dots$, of H . Let $B_j = \{\alpha; \alpha \in A_j, H \cap H_\alpha \neq \phi, j=i, i+1, \dots\}$. Let $g_{j+1,j} = f_{j+1,j}|_{B_{j+1}}, j=i, i+1, \dots$, and then $g_{j+1,j}$ is onto. Let B_0 be the aggregate of points $a = (\alpha_i, \alpha_{i+1}, \dots) \in \lim \{B_j (j=i, i+1, \dots), g_{j+1,j}\}$ such that $\bigcap_{j=1}^\infty (H \cap H_{\alpha_j}) \neq \phi$. Let g be a mapping of B_0 into H defined by $g(a) = \bigcap_{j=1}^\infty (H \cap H_{\pi_j(a)})$, where π_j is the projection of B_0 onto B_j , and then we can know by the same argument as in the proof of (B) $\rightarrow$ (C) of Theorem 3.2 that g is a closed continuous mapping of B_0 onto H with order $g \leq n+1$. Since it can easily be seen by (3.10) that $g(\pi_j^{-1}(\alpha)) = H \cap H_\alpha$ for every $\alpha \in B_j, j=i, i+1, \dots$, it holds that order $(x, g) \geq s$ for every point $x \in H$ by the inequality order $(x, H \cap \mathfrak{H}_j) \geq s$ for $x \in H$ and $j=i, i+1, \dots$. Since B_0 has of course $\dim B_0 \leq 0$, we get $\dim H \leq n+1 - (s-1) - 1 = n-s+1$ by Theorem 3.3 and the proof is completed.

THEOREM 3.6. *Let R be a metric space and $C_i, i=1, 2, \dots$, be closed sets of*

16) Theorems 3.5 and 3.6 have been proved independently by Prof. Morita. These are also deeply related to Morita [16; 18] and Kat'ov [11].

R with $\dim C_i < \infty$ and $\mathfrak{U}$ be an arbitrary open covering of R . Then there exist a sequence of locally finite closed coverings $\mathfrak{H}_i = \{H_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, and a sequence of mappings $f_{i+1,i}: A_{i+1} \rightarrow A_i$, $i=1, 2, \dots$, which satisfy the following conditions.

(3.12) $\mathfrak{H}_1$ refines $\mathfrak{U}$.

(3.13) $\text{mesh } \mathfrak{H}_i \rightarrow 0$ as $i \rightarrow \infty$.

(3.14) For any i and any j , $\text{order } \mathfrak{H}_i \cap C_j \leq \dim C_j + 1$.

(3.15) For any i and any $\alpha \in A_i$, $H_\alpha = \bigcup \{H_\beta; \beta \in f_{i+1,i}^{-1}(\alpha)\}$.

(3.16) For any i , $s(\leq \dim C_i + 2)$ and t , $\dim \bigcap_{j=1}^s \{H_{\alpha(j)} \cap C_i; \alpha(1), \dots, \alpha(s) \text{ are mutually different indices of } A_i\} \leq \dim C_i - s + 1$.

Moreover if a sequence of locally finite closed coverings $\mathfrak{H}_i$, $i=1, 2, \dots$, and $f_{i+1,i}$, $i=1, 2, \dots$, satisfy the conditions (3.13), (3.14), (3.15), they satisfy (3.16).

PROOF. First we construct by the induction a sequence of locally finite closed coverings $\mathfrak{F}_i = \{F_\alpha; \alpha \in B_i\}$, $i=1, 2, \dots$, and a sequence of locally finite open coverings $\mathfrak{U}_i = \{U_\alpha; \alpha \in B_i\}$, $i=1, 2, \dots$, which satisfy the following conditions.

(3.17) $\mathfrak{U} > \mathfrak{F}_1 > \mathfrak{U}_1 > \mathfrak{F}_2 > \mathfrak{U}_2 > \dots$.

(3.18) $\text{mesh } \mathfrak{F}_i < 1/i$.

(3.19) For any i and any $j \leq i$, $\text{order } \mathfrak{F}_i \cap C_j \leq \dim C_j + 1$.

Let $\mathfrak{B}_1 = \{V_\alpha; \alpha \in B_1\}$ be a locally finite open covering of R with $\text{mesh } \mathfrak{B}_1 < 1$ such that $\mathfrak{B}_1$ refines $\mathfrak{U}$. Let $\mathfrak{B}' = \{W_\alpha'; \alpha \in B_1\}$ be an open covering of C_1 such that $\text{order } W' \leq \dim C_1 + 1$ and $W_\alpha' \subset V_\alpha$ for every $\alpha \in B_1$. Let W_α'' , $\alpha \in B_1$, be an open set of R with $W_\alpha'' \cap C_1 = W_\alpha'$. Let $W_\alpha = (W_\alpha'' \cap V_\alpha) \cup (V_\alpha - C_1)$. Then $\mathfrak{B} = \{W_\alpha; \alpha \in B_1\}$ is a locally finite open covering of R and hence there exist a closed covering $\mathfrak{F}_1 = \{F_\alpha; \alpha \in B_1\}$ and an open covering $\mathfrak{U}_1 = \{U_\alpha; \alpha \in B_1\}$ such that $U_\alpha \subset F_\alpha \subset W_\alpha (\subset V_\alpha)$ for every $\alpha \in B_1$. Since $\mathfrak{B} \cap C_1 = \mathfrak{B}'$, $\text{order } \mathfrak{F}_1 \cap C_1 \leq \text{order } \mathfrak{B} \cap C_1 = \text{order } \mathfrak{B}' \leq \dim C_1 + 1$.

Now make the induction assumption that there exist locally finite closed coverings $\mathfrak{F}_1, \dots, \mathfrak{F}_{m-1}$ and locally finite open coverings $\mathfrak{U}_1, \dots, \mathfrak{U}_{m-1}$ which satisfy the conditions (3.17), (3.18), (3.19) for $i=1, \dots, m-1$. Let $\mathfrak{B}_m = \{V_\alpha; \alpha \in B_m\}$ be a locally finite open covering of R such that $\mathfrak{B}_m$ refines $\mathfrak{U}_{m-1}$ and $\text{mesh } \mathfrak{B}_m < 1/m$. Let $D_1, \dots, D_m$ be a system with $\dim D_1 \leq \dim D_2 \leq \dots \leq \dim D_m$ obtained by changing the order of $C_1, \dots, C_m$. By the same argument as in the above there exist open coverings $\mathfrak{B}_i = \{W_{i\alpha}; \alpha \in B_m\}$, $i=1, \dots, m$, such that $V_\alpha \supset W_{1\alpha} \supset W_{2\alpha} \supset \dots \supset W_{m\alpha}$ for every $\alpha \in B_m$ and $\text{order } \mathfrak{B}_i \cap D_i \leq \dim D_i + 1$ for $i=1, \dots, m$. Since $\text{order } \mathfrak{B}_i \cap D_j \leq \text{order } \mathfrak{B}_j \cap D_j$ for $j \leq i$, we get $\text{order } \mathfrak{B}_m \cap C_i \leq \text{order } \mathfrak{B}_{i_1} \cap C_i \leq \dim C_i + 1$ for $1, \dots, m$, where $C_i = D_{i_1}$. Since $\mathfrak{B}_m = \{W_{m\alpha}; \alpha \in B_m\}$ is locally finite, there exist a closed covering $\mathfrak{F}_m = \{F_\alpha; \alpha \in B_m\}$ and an open covering $\mathfrak{U}_m = \{U_\alpha; \alpha \in B_m\}$ such that $U_\alpha \subset F_\alpha \subset W_\alpha (\subset V_\alpha)$ for every $\alpha \in B_m$. It is obvious that i) $\text{order } \mathfrak{F}_m \cap C_i \leq \dim C_i + 1$ for $i \leq m$, ii) $\mathfrak{U}_{m-1} > \mathfrak{F}_m > \mathfrak{U}_m$, iii) $\text{mesh } \mathfrak{F}_m < 1/m$, which completes the induction.

Now define $f_{i+1,i}: B_{i+1} \rightarrow B_i$ in such a way that $f_{i+1,i}(\alpha) = \beta$ yields $F_\alpha \subset F_\beta$. Let B_0 be the aggregate of points $a = (\alpha_1, \alpha_2, \dots)$ of $\lim \{B_i (i=1, 2, \dots), f_{i+1,i}\}$ such that $\bigcap_{i=1}^\infty F_{\alpha_i} \neq \emptyset$. Let f be a mapping of B_0 into R defined by $f(a) = \bigcap_{i=1}^\infty F_{\pi_i(a)}$, where π_i is the projection of B_0 into B_i . Then we can know by the same argument as in the proof of (B) $\rightarrow$ (C) of Theorem 3.2 that f is a closed continuous mapping of B_0 onto R .

Let

$$(3.20) \quad \mathfrak{H}_i = \{H_\alpha = f(\pi_i^{-1}\alpha); \alpha \in A_i = \pi_i(B_0)\}, \quad i=1, 2, \dots,$$

and then they form a sequence of locally finite closed coverings which satisfies (3.12), (3.13), (3.15). Since $\mathfrak{H}_i$ refines $\mathfrak{F}_i$ for every i , we get i) order $\mathfrak{H}_i \cap C_j \leq$ order $\mathfrak{F}_i \cap C_j \leq \dim C_j + 1$ when $i \geq j$, ii) order $\mathfrak{H}_i \cap C_j \leq$ order $\mathfrak{H}_j \cap C_j \leq \dim C_j + 1$ when $i < j$ by (3.15) and (3.19). Thus the sequence (3.20) satisfies (3.14).

When a sequence of locally finite closed coverings satisfies (3.13), (3.14), (3.15), we can know that it satisfies also (3.16) by the same argument as in the proof of the preceding theorem. Thus the theorem is completely proved.

THEOREM 3.7. *Let R and S be metric spaces and f a closed continuous mapping of R onto S . If $f^{-1}(y)$ consists of exactly $k(<\infty)$ points for every point $y \in S$, then we have $\dim R = \dim S$.*

PROOF. Since it holds in general that $\dim R \leq \dim S + \sup\{\dim f^{-1}(y); y \in S\}$ by Morita [17, Theorem 2] or Nagami [20, Theorem 2], we get $\dim R \leq \dim S$.

Conversely let us prove that $\dim S \leq \dim R$. Let $\mathfrak{F}_i = \{F_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, be locally finite closed coverings of R such that $\text{mesh } \mathfrak{F}_i \rightarrow 0$ as $i \rightarrow \infty$. Then $f(\mathfrak{F}_i) = \{f(F_\alpha); \alpha \in A_i\}$ is a point-finite closed covering of S . Let y be an arbitrary point of S and B_i the subset of A_i such that $\alpha \in B_i$ if and only if $y \notin f(F_\alpha)$. Since $\bigcup\{F_\alpha; \alpha \in B_i\}$ is closed, $V = S - f(\bigcup\{F_\alpha; \alpha \in B_i\}) = S - \bigcup\{f(F_\alpha); \alpha \in B_i\}$ is an open neighborhood of y . If $f(F_\alpha) \cap V \neq \emptyset$, then $\alpha \notin B_i$. Since $A_i - B_i$ is finite, V intersects only a finite number of $f(F_\alpha)$, $\alpha \in A_i$. Thus $f(\mathfrak{F}_i)$ is locally finite.

Let y be an arbitrary point of S and $\{x_1, \dots, x_k\}$ the inverse image of y . Then there exists i such that when $x_s \in F_\alpha \in \mathfrak{F}_i$, $x_t \in F_\beta \in \mathfrak{F}_i$, $s \neq t$, then $F_\alpha \cap F_\beta = \emptyset$. Select $F_{\alpha(j)} \in \mathfrak{F}_i$, $j=1, \dots, k$, such that $x_j \in F_{\alpha(j)}$ and put $F(y) = \bigcap_{j=1}^k f(F_{\alpha(j)})$; then $F(y)$ is a closed set of S which contains y . Since $F_{\alpha(1)} \cap f^{-1}(F(y))$ is a closed set of R and $f|_{F_{\alpha(1)} \cap f^{-1}(F(y))}$ is a homeomorphism of $F_{\alpha(1)} \cap f^{-1}(F(y))$ onto $F(y)$, $\dim F(y) = \dim F_{\alpha(1)} \cap f^{-1}(F(y)) \leq \dim R$. Since $\{F(y); y \in S\}$ is a closed covering of S which is a countable sum of locally finite subfamilies, $\dim S = \sup\{\dim F(y); y \in S\} \leq \dim R$ by Morita [16, Theorem 5.2] or by Nagami [19, Theorem 2], which completes the proof.

A slight modification of the above theorem is the following.

THEOREM 3.8. *Let R and S be metric spaces and f a closed continuous mapping of R onto S . If the boundary of $f^{-1}(y)$ consists of exactly $k(<\infty)$ points for every point $y \in S$, then we have $\dim S \leq \dim R$.*

PROOF. When $R = \emptyset$, then $S = \emptyset$ and evidently $\dim S = \dim R$. Let us prove the theorem for the case when $R \neq \emptyset$. When $k=0$, select a point $x(y) \in f^{-1}(y)$ for every $y \in S$. Then it can easily be seen that $\{x(y); y \in S\}$ is a closed set of R and hence it is homeomorphic to S under $f|_{\{x(y); y \in S\}}$. Therefore $\dim S = \dim\{x(y); y \in S\} \leq \dim R$.

When $k>0$, $R_1 = R - \bigcup\{\text{the open kernel of } f^{-1}(y); y \in S\}$ is a closed set of R and $f_1 = f|_{R_1}$ is a closed continuous mapping of R_1 onto S such that $f_1^{-1}(y)$ consists of exactly k points for every point $y \in S$. Hence Theorem 3.7 can be applied and we get $\dim S = \dim R_1 \leq \dim R$, which completes the proof.

When a metric space R is the sum of countable subspaces R_i with $\dim R_i \leq 0$, we call $\dim R \leq \aleph_0$. Under this definition we get the following theorem.

THEOREM 3.9. *Let R and S be metric spaces and f a closed continuous mapping of R onto S . If the boundary of $f^{-1}(y)$ is finite for every point $y \in S$ and $\dim R \leq \aleph_0$, then we have $\dim S \leq \aleph_0$.*

PROOF. Let S_i , $i=0, 1, 2, \dots$, be the aggregate of points $y \in S$ such that the boundary of $f^{-1}(y)$ consists of exactly i points. Then $S = \bigcup_{i=0}^{\infty} S_i$ and $\dim S_i \leq \dim R \leq \aleph_0$ by Theorem 3.8 and hence $\dim S \leq \aleph_0$.

§ 4. Open mappings whose order are point-wise finite

THEOREM 4.1. *Let R and S be paracompact Hausdorff spaces and f an open continuous mapping of R onto S . If $f^{-1}(y)$ is finite for every point $y \in S$, then we have $\dim R = \dim S$. If moreover the paracompactness of R and S is hereditary, we have $\text{Ind } R = \text{Ind } S$.*

PROOF. Let S_i be the aggregate of points $y \in S$ such that $f^{-1}(y)$ consists of exactly i points; then $S = \bigcup_{i=1}^{\infty} S_i$. Let $\{x_1, \dots, x_i\}$ be the inverse image of $y \in S_i$, under f . Let V_j be an open neighborhood of x_j , $j=1, \dots, i$, such that $V_j \cap V_k = \emptyset$ if $j \neq k$. Since f is open, $V = \bigcap_{j=1}^i f(V_j)$ is an open neighborhood of y . Since for every point $y \in V$ $f^{-1}(y)$ consists of at least i points, $V \subset \bigcup_{k=i}^{\infty} S_k$. Thus

(4.1) $\bigcup_{k=1}^i S_k$ is closed in S for every i .

Let $R_i = f^{-1}(S_i)$ and consider $f_i = f|_{R_i}$ and then f_i is an open continuous mapping of R_i onto S_i such that $f_i^{-1}(y)$ consists of exactly i points for every point $y \in S_i$. Let $\{x_1, \dots, x_i\}$ be the inverse image of $y \in S_i$ under f_i . Let W_j be an open neighborhood of x_j , $j=1, \dots, i$, such that $W_j \cap W_k = \emptyset$ whenever $j \neq k$. Since f_i is open, $W = \bigcap_{j=1}^i f_i(W_j)$ is an open neighborhood of y . Since $W_j \cap f^{-1}(W)$ is homeomorphic to W under f_i , $j=1, \dots, i$, and the covering dimension of paracompact Hausdorff spaces is completely determined by the local covering dimension by C. H. Dowker [5] or Nagami [19, Theorem 1], we get $\dim R_i = \dim S_i$. Since $\dim R = \sup \dim R_i$ and $\dim S = \sup \dim S_i$ by (4.1) (cf. for instance [20, Lemma 3]), we get $\dim R = \dim S$.

When R and S are hereditarily paracompact Hausdorff spaces, we get $\text{Ind } R = \text{Ind } S$ since i) the large inductive dimension of such spaces is completely determined by the local large inductive dimension by C. H. Dowker [5], ii) $\text{Ind } R = \sup \text{Ind } R_i$ and $\text{Ind } S = \sup \text{Ind } S_i$ by C. H. Dowker [4]. Thus the theorem is completely proved.

§ 5. Cuts

DEFINITION 5.1. Let R and S be topological spaces. Let $\mathfrak{F} = \{F_\alpha; \alpha \in A\}$ and $\mathfrak{H} = \{H_\alpha; \alpha \in A\}$ be respectively locally finite closed coverings of R and S . Let f be a mapping of R onto S . Let r be a positive integer. If the following conditions are satisfied, $(R, \mathfrak{F}, f)$ is called the *cut of order r of $(S, \mathfrak{H})$* .

(5.1) For every $\alpha \in A$, $f|_{F_\alpha}$ is a homeomorphism of F_α onto H_α .

(5.2) If $\text{order}(y, \mathfrak{H})$, $y \in S$, is greater than r , $f^{-1}(y)$ consists of one and only one point.

(5.3) If $r_1 = \text{order}(y, \mathfrak{H})$, $y \in S$, is not greater than r , $f^{-1}(y)$ consists of exactly r_1 points.

R is called the *cut-space* of order r obtained from $(S, \mathfrak{H})$, $\mathfrak{F}$ the *derived covering* of order r and f the *cut-mapping*.

It is to be noted that f is a closed continuous mapping by the condition (5.1) and the local finiteness of $\mathfrak{F}$ and $\mathfrak{H}$.

THEOREM 5.2. *Let S be an arbitrary topological space, $\mathfrak{H}=\{H_\alpha; \alpha \in A\}$ an arbitrary locally finite closed covering of S and r an arbitrary positive integer. Then there exists a cut $(R, \mathfrak{F}=\{F_\alpha; \alpha \in A\}, f)$ of order r of $(S, \mathfrak{H})$. Moreover the cut is essentially unique: If $(T, \mathfrak{K}=\{K_\alpha; \alpha \in A\}, g)$ is another cut of order r of $(S, \mathfrak{H})$, there exists a homeomorphism h of R onto T such that $h(F_\alpha)=K_\alpha$ and $gh|F_\alpha=f|F_\alpha$ for every $\alpha \in A$.*

PROOF. First let us construct a cut $(R, \mathfrak{F}, f)$ of order r of $(S, \mathfrak{H})$. Let R_0 be the mutually disjoint union of C_α , $\alpha \in A$, where every C_α is a copy of H_α . Let $f_{0\alpha}: C_\alpha \rightarrow H_\alpha$, $\alpha \in A$, be a copy-mapping.¹⁷⁾ Let f_0 be a mapping of R_0 onto S such that for every $\alpha \in A$ $f_0|C_\alpha=f_{0\alpha}$. Define the topology of R_0 as follows: A subset B of R_0 is closed if and only if for every $\alpha \in A$ $f_0(B \cap C_\alpha)$ is closed in H_α (and hence closed in S). It can easily be seen that R_0 with this topology is a topological space. Then f_0 is a closed continuous mapping such that for every $\alpha \in A$ $f_0|C_\alpha$ is a homeomorphism of C_α onto H_α . Let $H=\{x; x \in S, \text{ order}(x, \mathfrak{H}) > r\}$; then H is a closed subset of S by the local finiteness of $\mathfrak{H}$. Let R_1 be a copy of H and $f_1: R_1 \rightarrow H$ a copy-mapping. Let R_2 be a copy of $R_0 - f_0^{-1}(H)$ and $f_2: R_2 \rightarrow R_0 - f_0^{-1}(H)$ a copy-mapping. Let R be the disjoint union of R_1 and R_2 . Let f_3 be a mapping of R_0 onto R such that i) when $x \in f_0^{-1}(H)$, $f_3(x)=f_1^{-1}(f_0(x))$, ii) when $x \in R_0 - f_0^{-1}(H)$, $f_3(x)=f_2^{-1}(x)$. Let f be a mapping of R onto S such that i) when $x \in R_1$, $f(x)=f_1(x)$, ii) when $x \in R_2$, $f(x)=f_0(f_2(x))$. Let $F_\alpha=f_3(C_\alpha)$, $\alpha \in A$; then $\mathfrak{F}=\{F_\alpha; \alpha \in A\}$ is a covering of R such that $f(F_\alpha)=H_\alpha$ for every $\alpha \in A$.

Define the topology of $R^{(18)}$ as follows: A subset B of R is closed if and only if for every $\alpha \in A$ $f(B \cap F_\alpha)$ is closed in H_α (and hence closed in S). Then it can easily be seen that R is a topological space.

Let B be a closed subset of R . Since $f(B)=f(\bigcup_\alpha (B \cap F_\alpha))=\bigcup_\alpha f(B \cap F_\alpha)$, $f(B)$ is closed in S , which shows the closedness of f . Let C be a closed set of S ; then $f^{-1}(C)$ is closed in R by the equality $f(f^{-1}(C) \cap F_\alpha)=C \cap H_\alpha$, which shows the continuity of f . Thus f is a closed continuous mapping of R onto S .

Let F_α and F_β be arbitrary elements of $\mathfrak{F}$; then $f(F_\alpha \cap F_\beta)$ is H_α or $H_\alpha \cap$

17) When there is a one-to-one mapping $f_{0\alpha}: C_\alpha \rightarrow H_\alpha$, C_α is called a copy of H_α and $f_{0\alpha}$ a copy-mapping for the brevity.

18) It is to be noted that the quotient topology Q of R obtained from $f_3: R_0 \rightarrow R$ is the same as the topology P defined in the proof: i) Let F be a closed subset of R with the topology Q . Then by the definition of Q $f_3^{-1}(F)$ is closed and hence $f_3^{-1}(F) \cap C_\alpha$ is closed for any $\alpha \in A$. Since $f_0(f_3^{-1}(F) \cap C_\alpha)=f(f_3(f_3^{-1}(F) \cap C_\alpha))=f(F \cap f_3(C_\alpha))=f(F \cap F_\alpha)$, $f(F \cap F_\alpha)$ is closed by the closedness of f_0 . Thus F is closed in R with the topology P . ii) Conversely let F be a closed subset of R with the topology P ; then $f(F \cap F_\alpha)=f_0(f_3^{-1}(F) \cap C_\alpha)$ is closed for any $\alpha \in A$. Since $f_0|C_\alpha: C_\alpha \rightarrow H_\alpha$ is a homeomorphism, $f_3^{-1}(F) \cap C_\alpha$ is closed in C_α . Hence $f_3^{-1}(F) \cap C_\alpha$ is closed in R_0 . By the local finiteness of $\{C_\alpha; \alpha \in A\}$, $f_3^{-1}(F)$ is closed in R_0 , which shows the closedness of F in the topology Q .

It is also to be noted that the topology of S is the same as the quotient topology obtained from $f_0: R_0 \rightarrow S$. This fact is a trivial consequence of the closedness and the continuity of $f_0: R_0 \rightarrow S$ with the preassigned topology.

$H_\beta \cap H$ according to $\alpha = \beta$ or $\alpha \neq \beta$. In both cases $f(F_\alpha \cap F_\beta)$ is closed, which shows the closedness of F_α .

Let x be an arbitrary point of R and V a neighborhood of $f(x)$ which meets only a finite number of elements of $\mathfrak{H} = \{H_\alpha; \alpha \in A\}$. It can easily be seen that $V \cap H_\alpha \neq \emptyset$ if and only if $f^{-1}(V) \cap F_\alpha \neq \emptyset$. Hence $f^{-1}(V)$ meets only a finite number of elements of $\mathfrak{F} = \{F_\alpha; \alpha \in A\}$. Since f is continuous, $f^{-1}(V)$ is a neighborhood of x . Thus the local finiteness of $\mathfrak{F}$ is proved.

Since it is evident that $(R, \mathfrak{F}, f)$ satisfies (5.1), (5.2), (5.3), $(R, \mathfrak{F}, f)$ is the desired cut of $(S, \mathfrak{H})$.

Let $(T, \mathfrak{K} = \{K_\alpha; \alpha \in A\}, g)$ be another cut of order r of $(S, \mathfrak{H})$. Let $f_\alpha = f|F_\alpha$ and $g_\alpha = g|K_\alpha$, $\alpha \in A$; then $h_\alpha = g_\alpha^{-1}f_\alpha$ is a homeomorphism of F_α onto K_α . Let α and β be different indices of A with $F_\alpha \cap F_\beta \neq \emptyset$. Then $F_\alpha \cap F_\beta \subset R_1$. Let $T_1 = g^{-1}(H)$ and $g_1 = g|T_1$; then g_1 is a homeomorphism of T_1 onto H . Since $h_\alpha|F_\alpha \cap F_\beta = g_1^{-1}f_1|F_\alpha \cap F_\beta$ and $h_\beta|F_\alpha \cap F_\beta = g_1^{-1}f_1|F_\alpha \cap F_\beta$, we can define a mapping h such that $h|F_\alpha = h_\alpha$ for every $\alpha \in A$. Since $\mathfrak{F}$ is locally finite and h is one-to-one onto and $h|F_\alpha = h_\alpha$ is a homeomorphism, h is a homeomorphism of R onto T . Evidently $h(F_\alpha) = K_\alpha$ and $gh|F_\alpha = f|F_\alpha$ for every $\alpha \in A$ and the theorem is completely proved.

REMARK 5.3. It is to be noted that when S is a metric space, R is also metrizable by J. Nagata's metrization theorem [25, Theorem 2]: A topological space is metrizable if it is covered by a locally finite closed covering each of whose elements is metrizable.

REMARK 5.4. Let y be arbitrary point of S ; then $f^{-1}(y) \cap F_\alpha$ is a single point or the empty set for every $\alpha \in A$ by the condition (5.1). Hence if $r_1 = \text{order}(y, \mathfrak{H}) \leq r$, then $\text{order}(x, \mathfrak{F}) = 1$ for any point $x \in f^{-1}(y)$ by the condition (5.3).

The following lemma will be used in the succeeding paragraph.

LEMMA 5.5. Let R and S be topological spaces, $\mathfrak{F} = \{F_\alpha; \alpha \in A\}$ and $\mathfrak{H} = \{H_\alpha; \alpha \in A\}$ respectively locally finite closed coverings of R and S , and φ a closed continuous mapping of R onto S such that $\varphi(F_\alpha) = H_\alpha$ for every $\alpha \in A$. Let $(R_1, \mathfrak{F}_1 = \{F_{1\alpha}; \alpha \in A\}, f)$ and $(S_1, \mathfrak{H}_1 = \{H_{1\alpha}; \alpha \in A\}, g)$ be respectively the cuts of order r of $(R, \mathfrak{F})$ and $(S, \mathfrak{H})$. Then there is one and only one closed continuous mapping ψ of R_1 onto S_1 such that $\varphi f = g \psi$.

PROOF: First let us prove the existence of ψ . Let, for every $\alpha \in A$, $\varphi_\alpha = \varphi|F_\alpha$, $f_\alpha = f|F_{1\alpha}$ and $g_\alpha = g|H_{1\alpha}$. Let α and β be different indices of A with $F_{1\alpha} \cap F_{1\beta} \neq \emptyset$ and x an arbitrary point of $F_{1\alpha} \cap F_{1\beta}$. Then $\text{order}(f(x), \mathfrak{F}) > r$ and hence $\text{order}(\varphi(f(x)), \mathfrak{H}) > r$. Therefore $g^{-1}(\varphi(f(x)))$ is a single point and there exists $y \in S_1$ such that $y = g^{-1}(\varphi(f(x)))$. Let $f_0 = f|F_{1\alpha} \cap F_{1\beta}$, $\varphi_0 = \varphi|f(F_{1\alpha} \cap F_{1\beta})$ and $h = g_\alpha^{-1}|\varphi(f(F_{1\alpha} \cap F_{1\beta}))$. Then $g_\alpha^{-1}\varphi_\alpha f_\alpha|F_{1\alpha} \cap F_{1\beta} = h\varphi_0 f_0 = g_\beta^{-1}\varphi_\beta f_\beta|F_{1\alpha} \cap F_{1\beta}$. Thus we can define $\psi: R_1 \rightarrow S_1$ in such a way that $\psi|F_{1\alpha} = g_\alpha^{-1}\varphi_\alpha f_\alpha$ for every $\alpha \in A$. Let $\psi_\alpha = \psi|F_{1\alpha}$, $\alpha \in A$; then $\psi_\alpha = g_\alpha^{-1}\varphi_\alpha f_\alpha$. Since g_α is a homeomorphism, $g_\alpha \psi_\alpha = \varphi_\alpha f_\alpha$ for every $\alpha \in A$. In other words $g\psi|F_{1\alpha} = \varphi f|F_{1\alpha}$ for every $\alpha \in A$. Hence $g\psi = \varphi f$. Since ψ_α is closed continuous for every $\alpha \in A$, ψ is closed continuous by the local finiteness of $\mathfrak{F}_1$ and $\mathfrak{H}_1$.

Let ϕ_1 be another closed continuous mapping of R_1 onto S_1 such that $g\phi_1 = \varphi f$. Then for every $\alpha \in A$ $g\phi_1|F_{1\alpha} = \varphi f|F_{1\alpha}$ and hence $\phi_1|F_{1\alpha} = g_\alpha^{-1}\varphi_\alpha f_\alpha$, which proves $\phi_1 = \psi$.

Let $(S_{i1}, \mathfrak{H}_{i11} = \{H(i, i, \alpha_i); \alpha_i \in A_i\}, h_{i1})$ is the cut of order $r = n - m + 1$ of $(R_0, \mathfrak{F}_{01})$. Let

$$(6.5) \quad H(i, i, \alpha_j) = \bigcup \{H(i, i, \alpha_i); f_{ij}(\alpha_i) = \alpha_j\} \text{ if } j < i,$$

and

$$(6.6) \quad H(i, i, \alpha_j) = h_{i1}^{-1}(F(0, \alpha_j)) \cap H(i, i, f_{ji}(\alpha_j)) \text{ if } j > i,$$

where $f_{ji}, j > i$, is $f_{i+1, i} \cdots f_{j-1, j-2} f_{j, j-1}$. Then $\mathfrak{H}_{i1j} = \{H(i, i, \alpha_j); \alpha_j \in A_j\}, j = 1, 2, \dots$, is a sequence of locally finite closed coverings of S_{i1} which satisfies the following conditions.

(6.7) For any point p of S_{i1} and any neighborhood U of p , there exists $\mathfrak{H}_{i1j}$ such that $\text{Star}(p, \mathfrak{H}_{i1j}) \subset U$.

(6.8) For any j and any $\alpha_j \in A_j$, $H(i, i, \alpha_j) = \bigcup \{H(i, i, \alpha_{j+1}); f_{j+1, j}(\alpha_{j+1}) = \alpha_j\}$.

(6.9) For any $j \geq i$ and any $\alpha_j \in A_j$, $h_{i1} | H(i, i, \alpha_j)$ is a homeomorphism of $H(i, i, \alpha_j)$ onto $F(0, \alpha_j)$.

iii) We construct by a successive process the cuts $(S_{ij}, \mathfrak{H}_{i, i-j+1, i-j+1}, h_{ij})$, $j = 2, \dots, i$, and sequences of locally finite closed coverings $\mathfrak{H}_{i, i-j+1, k} = \{H(i, i-j+1, \alpha_k); \alpha_k \in A_k\}, k = 1, 2, \dots$, of $S_{ij}, j = 2, \dots, i$, which satisfy the following conditions.

(6.10) $(S_{ij}, \mathfrak{H}_{i, i-j+1, i-j+1}, h_{ij})$ is the cut of order r of $(S_{i, i-j}, \mathfrak{H}_{i, i-j+2, i-j+1})$.

(6.11) $H(i, i-j+1, \alpha_k) = \bigcup \{H(i, i-j+1, \alpha_{k-j+1}); f_{i-j+1, k}(\alpha_{i-j+1}) = \alpha_k\}$ if $k < i-j+1$, and $H(i, i-j+1, \alpha_k) = h_{ij}^{-1}(H(i, i-j+2, \alpha_k)) \cap H(i, i-j+1, f_{k, i-j+1}(\alpha_k))$ if $k > i-j+1$. Then the sequence $\mathfrak{H}_{i, i-j+1, k}, k = 1, 2, \dots$, satisfies the following conditions.

(6.12) For any point p of S_{ij} and any neighborhood U of p , there exists k such that $\text{Star}(p, \mathfrak{H}_{i, i-j+1, k}) \subset U$.

(6.13) For any k and any $\alpha_k \in A_k$, $H(i, i-j+1, \alpha_k) = \bigcup \{H(i, i-j+1, \alpha_{k+1}); f_{k+1, k}(\alpha_{k+1}) = \alpha_k\}$.

(6.14) For any k with $k \geq i-j+1$ and any $\alpha_k \in A_k$, $h_{ij} | H(i, i-j+1, \alpha_k)$ is a homeomorphism of $H(i, i-j+1, \alpha_k)$ onto $H(i, i-j+2, \alpha_k)$.

iv) Let $h_i = h_{i1} h_{i2} \cdots h_{ii}$; then h_i is a closed continuous mapping of S_{ii} onto R_0 such that $h_i | H(i, i, \alpha_i)$ is a homeomorphism of $H(i, 1, \alpha_i)$ onto $F(0, \alpha_i)$ for every $\alpha_i \in A_i$. Let $s = s(x), x \in R_0$, be the greatest integer such that order $(x, \mathfrak{F}_{0s}) \leq r$. When order $(x, \mathfrak{F}_{01}) > r$, let $s(x) = 0$.

First we consider the case $s(x) = 0$. Then order $(x, \mathfrak{F}_{0i}) > r$ for every i . Hence order $(x, h_{i1} h_{i2} \cdots h_{ij}) = 1$ for every j . Especially order $(x, h_i) = 1$.

Next we consider the case $0 < s(x) \leq i$. Then order $(x, h_{i1} \cdots h_{i, i-s})^{20)} = 1$ and order $(x, h_{i1} \cdots h_{i, i-s+1}) = \text{order}(x, \mathfrak{F}_{0s})$. Let y be an arbitrary point of $(h_{i1} \cdots h_{i, i-s+1})^{-1}(x)$ and then order $(y, \mathfrak{H}_{i, i-s+1}) = 1$ by Remark 5.4. Hence order $(y, h_{i, i-s+2} \cdots h_{ii}) = 1$. Therefore order $(x, h_i) = \text{order}(x, \mathfrak{F}_{0s})$.

Last we consider the case $i < s(x)$. Then order $(x, \mathfrak{F}_{0i}) \leq \text{order}(x, \mathfrak{F}_{0s}) \leq r$. Hence order $(x, h_{i1}) = \text{order}(x, \mathfrak{F}_{0i})$. By the same argument as in the case of $0 < s(x) \leq i$ we get order $(x, h_i) = \text{order}(x, \mathfrak{F}_{0i})$.

In any case of the above, we get order $(x, h_i) \leq r$ and hence

$$(6.15) \quad \text{order } h_i \leq r.$$

Diagram 2.

20) When $s(x) = i$, there is no necessity for considering this.

Let $h_{i+1,1}=g_{i+1,0}$. Since $g_{i+1,0}(H(i+1, i+1, \alpha_i))=F(0, \alpha_i)$ for every $\alpha_i \in A_i$, there exists $g_{i+1,1}: S_{i+1,2} \rightarrow S_{i1}$ with $g_{i+1,0}h_{i+1,2}=h_{i1}g_{i+1,1}$ by Lemma 5.5. By the successive applications of Lemma 5.5 we get $g_{i+1,j}: S_{i+1,j+1} \rightarrow S_{ij}$ with

$$(6.16) \quad g_{i+1,j-1}h_{i+1,j+1}=h_{ij}g_{i+1,j}, \quad j=0, 1, \dots, i,$$

since $g_{i+1,j+1}(H(i+1, i+1-(j-1), \alpha_{i-(j-1)}))=H(i, i-(j-1)+1, \alpha_{i-j+1})^{21)}$ for every $\alpha_{i-j+1} \in A_{i-j+1}$. Therefore the equality $h_{i1}h_{i2} \cdots h_{ii}g_{i+1,i}=g_{i+1,0}h_{i+1,2}h_{i+1,3} \cdots h_{i+1,i+1}$ holds. Since $g_{i+1,0}=h_{i+1,1}$, we get

$$(6.17) \quad h_i g_{i+1,i} = h_{i+1}.$$

Therefore

$$(6.18) \quad g_{i0}g_{21} \cdots g_{i,i-1} = h_i.$$

Let α_{i+1} be an arbitrary element of A_{i+1} and $\alpha_i = f_{i+1,i}(\alpha_{i+1})$. Let $h_{i+1,2,\alpha_i} = h_{i+1,2} | H(i+1, i, \alpha_i)$, $g_{i+1,0,\alpha_i} = g_{i+1,0} | h_{i+1,2}(H(i+1, i, \alpha_i)) = g_{i+1,0} | H(i+1, i+1, \alpha_i)$ and $h_{i+1,1,\alpha_i} = h_{i1} | H(i, i, \alpha_i)$. Then by the construction of $g_{i+1,1}$ (cf. the proof of Lemma 5.5) $g_{i+1,1} | H(i+1, i, \alpha_i) = h_{i+1,1,\alpha_i}^{-1} g_{i+1,0,\alpha_i} h_{i+1,2,\alpha_i}$. Since $h_{i+1,2}(H(i+1, i, \alpha_{i+1})) = H(i+1, i+1, \alpha_{i+1})$ and $h_{i1}(H(i, i, \alpha_{i+1})) = F(0, \alpha_{i+1}) = g_{i+1,0}(H(i+1, i+1, \alpha_{i+1}))$, we get $g_{i+1,1} | H(i+1, i, \alpha_{i+1}) = h_{i+1,1,\alpha_i}^{-1} g_{i+1,0,\alpha_i} h_{i+1,2,\alpha_i} | H(i+1, i, \alpha_{i+1})$. Thus $g_{i+1,1} | H(i+1, i, \alpha_{i+1})$ is a homeomorphism of $H(i+1, i, \alpha_{i+1})$ onto $H(i, i, \alpha_{i+1})$. By the same argument as above we can know succeedingly that for every $\alpha_{i+1} \in A_{i+1}$ $g_{i+1,j} | H(i+1, i-j+1, \alpha_{i+1})$ is a homeomorphism of $H(i+1, i-j+1, \alpha_{i+1})$ onto $H(i, i-j+1, \alpha_{i+1})$, $j=1, \dots, i$. Especially $g_{i+1,i} | H(i+1, 1, \alpha_{i+1})$ is a homeomorphism of $H(i+1, 1, \alpha_{i+1})$ onto $H(i, 1, \alpha_{i+1})$ for every $\alpha_{i+1} \in A_{i+1}$.

Figure 1.

21) More precisely we have to prove this equality as well as (6.16) by the induction on j . The proof is rather simple and is left to the reader.

vi) Let $R_i = S_{ii}$, $F(i, \alpha_j) = H(i, 1, \alpha_j)$ and $\mathfrak{F}_{ij} = \{F(i, \alpha_j); \alpha_j \in A_j\}$, $i=1, 2, \dots$, $j=1, 2, \dots$. Let $T = \lim \{R_i, g_{i+1,i}(i=0, 1, 2, \dots)\}$ and π_i the projection of T onto R_i . Since every R_i is metrizable by Remark 5.3, T is also metrizable. Let $K(\alpha_i) = \pi_i^{-1}(F(i, \alpha_i))$; then it can easily be seen from iii) and v) that $\mathfrak{R}_i = \{K(\alpha_i); \alpha_i \in A_i\}$, $i=1, 2, \dots$, is a sequence of locally finite closed coverings of T such that

(6.19) $\mathfrak{R}_{i+1}$ refines $\mathfrak{R}_i$ for every i ,

(6.20) for any point ξ of T and neighborhood U of ξ there exists i with $\text{Star}(\xi, \mathfrak{R}_i) \subset U$.

vii) The projection $\pi_0: T \rightarrow R_0$ is clearly continuous. To prove order $\pi_0 \leq r = n - m + 1$, let us place the assumption that order $\pi_0 > r$. Then there exist $r+1$ points $\xi_1, \dots, \xi_{r+1}$ of T with $\pi_0(\xi_j) = \pi_0(\xi_1)$, $j=2, \dots, r+1$. Since ξ_j , $j=1, \dots, r+1$, are mutually different, there exists i such that $\pi_i(\xi_j)$, $j=1, \dots, r+1$, are mutually different points of R_i . Since $g_{10}g_{21} \dots g_{i,i-1}\pi_i(\xi_j) = \pi_0(\xi_1)$, $j=1, \dots, r+1$, we get $h_i\pi_i(\xi_j) = \pi_0(\xi_1)$, $j=1, \dots, r+1$, by (6.18). Thus we get order $h_i > r$. On the other hand order $h_i \leq r$ by (6.15), which is a contradiction. Therefore we get order $\pi_0 \leq r$.

viii) Now let us prove the closedness of π_0 . Let i be an arbitrary positive integer and α_i an arbitrary element of A_i . Since a) $\pi_i(K(\alpha_i)) = \pi_i(\pi_i^{-1}(F(i, \alpha_i))) = F(i, \alpha_i)$, b) $h_i(F(i, \alpha_i)) = h_i(H(i, 1, \alpha_i)) = F(0, \alpha_i)$ by iv) and c) $h_i = g_{10} \dots g_{i,i-1}$ by (6.18), we get $\pi_0(K(\alpha_i)) = g_{10} \dots g_{i,i-1}\pi_i(K(\alpha_i)) = h_i(F(i, \alpha_i)) = F(0, \alpha_i)$. Let F be a non-empty closed subset of T and x_0 an arbitrary point of $\overline{\pi_0(F)}$. Then $\pi_0(F \cap K(\alpha_1)) \subset F(0, \alpha_1)$ and hence $\overline{\pi_0(F)} = \bigcup \{\pi_0(F \cap K(\alpha_1)); \alpha_1 \in A_1\}$ by the local finiteness of $\mathfrak{F}_{01} = \{F(0, \alpha_1); \alpha_1 \in A_1\}$. Therefore there exists $\alpha_1(x_0) \in A_1$ such that $x_0 \in \overline{\pi_0(F \cap K(\alpha_1(x_0)))}$. By (6.19) and the local finiteness of $\mathfrak{F}_{02}$, $\overline{\pi_0(F \cap K(\alpha_1(x_0)))} = \bigcup \{\pi_0(F \cap K(\alpha_1(x_0)) \cap K(\alpha_2)) = \pi_0(F \cap K(\alpha_2)); \alpha_2 \in A_2, f_{21}(\alpha_2) = \alpha_1(x_0)\}$. Hence there exists $\alpha_2(x_0) \in A_2$ with $f_{21}(\alpha_2(x_0)) = \alpha_1(x_0)$ such that $x_0 \in \overline{\pi_0(F \cap K(\alpha_2(x_0)))}$. Repeating this argument we get $\alpha_i(x_0) \in A_i$ with $f_{i,i-1}(\alpha_i(x_0)) = \alpha_{i-1}(x_0)$, $i=2, 3, \dots$, such that $x_0 \in \overline{\pi_0(F \cap K(\alpha_i(x_0)))}$.

Let $r_1 = \text{order}(x_0, \pi_0)$ and $\{\xi_1, \dots, \xi_{r_1}\} = \pi_0^{-1}(x_0)$. By (6.20) there exists i_0 such that $\text{Star}(\xi_i, \mathfrak{R}_{i_0}) \cap \text{Star}(\xi_j, \mathfrak{R}_{i_0}) = \emptyset$ whenever $i \neq j$. Then evidently $\pi_{i_0}(\xi_i)$, $i=1, \dots, r_1$, are mutually different points of R_{i_0} such that $\text{Star}(\pi_{i_0}(\xi_i), \mathfrak{F}_{i_0, i_0}) \cap \text{Star}(\pi_{i_0}(\xi_j), \mathfrak{F}_{i_0, i_0}) = \emptyset$ whenever $i \neq j$. Since a) $x_0 \in \overline{\pi_0(F \cap K(\alpha_{i_0}(x_0)))} \subset \pi_0(K(\alpha_{i_0}(x_0))) = F(0, \alpha_{i_0}(x_0))$, b) $\pi_0(K(\alpha_{i_0}(x_0))) = h_{i_0}\pi_{i_0}(K(\alpha_{i_0}(x_0))) = h_{i_0}(F(i_0, \alpha_{i_0}(x_0)))$, and c) $h_{i_0}^{-1}(x_0) = \{\pi_{i_0}(\xi_i); i=1, \dots, r_1\}$, there exists one and only one ξ_i such that $\pi_{i_0}(\xi_i) \in F(i_0, \alpha_{i_0}(x_0))$. We can assume without loss of generality that $\xi_1 = \xi_i$. Since order $(\pi_{i_0}(\xi_1), g_{i_0+1, i_0}g_{i_0+2, i_0+1} \dots g_{j-1, j-1}) = 1$ whenever $j > i_0$, we get $\pi_{i_0}^{-1}\pi_{i_0}(\xi_1) = \xi_1$. Hence $\xi_1 \in \pi_{i_0}^{-1}(F(i_0, \alpha_{i_0}(x_0))) = K(\alpha_{i_0}(x_0))$. In other words $\pi_0^{-1}(x_0) \cap K(\alpha_{i_0}(x_0)) = \xi_1$. When $j > i_0$, $K(\alpha_j(x_0)) \subset K(\alpha_{i_0}(x_0))$ and hence $\pi_0^{-1}(x_0) \cap K(\alpha_j(x_0)) = \xi_1$ by $x_0 \in \pi_0(K(\alpha_j(x_0)))$. Thus $\xi_1 \in \bigcap_{i=1}^{\infty} K(\alpha_i(x_0))$ and by (6.20) $\xi_1 = \bigcap_{i=1}^{\infty} K(\alpha_i(x_0))$. Let U be an arbitrary neighborhood of ξ_1 and there exists i_1 such that $\text{Star}(\xi_1, \mathfrak{R}_{i_1}) \subset U$. Since $U \cap F \supset \text{Star}(\xi_1, \mathfrak{R}_{i_1}) \cap F \supset K(\alpha_{i_1}(x_0)) \cap F \neq \emptyset$, $\xi_1 \in F$ and hence $x_0 = \pi_0(\xi_1) \in \pi_0(F)$. Since x_0 is an arbitrary point of $\overline{\pi_0(F)}$, we get $\overline{\pi_0(F)} \subset \pi_0(F)$, which shows the closedness of $\pi_0(F)$. Thus the closedness of π_0 is proved.

ix) Finally let us show $\dim T = m$. We have already known in vii) and viii) that π_0 is a closed continuous mapping of T onto R_0 with order $\pi_0 \leq r$. Hence Morita [18, Theorem 4] can be applied and $\dim T \geq \dim R_0 - r + 1 = n - (n - m + 1) + 1 = m$.

Now we proceed to prove $\dim T \leq m$. Let i be an arbitrary positive integer and x an arbitrary point of R_i . If $\text{order}(x, \mathfrak{F}_{ii}) > 1$ there exist different indices α_i and β_i of A_i such that $x \in F(i, \alpha_i) \cap F(i, \beta_i)$. Then $h_{i2}h_{i3}\cdots h_{ii}(x) \in h_{i2}\cdots h_{ii}(F(i, \alpha_i) \cap F(i, \beta_i)) \subset h_{i2}\cdots h_{ii}(F(i, \alpha_i)) \cap h_{i2}\cdots h_{ii}(F(i, \beta_i)) = H(i, i, \alpha_i) \cap H(i, i, \beta_i)$ by (6.14). Since for every point $y \in S_{i1}$ either $\text{order}(y, \mathfrak{F}_{ii}) = 1$ or $> r$ holds, we get $\text{order}(h_{i2}\cdots h_{ii}(x), \mathfrak{F}_{ii}) > r$. Hence $\text{order}(h_i(x), \mathfrak{F}_{0i}) > r$.

Let

$$(6.21) \quad P_i = \bigcup \{ \bigcap_{j=1}^{r+1} F(0, \alpha(j)); \alpha(1), \dots, \alpha(r+1) \text{ are mutually different indices of } A_i \};$$

then by (6.4) and the local finiteness of $\mathfrak{F}_{0i}$, P_i is a closed subset of R_0 with $\dim P_i \leq n - (r+1) + 1 = n - (n-m+1) = m-1$ (cf. Morita [15, Theorem 3.2]). Let α_i be an arbitrary element of A_i and

$$(6.22) \quad Q_i = \{x; x \in R_i, \text{order}(x, \mathfrak{F}_{ii}) > 1\}$$

and then Q_i is a closed subset of R_i by the local finiteness of $\mathfrak{F}_{ii}$ and $P_i \cap F(0, \alpha_i) \supset h_i(Q_i \cap F(i, \alpha_i))$. Since $h_i|F(i, \alpha_i)$ is a homeomorphism, we get $\dim Q_i \cap F(i, \alpha_i) \leq m-1$. Then by the local finiteness of $\mathfrak{F}_{ii}$ we get

$$(6.23) \quad \dim Q_i \leq m-1.$$

Let x be an arbitrary point of Q_i and γ_i and δ_i be elements of A_i with $\gamma_i \neq \delta_i$ such that $x \in F(i, \gamma_i) \cap F(i, \delta_i)$. Let $s = s(h_i(x))$, where s is the function of R_0 defined in iv). Let us consider three cases. a) When $s=0$, we get $\text{order}(h_i(x), h_{i+1}) = 1$ by iv). Since $h_i g_{i+1, i} = h_{i+1}$ by (6.17), $g_{i+1, i}^{-1}(x)$ is a single point. Since $g_{i+1, i}(F(i+1, \alpha_i)) = F(i, \alpha_i)$ for every $\alpha_i \in A_i$, we get $g_{i+1, i}^{-1}(x) \in F(i+1, \gamma_i) \cap F(i+1, \delta_i)$. b) When $s \geq i$, then by iv) $\text{order}(h_i(x), h_i) = \text{order}(h_i(x), \mathfrak{F}_{0i}) \leq \text{order}(h_i(x), \mathfrak{F}_{0s}) \leq r$. Then by the construction of R_i and Remark 5.4, we get $\text{order}(x, \mathfrak{F}_{ii}) = 1$, which is a contradiction. Therefore the case $s \geq i$ does not happen. c) When $s < i$, we have already known in iv) that $\text{order}(h_i(x), h_i) = \text{order}(h_i(x), h_{i+1}) = \text{order}(h_i(x), \mathfrak{F}_{0s})$. Hence $g_{i+1, i}|h_{i+1}^{-1}(h_i(x))$ is a homeomorphism of $h_{i+1}^{-1}(h_i(x))$ onto $h_i^{-1}(h_i(x))$ by the relation (6.16). Since $|h_{i+1}^{-1}(h_i(x))| = |h_i^{-1}(h_i(x))|$, $g_{i+1, i}^{-1}(x)$ is a single point. Therefore $g_{i+1, i}^{-1}(x) \in F(i+1, \gamma_i) \cap F(i+1, \delta_i)$.

Thus we can conclude that for every i $g_{i+1, i}^{-1}(Q_i) \subset L_{i+1} = \{x; x \in R_{i+1}, \text{order}(x, \mathfrak{F}_{i+1, i}) > 1\}$. Since evidently $g_{i+1, i}(L_{i+1}) \subset Q_i$, we get $g_{i+1, i}^{-1}(Q_i) \subset L_{i+1}$ and hence $g_{i+1, i}|L_{i+1}$ is a homeomorphism of L_{i+1} onto Q_i . Since $L_j \subset Q_j$ for every j , $g_{i+1, i} \cdots g_{i+j, i+j-1}|(g_{i+1, i} \cdots g_{i+j, i+j-1})^{-1}(Q_i)$ is a homeomorphism of $(g_{i+1, i} \cdots g_{i+j, i+j-1})^{-1}(Q_i)$ onto Q_i for every positive integer j . Therefore

$$(6.24) \quad K_i = \{\xi; \xi \in T, \text{order}(\xi, \mathfrak{R}_i) > 1\}$$

is homeomorphic to Q_i and hence by (6.23) we get

$$(6.25) \quad \dim K_i \leq m-1.$$

Let

$$(6.26) \quad T_0 = T - \bigcup_{i=1}^{\infty} K_i$$

and

$$(6.27) \quad U(\alpha_i) = T_0 \cap K(\alpha_i), \alpha_i \in A_i,$$

and then $\mathfrak{U}_i = \{U(\alpha_i); \alpha_i \in A_i\}$, $i=1, 2, \dots$, is as can easily be seen a sequence of locally finite closed coverings of T_0 which satisfies the following conditions.

$$(6.28) \quad \mathfrak{U}_{i+1} \text{ refines } \mathfrak{U}_i \text{ for every } i.$$

$$(6.29) \quad \text{order } \mathfrak{U}_i \leq 1 \text{ for every } i.$$

$$(6.30) \quad \text{For any point } \xi_0 \text{ of } T_0 \text{ and any neighborhood } G_0 \text{ of } \xi_0 \text{ there exists } i \text{ such that } \text{Star}(\xi_0, \mathfrak{U}_i) \subset G_0.$$

Therefore by Theorem 3.2 we get

$$(6.31) \quad \dim T_0 \leq 0.$$

Since K_i is closed in T and $\dim K_i \leq m-1$ by (6.25) we get by the sum theorem

$$(6.32) \quad \dim \bigcup_{i=1}^{\infty} K_i \leq m-1.$$

Hence by Theorem 3.2 T is the sum of $m+1$ subspaces T_i , $i=0, 1, \dots, m$, with $\dim T_i \leq 0$ and hence we get at last

$$(6.33) \quad \dim T \leq m.$$

Thus our construction is completed.

Concerning the dimension of an inverse limiting space we get the following theorem.

THEOREM 6.1. *Let $R = \lim \{R_i, f_{ij}\}$ where R_i is a metric space and $f_{ij}: R_i \rightarrow R_j$ ($i > j$) is continuous, for every i . Then $\dim R \leq \liminf \dim R_i$.*

PROOF. When $\liminf \dim R_i = \infty$, the theorem is evidently true. When $\liminf \dim R_i = n (< \infty)$, R is homeomorphic to $\lim \{R_{i'}, f_{i'j'}\}$ where $\{R_{i'}, f_{i'j'}\}$ is a cofinal subsystem of $\{R_i, f_{ij}\}$ and $\dim R_{i'} \leq n$ for every i' . Hence we can assume with no loss of generality that $\dim R_i \leq n$ for every i . It is easy to construct by the induction a sequence of open coverings $\mathfrak{U}_{ji}$, $i=1, 2, \dots$, of R_j , $j=1, 2, \dots$, which satisfies the following conditions.

$$(6.34) \quad \text{mesh } \mathfrak{U}_{ji} < 1/i \text{ for every } j \text{ and } i.$$

$$(6.35) \quad \mathfrak{U}_{j,i+1} \text{ refines } \mathfrak{U}_{ji} \text{ for every } j \text{ and } i.$$

$$(6.36) \quad \mathfrak{U}_{ji} \text{ refines } f_{j,j-1}^{-1}(\mathfrak{U}_{j-1,i}) = \{f_{j,j-1}^{-1}(U); U \in \mathfrak{U}_{j-1,i}\} \text{ for every } j \text{ and } i.$$

It is also easy to construct by the induction a locally finite open covering $\mathfrak{V}_i$ of R_i , $i=1, 2, \dots$, which satisfies the following conditions.

$$(6.37) \quad \mathfrak{V}_{i+1} = \{\bar{V}; V \in \mathfrak{V}_{i+1}\} \text{ refines } \mathfrak{U}_{i+1,i+1} \text{ as well as } f_{i+1,i}^{-1}(\mathfrak{V}_i).$$

$$(6.38) \quad \text{order } \mathfrak{V}_i \leq n+1.$$

Let

$$(6.39) \quad \mathfrak{F}_i = \{\pi_i^{-1}(\bar{V}); V \in \mathfrak{V}_i\}, \quad i=1, 2, \dots,$$

where $\pi_i: R \rightarrow R_i$ is the projection. Then $\mathfrak{F}_i$, $i=1, 2, \dots$, is a sequence of locally finite closed coverings of R such that

$$(6.40) \quad \mathfrak{F}_{i+1} \text{ refines } \mathfrak{F}_i \text{ for every } i,$$

$$(6.41) \quad \text{order } \mathfrak{F}_i \leq n+1 \text{ for every } i.$$

Let ξ be an arbitrary point of R and W an arbitrary neighborhood of ξ and then there exist i and a neighborhood W_i of $\pi_i(\xi)$ such that $\pi_i^{-1}(W_i) \subset W$. Then by (6.34) there exists j such that $\text{Star}(\pi_i(\xi), \mathfrak{U}_{ij}) \subset W_i$. By (6.35) we can assume with no loss of generality that $i \leq j$. By (6.36) and (6.37) we get

$$(6.42) \quad \text{Star}(\xi, \mathfrak{F}_j) \subset \pi_i^{-1}(W_i) \subset W.$$

Since $\mathfrak{F}_i$, $i=1, 2, \dots$, satisfy the conditions (6.40), (6.41), (6.42), we get $\dim R \leq n$ by Theorem 3.2. Thus the theorem is proved.

REMARK 6.2. Even if every f_{ij} in the above theorem is onto and $\dim R_i = n$ for every i , we cannot assert that $\dim R = n$. The space T which has been constructed in this paragraph is such an example.

§ 7. Characterization of dimension by means of a sequence of coverings

The following lemma is a slight modification of Morita's metrization theorem [18, Theorem 1].

LEMMA 7.1. *In order that a topological space R be metrizable it is necessary*

and sufficient that there exists a sequence of locally finite coverings $\mathfrak{F}_i$, $i=1, 2, \dots$, of R such that

(7.1) for any neighborhood U of any point x of R there exists i with $\text{Star}(x, \mathfrak{F}_i) \subset U$.

Now we proceed to get a weaker condition which characterizes dimension than that of (B) of § 3.

THEOREM 7.2. *In order that a topological space R be a metrizable space with $\dim R \leq n$ it is necessary and sufficient that there exists a sequence of locally finite coverings $\mathfrak{F}_i = \{F_{\alpha_i}; \alpha_i \in A_i\}$, $i=1, 2, \dots$, which satisfies the following conditions.*

(7.2) $\mathfrak{F}_{i+1}$ refines $\mathfrak{F}_i$ for every i .

(7.3) $\liminf \text{order}(x, \mathfrak{F}_i) \leq n+1$ for every $x \in R$.

(7.4) For any point $x \in R$ and any neighborhood U of x there exists i with $\text{Star}(x, \mathfrak{F}_i) \subset U$.

PROOF. The necessity of the theorem is obvious from the implication (A) $\rightarrow$ (B) of § 3.

Let us assume the existence of a sequence satisfying the conditions of the proposition. Then R is metrizable by Lemma 7.1. For every i let us define a single-valued mapping $f_{i+1, i}$ of A_{i+1} into A_i in such a way that $f_{i+1, i}(\alpha_{i+1}) = \alpha_i$ yields $\bar{F}_{\alpha_{i+1}} \subset F_{\alpha_i}$.

Consider A_i as a topological space with discrete topology, $i=1, 2, \dots$. Let A_0 be the aggregate of points $a = (\alpha_1, \alpha_2, \dots)$ of $\lim \{A_i, f_{i+1, i}\}$ such that $\bigcap_{i=1}^{\infty} F_{\alpha_i} \neq \emptyset$ and f a mapping of A_0 into R defined by $f(a) = f((\alpha_1, \alpha_2, \dots)) = \bigcap_{i=1}^{\infty} F_{\alpha_i}$. Then we can verify by the same argument as in the proof of (B) $\rightarrow$ (C) of § 3 that f is a closed continuous mapping of A_0 onto R .

To prove $\text{order } f \leq n+1$ assume that $\text{order } f > n+1$. Then there exists a point x_0 of R with $\text{order}(x_0, f) > n+1$. Let a_i , $i=1, \dots, n+2$, be mutually different points of A_0 with $f(a_i) = x_0$ for $i=1, \dots, n+2$. Let j be an integer such that $\pi_j(a_i)$, $i=1, \dots, n+2$, are mutually different points of A_j , where π_j is the projection of A_0 into A_j . By the condition (7.3) there exists k with $k \geq j$ such that $\text{order}(x_0, \mathfrak{F}_k) \leq n+1$. Since $k \geq j$, $\pi_k(a_i)$, $i=1, \dots, n+2$, are mutually different points of A_k . Hence $\text{order}(x_0, \mathfrak{F}_k) > n+1$, which is a contradiction. Thus $\text{order } f \leq n+1$ is true. Evidently $\dim A_0 \leq 0$. Hence the condition (C) of § 3 is satisfied and we get $\dim R \leq n$. The proof is completed.

The following proposition is a trivial consequence of Theorem 7.2.

COROLLARY 7.3 (C. H. Dowker and W. Hurewicz [6]). *A metric space R has dimension $\leq n$ if and only if there exists a sequence of locally finite open coverings $\mathfrak{U}_i$, $i=1, 2, \dots$, of R which satisfies the following conditions.*

(7.5) $\mathfrak{U}_{i+1}$ refines $\mathfrak{U}_i$ for every i .

(7.6) $\text{order } \mathfrak{U}_i \leq n+1$ for every i .

(7.7) $\text{mesh } \mathfrak{U}_i \rightarrow 0$ as $i \rightarrow \infty$.

The following theorem of P. Alexandroff and P. Urysohn is well known.

LEMMA 7.4. *In order that a topological space R be metrizable it is necessary and sufficient that there exists a sequence of open coverings $\mathfrak{U}_i$, $i=1, 2, \dots$, which satisfies the following conditions.*

(7.8) $\mathfrak{U}_{i+1}^*$ refines $\mathfrak{U}_i$ for every i .²²⁾

²²⁾ $\mathfrak{U}_j^* = \{\text{Star}(U, \mathfrak{U}_j); U \in \mathfrak{U}_j\}$.

(7.9) For any point x of R and any neighborhood U of x there exists i with $\text{Star}(x, \mathfrak{U}_i) \subset U$.

THEOREM 7.5. *In order that a topological space R be a metrizable space with $\dim R \leq n$ it is necessary and sufficient that there exists a sequence of open coverings $\mathfrak{U}_i$, $i=1, 2, \dots$, of R which satisfies the following conditions.*

(7.10) $\mathfrak{U}_{i+1}^*$ refines $\mathfrak{U}_i$ for every i .

(7.11) $\liminf \text{order}(x, \mathfrak{U}_i) \leq n+1$ for every point $x \in R$.

(7.12) For any point $x \in R$ and any neighborhood U of x there exists i with $\text{Star}(x, \mathfrak{U}_i) \subset U$.

PROOF. Let R be a metrizable space with $\dim R \leq n$. Let $\mathfrak{B}_i$, $i=1, 2, \dots$, be a sequence of open coverings of R with $\text{mesh } \mathfrak{B}_i^{23) < 1/i}$. Let $\mathfrak{U}_1$ be an open covering of R with order $\mathfrak{U}_1 \leq n+1$ such that $\mathfrak{U}_1$ refines $\mathfrak{B}_1$. Now make the induction assumption that there exist open coverings $\mathfrak{U}_i$, $i=1, \dots, m$, of R with $\text{mesh } \mathfrak{U}_i < 1/i$ such that $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_3^* > \dots > \mathfrak{U}_m^*$ and order $\mathfrak{U}_i \leq n+1$, $i=1, \dots, m$. Let $\mathfrak{U}_{m+1}$ be an open covering of R with order $\mathfrak{U}_{m+1} \leq n+1$ such that $\mathfrak{U}_{m+1}^*$ refines $\mathfrak{U}_m$ as well as $\mathfrak{B}_{m+1}$. Thus there exists by the induction a sequence of open coverings $\mathfrak{U}_i$, $i=1, 2, \dots$, of R such that a) $\text{mesh } \mathfrak{U}_i < 1/i$ for every i , b) order $\mathfrak{U}_i \leq n+1$ for every i , and c) $\mathfrak{U}_{i+1}^*$ refines $\mathfrak{U}_i$ for every i . Then the conditions (7.10), (7.11) and (7.12) are evidently satisfied.

Conversely let us assume the existence of a sequence of open coverings $\mathfrak{U}_i = \{U_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, satisfying the conditions of the proposition. Then R is metrizable by Lemma 7.4. We shall prove $\dim R \leq n$. Define a single-valued mapping $f_{i+1, i} : A_{i+1} \rightarrow A_i$, $i=1, 2, \dots$, in such a way that $f_{i+1, i}(\alpha_{i+1}) = \alpha_i$ yields $\text{Star}(U_{\alpha_{i+1}}, \mathfrak{U}_{i+1}) \subset U_{\alpha_i}$. For any i and j with $i > j$ let $f_{ij} = f_{j+1, j} \dots f_{i-1, i-2} f_{i, i-1}$. Consider A_i as a topological space with discrete topology. Let R_0 be the aggregate of points a of $\lim \{A_i, f_{ij}\}$ with $\bigcap_{i=1}^{\infty} U_{\pi_i(a)} \neq \emptyset$, where π_j , $j=1, 2, \dots$, is the projection of $\lim A_i$ into A_j . When $\bigcap_{i=1}^{\infty} U_{\pi_i(a)} \neq \emptyset$, it consists of one and only one point by the condition (7.12).

Let $f : R_0 \rightarrow R$ be a mapping such that $f(a) = \bigcap_{i=1}^{\infty} U_{\pi_i(a)}$. Let us show that f is onto. Let x be an arbitrary point of R ; then by the condition (7.11) there exists a sequence of positive integers m_i , $i=1, 2, \dots$, with $m_1 < m_2 < \dots$ such that $\text{order}(x, \mathfrak{U}_{m_i}) \leq n+1$ for every i . Let $B_{m_i} = \{\alpha; \alpha \in A_{m_i}, x \in U_\alpha\}$; then B_{m_i} , $i=1, 2, \dots$, is a finite set such that $f_{m_{i+1}, m_i}(B_{m_{i+1}}) \subset B_{m_i}$ holds for every i . Since an inverse limiting space of non-empty compact Hausdorff spaces is not empty, we get $\lim \{B_{m_i}, f_{m_{i+1}, m_i}\} \neq \emptyset$. Let b be an arbitrary point of $\varprojlim B_{m_i}$ and a a point of $\lim \{A_i, f_{ij}\}$ such that $\pi_{m_i}(b) = \pi_{m_i}(a)$ for every i . Since $\bigcap_{i=1}^{\infty} U_{\pi_i(a)} = \bigcap_{i=1}^{\infty} U_{\pi_{m_i}(b)} = x$, a is a point of R_0 such that $f(a) = x$. Thus f is onto.

Let

$$(7.13) \quad \mathfrak{F}_i = \{F_{\alpha_i} = f(\pi_i^{-1}(\alpha_i) \cap R_0); \alpha_i \in A_i\};$$

then $\mathfrak{F}_i$ is a covering of R since f is onto. Let us prove that $\mathfrak{F}_i$ is locally finite. Let x be an arbitrary point of R and there exist positive integers $m_j(x)$, $j=1, 2, \dots$, such that $m_1(x) < m_2(x) < \dots$ and $\text{order}(x, \mathfrak{U}_{m_j(x)}) \leq n+1$ for every j . For every $\alpha \in A_{m_{j+1}(x)}$ let

23) This mesh is one with respect to an arbitrary but fixed metric on R agreeing with the preassigned topology of R .

(7.14) V_α be the maximal open set with $\text{Star}(V_\alpha, \mathfrak{U}_{m_{j+1}(x)}) \subset U_\alpha$
and

(7.15) $\mathfrak{B}_{m_j(x)} = \{V_\alpha; \alpha \in A_{m_{j+1}(x)}\}$.

Since $\mathfrak{U}_{m_j(x)} > \mathfrak{U}_{m_{j+1}(x)}^*$, it holds that $\mathfrak{U}_{m_j(x)} > \mathfrak{B}_{m_j(x)} > \mathfrak{U}_{m_{j+1}(x)}$ and hence $\mathfrak{B}_{m_j(x)}$ is an open covering of R . Let U_β be an arbitrary element of $\mathfrak{U}_{m_{j+1}(x)}$ with $x \in U_\beta$. When $U_\beta \cap V_\alpha \neq \emptyset$ where $V_\alpha \in \mathfrak{B}_{m_j(x)}$, U_α contains x . Therefore a number of elements of $\mathfrak{B}_{m_j(x)}$ which meet U_β is at most $n+1$ by the inequality order $(x, \mathfrak{U}_{m_j(x)}) \leq n+1$. Thus $\mathfrak{B}_{m_j(x)}$ is locally finite at x and order $(x, \mathfrak{B}_{m_j(x)}) \leq n+1$, $j=1, 2, \dots$. Take $m_j(x)$ with $m_j(x) \geq i$ as well as $j \geq 2$. Let α be an arbitrary element of $A_{m_j(x)}$. Since $F_\alpha \subset \bigcup \{U_\beta; \beta \in A_{m_j(x)}, f_{m_{j+1}(x), m_j(x)}(\beta) = \alpha\} \subset \bigcup \{U_\beta; \beta \in A_{m_{j+1}(x)}, \text{Star}(U_\beta, \mathfrak{U}_{m_{j+1}(x)}) \subset U_\alpha\} \subset V_\alpha$ and $\mathfrak{B}_{m_{j-1}(x)} = \{V_\alpha; \alpha \in A_{m_j(x)}\}$ is locally finite at x , $\mathfrak{F}_{m_j(x)} = \{F_\alpha; \alpha \in A_{m_j(x)}\}$ is locally finite at x . Let F_γ be an arbitrary element of $\mathfrak{F}_i$ and then $F_\gamma = \bigcup \{F_\beta; f_{m_j(x), i}(\beta) = \gamma\}$. Hence $\mathfrak{F}_i$ is locally finite at x . Since x was an arbitrary point of R , $\mathfrak{F}_i$ is locally finite at every point of R .

Next we shall prove that each element of $\mathfrak{F}_i$ is a closed subset of R . Since $\mathfrak{F}_j$ is locally finite and $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$ for every $\alpha \in A_j$, there exists a locally finite open covering $\mathfrak{G}_j = \{G_\alpha; \alpha \in A_j\}$ of R such that $\bar{F}_\alpha \subset G_\alpha \subset U_\alpha$ for every $\alpha \in A_j$ by [15, Lemma on page 22]. Let α_i be an arbitrary element of A_i . Since $\bar{F}_{\alpha_i} = \bigcup \{\bar{F}_{\alpha_{i+j}}; f_{i+j, i}(\alpha_{i+j}) = \alpha_i\}$ for any positive integer j , $\bar{F}_{\alpha_i} \subset \bigcap_{j=1}^{\infty} (\bigcup \{G_{\alpha_{i+j}}; f_{i+j, i}(\alpha_{i+j}) = \alpha_i\}) \subset \bigcap_{j=1}^{\infty} (\bigcup \{U_{\alpha_{i+j}}; f_{i+j, i}(\alpha_{i+j}) = \alpha_i\}) = F_{\alpha_i}$. Thus $\mathfrak{F}_i$ is a locally finite closed covering of R . Evidently a sequence $\mathfrak{F}_i$, $i=1, 2, \dots$, satisfies the conditions (7.2), (7.3) and (7.4). Therefore we get $\dim R \leq n$ by Theorem 7.2. Thus the theorem is completely proved.

As a direct consequence of this theorem we get the following theorem.

COROLLARY 7.6 (J. Nagata [24, Theorem 3]). *In order that a topological space R be a metrizable space with $\dim R \leq n$ it is necessary and sufficient that there exists a sequence of open coverings $\mathfrak{U}_i$, $i=1, 2, \dots$, of R which satisfies the following conditions.*

(7.16) $\mathfrak{U}_{i+1}^*$ refines $\mathfrak{U}_i$ for every i .

(7.17) order $\mathfrak{U}_i \leq n+1$ for every i .

(7.18) For any point x of R and any neighborhood U of x there exists i with $\text{Star}(x, \mathfrak{U}_i) \subset U$.

§ 8. Subordinate metric spaces

Let R be a normal space and $\mathfrak{U}_i$, $i=1, 2, \dots$, a sequence of open coverings of R such that $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \dots$. For any pair of points x, y of R define a non-negative function $D(x, y)$ as follows (cf. J. W. Tukey [29, Chap. 5]).

(8.1) When $x \notin \text{Star}(y, \mathfrak{U}_1)$, $D(x, y) = 1$.

(8.2) When there exists i such that $x \in \text{Star}(y, \mathfrak{U}_i)$ and $x \notin \text{Star}(y, \mathfrak{U}_{i+1})$, $D(x, y) = 2^{-i}$.

(8.3) When $x \in \text{Star}(y, \mathfrak{U}_i)$ for every i , $D(x, y) = 0$.

Then D satisfies the following conditions for any points $x, y, z \in R$.

(8.4) $D(x, x) = 0$.

(8.5) $D(x, y) = D(y, x) \geq 0$.

(8.6) $D(x, y) \leq 2 \max(D(x, z), D(z, y))$.

(8.7) For any positive number ε , $\{y; D(x, y) < \varepsilon\}$ is an open set of R .

In other words D is a so-called *D-function*.

With the aid of the function D define a function $r(x, y)$ for any pair of points x, y of R as follows (cf. A. H. Frink [7]).

(8.8) $r(x, y) = \inf \sum_{i=0}^m D(x_i, x_{i+1})$, where $x_0 = x, x_{m+1} = y$ and the infimum is taken over all $m \geq 0$ and sets $x_1, \dots, x_m$ of points of R .

Then the inequalities $(1/4)D(x, y) \leq r(x, y) \leq D(x, y)$ hold for any pair of points x, y , of R . Moreover r satisfies the following conditions for any points x, y, z of R .

$$(8.9) \quad r(x, x) = 0.$$

$$(8.10) \quad r(x, y) = r(y, x) \geq 0.$$

$$(8.11) \quad r(x, y) \leq r(x, z) + r(z, y).$$

$$(8.12) \quad \text{For any positive number } \varepsilon, \{y; r(x, y) < \varepsilon\} \text{ is an open set of } R.$$

In other words r is a *pseudo-metric* on R .

Let $F_x = \{y; r(x, y) = 0\}$; then F_x is a closed set of R . Evidently for any point $y \in F_x$ we get $F_x = F_y$. Thus R can be decomposed into a sum of mutually disjoint closed sets $\{F_x; x \in R\}$. Let $R^* = \{x^*\}$ be a set of points x^* such that there is a one-to-one correspondence: $F_x \leftrightarrow x^*$.

Let

$$(8.13) \quad r^*(x^*, y^*) = r(x', y'), \text{ where } x' \in F_x \text{ and } y' \in F_y;$$

then r^* is independent of the special choice of $x' \in F_x$ and $y' \in F_y$ and hence uniquely defined. r^* satisfies the following conditions for any points x^*, y^*, z^* of R^* .

$$(8.14) \quad r^*(x^*, y^*) = 0 \text{ if and only if } x^* = y^*.$$

$$(8.15) \quad r^*(x^*, y^*) = r^*(y^*, x^*) \geq 0.$$

$$(8.16) \quad r^*(x^*, y^*) \leq r^*(x^*, z^*) + r^*(z^*, y^*).$$

Thus R^* is a metric space with the metric r^* . Define a mapping $\varphi: R \rightarrow R^*$ as $\varphi(x) = x^*$. Then φ is continuous.

DEFINITION 8.1. (R^*, r^*) is called the *subordinate metric space* of R with respect to a (normal) sequence $\mathfrak{U}_i, i=1, 2, \dots$.

THEOREM 8.2. Let R be a normal space and $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \dots$, be a normal sequence of open coverings of R which satisfies the following conditions.²⁴⁾

(8.17) Any element of any $\mathfrak{U}_{i+1}$ intersects at most a finite number of elements of $\mathfrak{U}_i$.

(8.18) order $\mathfrak{U}_i \leq n+1$ for every i .

Let (R^*, r^*) be the subordinate metric space of R with respect to $\mathfrak{U}_i, i=1, 2, \dots$. Then we get $\dim R^* \leq n$.

PROOF. Let $\mathfrak{U}_i = \{U_\alpha; \alpha \in A_i\}$ and

$$(8.19) \quad \mathfrak{B}_i = \{U_\alpha^* = \{x^*; F_x \subset U_\alpha\}; \alpha \in A_i\}.$$

Then

$$(8.20) \quad \text{order } \mathfrak{B}_i \leq \text{order } \mathfrak{U}_i \leq n+1 \text{ for every } i.$$

To prove that $\mathfrak{B}_i$ is a covering of R^* let x^* be an arbitrary point of R^* and y be an arbitrary point of F_x . Take $U_\alpha \in \mathfrak{U}_i$ such that $\text{Star}(y, \mathfrak{U}_{i+1}) \subset U_\alpha$. Let z be an arbitrary point of $R - U_\alpha$ and then $z \notin \text{Star}(y, \mathfrak{U}_{i+1})$ and hence $D(y, z) \geq 2^{-i}$. Therefore $r(y, z) \geq 2^{-i-2}$ and $z \notin F_y = F_x$, which proves $F_x \subset U_\alpha$ and hence $x^* \in U_\alpha^*$.

²⁴⁾ It is to be noted that when R is a normal space with $\dim R \leq n$, there is for any normal open covering $\mathfrak{U}$ of R a normal sequence which satisfies the conditions of the proposition such that $\mathfrak{U}_1 < \mathfrak{U}$.

Thus $\mathfrak{B}_i$ covers R^* .

Next let us prove that

(8.21) $\overline{\mathfrak{B}}_{i+1}$ refines $\mathfrak{B}_i$.

Let U_α^* be an arbitrary element of $\mathfrak{B}_{i+1}$ and U_β an element of $\mathfrak{U}_i$ such that $\text{Star}(U_\alpha, \mathfrak{U}_{i+1}) \subset U_\beta$. When $U_\beta^* = R^*$, it evidently holds that $\overline{U_\alpha^*} \subset U_\beta^*$. Consider the case of $U_\beta^* \neq R^*$. Let y^* be an arbitrary point of $R^* - U_\beta^*$; then $F_y - U_\beta$ is not empty. Hence there exists $x \notin U_\beta$ with $y^* = x^*$. Evidently $\text{Star}(x, \mathfrak{U}_{i+1}) \cap U_\alpha = \phi$. Let z be an arbitrary point of $\{z; r(x, z) < 2^{-i-2}\}$; then $D(x, z) < 2^{-i}$ by the inequality $(1/4)D(x, z) \leq r(x, z)$. Hence $z \in \text{Star}(x, \mathfrak{U}_{i+1})$ and we get $\{z; r(x, z) < 2^{-i-2}\} \cap U_\alpha = \phi$. Therefore $\{z^*; r^*(x^*, z^*) < 2^{-i-2}\} \cap U_\alpha^* = \phi$, which proves $x^* = y^* \notin \overline{U_\alpha^*}$. Thus we can conclude that $\overline{U_\alpha^*} \subset U_\beta^*$, which proves (8.21).

Now we proceed to prove that $\mathfrak{B}_i$ is locally finite. Let x^* be an arbitrary point of R^* and U_α an element of $\mathfrak{U}_{i+1}$ such that $\text{Star}(x, \mathfrak{U}_{i+2}) \subset U_\alpha$. Let y be an arbitrary point of $\{y; r(x, y) < 2^{-i-3}\}$ and then $D(x, y) \leq 4r(x, y) < 2^{-i-1}$. Hence $y \in \text{Star}(x, \mathfrak{U}_{i+2})$. If $r(y, z) = 0$, $r(x, z) \leq r(x, y) + r(y, z) = r(x, y)$ and hence $z \in \text{Star}(x, \mathfrak{U}_{i+2})$. Thus $F_y \subset U_\alpha$. Therefore $U(x^*) = \{y^*; r^*(x^*, y^*) < 2^{-i-3}\} \subset U_\alpha^*$. Since U_α intersects at most a finite number of elements of $\mathfrak{U}_i$, U_α^* intersects at most a finite number of elements of $\mathfrak{B}_i$. Therefore $U(x^*)$ is an open neighborhood of x^* which intersects at most a finite number of elements of $\mathfrak{B}_i$, which proves the local finiteness of $\mathfrak{B}_i$.

Finally let us prove that a sequence $\mathfrak{B}_i$, $i=1, 2, \dots$, satisfies the following condition.

(8.22) For any neighborhood U^* of any point x^* of R^* there exists i such that $\text{Star}(x^*, \mathfrak{B}_i) \subset U^*$.

Let i be a positive integer such that $S^* = \{y^*; r^*(x^*, y^*) < 2^{-i+1}\} \subset U^*$. Let $S = \{y; r(x, y) < 2^{-i+1}\}$ and then $\varphi^{-1}(S^*) = S$. Since $r(x, y) \leq D(x, y)$, $\{y; D(x, y) < 2^{-i+1}\} \subset S$ and hence $\text{Star}(x, \mathfrak{U}_i) \subset S$. If $x^* \in U_\alpha^* \in \mathfrak{B}_i$, $F_x \subset U_\alpha \in \mathfrak{U}_i$ and hence $x \in U_\alpha \in \mathfrak{U}_i$. Thus $U_\alpha \subset \text{Star}(x, \mathfrak{U}_i)$. Hence $\varphi^{-1}(\text{Star}(x^*, \mathfrak{B}_i)) \subset \text{Star}(x, \mathfrak{U}_i)$ by $\varphi^{-1}(U_\alpha^*) \subset U_\alpha$. Therefore $U^* \supset S^* = \varphi(S) \supset \varphi(\text{Star}(x, \mathfrak{U}_i)) \supset \text{Star}(x^*, \mathfrak{B}_i)$.

At last we can conclude by Theorem 7.2 that $\dim R^* \leq n$ and the proof is completed.

§ 9. Dimension-preserving completion

Let R be a metrizable space with $\dim R \leq n$. Let $\mathfrak{U}_1$ be a locally finite open covering of R such that $\text{order } \mathfrak{U}_1 \leq n+1$ and $\text{mesh } \mathfrak{U}_1 < 2^{-1}$. Let $\mathfrak{U}_2$ be a locally finite open covering of R such that i) $\text{order } \mathfrak{U}_2 \leq n+1$, ii) $\text{mesh } \mathfrak{U}_2 < 2^{-2}$, iii) $\mathfrak{U}_2^*$ refines $\mathfrak{U}_1$ and iv) $\mathfrak{U}_2$ refines $\{U(x) = R - \bigcup \{\bar{U}_\alpha; x \notin \bar{U}_\alpha, U_\alpha \in \mathfrak{U}_1\}; x \in R\}$. Then each element of $\mathfrak{U}_2$ intersects at most $n+1$ elements of $\mathfrak{U}_1$. Proceeding this procedure we get a sequence of open coverings $\mathfrak{U}_i$, $i=1, 2, \dots$, such that ²⁵⁾

(9.1) $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \dots$,

(9.2) for any neighborhood U of any point x of R there exists i with $\text{Star}(x, \mathfrak{U}_i) \subset U$,

(9.3) each element of $\mathfrak{U}_{i+1}$ intersects at most $n+1$ elements of $\mathfrak{U}_i$.

THEOREM 9.1. *Let R be a metrizable space with $\dim R \leq n$. Let $\mathfrak{U}_i =$*

25) The existence of the sequence of this property was for the first time proved by J. Nagata [24, Theorem 2].

$\{U_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots$, be a sequence of open coverings of R satisfying the conditions (9.1), (9.2) and (9.3). Let S be a completion of R with respect to $\{\mathfrak{U}_i; i=1, 2, \dots\}$. Then we get $\dim S \leq n$.

PROOF. For any open set G of R let $G^* = S - (\text{the closure of } R - G \text{ in } S)$.²⁶⁾ Then it holds by Morita [31, Theorems 3, 4 and Lemma 7] that $\mathfrak{B}_i = \{U_\alpha^*; \alpha \in A_i\}$, $i=1, 2, \dots$, satisfy the following conditions.

(9.4) $\{\mathfrak{B}_i; i=1, 2, \dots\}$ gives a uniformity of S which agrees with the topology of S .

(9.5) $\mathfrak{B}_{i+1}$ is a star-refinement of $\mathfrak{B}_i$, $i=1, 2, \dots$.

(9.6) Each element of $\mathfrak{B}_{i+1}$ meets at most $n+1$ elements of $\mathfrak{B}_i$.

Hence by Nagata [24, Theorem 2] or Theorem 7.5 we get $\dim S \leq n$.

REMARK 9.2. Another expression of Theorem 9.1 is as follows: Let R be a metrizable space with $\dim R \leq n$. Let $\mathfrak{U}_i$, $i=1, 2, \dots$, be a sequence of open coverings of R which satisfies the conditions (9.1), (9.2) and (9.3). Let r be a metric on R defined in § 8 with the usage of $\mathfrak{U}_i$, $i=1, 2, \dots$. Then a completion S of R with respect to r has dimension $\leq n$.

DEFINITION 9.3. Recently J. Nagata [24, Theorem 5] has proved the following theorem: In order that $\dim R \leq n$ for a metrizable space R it is necessary and sufficient that we can define a metric $\rho(x, y)$ agreeing with the topology of R such that for every $\varepsilon > 0$ and for every point x of R , $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ ($i=1, \dots, n+2$) imply $\rho(y_i, y_j) < \varepsilon$ for some i, j with $i \neq j$, where $S_{\varepsilon/2}(x) = \{y; \rho(x, y) < \varepsilon/2\}$. We call a metric ρ with this property *J. Nagata's metric*.

THEOREM 9.4. Let R be a metrizable space with $\dim R \leq n$. Then a completion S of R with respect to *J. Nagata's metric* ρ on R has dimension $\leq n$.

PROOF. Let M_1 be a maximal subset of R such that $x, y \in M_1$ and $x \neq y$ imply $\rho(x, y) \geq 1/2$. By the maximal property of M_1 $\mathfrak{B}_1 = \{S_{1/2}(x); x \in M_1\}$ is evidently an open covering of R . Let $\mathfrak{U}_1 = \{\bigcup \{S_{1/2^3}(y); y \in S_{1/2}(x)\}; x \in M_1\}$; then $\mathfrak{U}_1$ is also an open covering of R . Let x be an arbitrary point of R . If $S_{1/2^3}(x) \cap (\bigcup \{S_{1/2^3}(y); y \in S_{1/2}(x_i)\}) \neq \emptyset$, $i=1, \dots, n+2$, for $x_i \in M_1$, then $\rho(S_{1/2^2}(x), x_i) < 1/2$. Hence $\rho(x_i, x_j) < 1/2$ for some x_i and x_j . This implies $x_i = x_j$ by the property of M_1 . Therefore $S_{1/2^3}(x)$ for an arbitrary point x of R intersects at most $n+1$ sets of $\mathfrak{U}_1$. Let M_2 be a maximal subset of R such that $x, y \in M_2$ and $x \neq y$ imply $\rho(x, y) \geq 1/2^6$. By the maximal property of M_2 $\mathfrak{B}_2 = \{S_{1/2^6}(x); x \in M_2\}$ is an open covering of R . Let $\mathfrak{U}_2 = \{\bigcup \{S_{1/2^8}(y); y \in S_{1/2^6}(x)\}; x \in M_2\}$; then $\mathfrak{U}_2$ is an open covering of R . Since $\mathfrak{U}_1 > \{S_{1/2^3}(x); x \in R\} > \{S_{1/2^5}(x); x \in R\}^* > \{S_{1/2^5}(x); x \in R\} > \mathfrak{U}_2$, $\mathfrak{U}_2^*$ refines $\mathfrak{U}_1$ and each element of $\mathfrak{U}_2$ intersects at most $n+1$ elements of $\mathfrak{U}_1$.

In general let M_i be a maximal subset of R such that $x, y \in M_i$ and $x \neq y$ imply $\rho(x, y) \geq 2^{-1-5(i-1)}$. Then $\mathfrak{B}_i = \{S_{2^{-1-5(i-1)}}(x); x \in M_i\}$ is an open covering of R . Let $\mathfrak{U}_i = \{\bigcup \{S_{2^{-3-5(i-1)}}(y); y \in S_{2^{-1-5(i-1)}}(x)\}; x \in M_i\}$; then by the same argument as in the above $\mathfrak{U}_i^*$ refines $\mathfrak{U}_{i-1}$ and each element of $\mathfrak{U}_i$ intersects at most $n+1$ elements of $\mathfrak{U}_{i-1}$, $i=2, 3, \dots$. It is to be noted that the identity mapping of R onto R is a homeomorphism of $\{R, \mathfrak{U}_i (i=1, 2, \dots)\}$ onto $\{R, \rho\}$.

Since mesh $\mathfrak{U}_i$ with respect to ρ converges to 0 as $i \rightarrow \infty$, every Cauchy

26) cf. Morita [32, Lemma 1].

sequence of $\{R, \mathfrak{U}_i (i=1, 2, \dots)\}$ is a Cauchy sequence with respect to ρ .

Conversely let us show that every Cauchy sequence with respect to ρ is a Cauchy sequence of $\{R, \mathfrak{U}_i (i=1, 2, \dots)\}$. Let $\{x_i\} = \{x_i; i=1, 2, \dots\}$ be an arbitrary Cauchy sequence of $\{R, \rho\}$ and j an arbitrary positive integer; then there exists k such that for any $m \geq k$, $x_m \in S_{2^{-3-5(j-1)}}(x_k)$. Since $\mathfrak{B}_j$ covers R , there exists a point $x_0 \in M_i$ such that $x_k \in S_{2^{-1-5(j-1)}}(x_0)$. Since $S_{2^{-3-5(j-1)}}(x_k) \subset \bigcup \{S_{2^{-3-5(j-1)}}(y); y \in S_{2^{-1-5(j-1)}}(x_0)\} \in \mathfrak{U}_j$, we get $S_{2^{-3-5(j-1)}}(x_k) \subset \text{Star}(x_k, \mathfrak{U}_j)$. Therefore $x_m \in \text{Star}(x_k, \mathfrak{U}_j)$ for any $m \geq k$, which proves that $\{x_i\}$ is a Cauchy sequence of $\{R, \mathfrak{U}_i (i=1, 2, \dots)\}$. Thus we can conclude that a completion S of $\{R, \rho\}$ is essentially the same as a completion S_1 of $\{R, \mathfrak{U}_i (i=1, 2, \dots)\}$. Since a sequence of open coverings $\mathfrak{U}_i, i=1, 2, \dots$, of R clearly satisfies the conditions (9.1), (9.2) and (9.3), we get $\dim S_1 \leq n$ by Theorem 9.1. Therefore we get $\dim S \leq n$, which proves the theorem.

REMARK 9.5. The following is Morita's simple proof of Theorem 9.4 making no use of Theorem 9.1: Let $\bar{\rho}$ be a metric on S generated by ρ on R . Then it can be proved that $\bar{\rho}$ is Nagata's metric.

§ 10. Length of coverings

DEFINITION 10.1. We call a covering $\mathfrak{U}$ of a space a *multiplicative covering* if every non-empty intersection $\bigcap_{i=1}^k U_i$ of elements $U_i, i=1, \dots, k$, of $\mathfrak{U}$ is also an element of $\mathfrak{U}$. The maximal number n such that there exists a sequence $U_1 \supsetneq U_2 \supsetneq \dots \supsetneq U_n \neq \emptyset$ of elements of a multiplicative covering $\mathfrak{U}$ is called the *length* of $\mathfrak{U}$, *length* $\mathfrak{U}$. We mean by the *rank* of an element U of $\mathfrak{U}$ the maximal number r such that there exists a sequence $U = U_1 \supsetneq U_2 \supsetneq \dots \supsetneq U_r$ of elements of $\mathfrak{U}$.

Under these terminologies J. Nagata's theorem [24, Theorem 4] is as follows: In order that a topological space R be a metrizable space with $\dim R \leq n$ it is necessary and sufficient that there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \dots$ of multiplicative open coverings with *length* $\leq n+1$ such that $\{\text{Star}(p, \mathfrak{U}_m); m=1, 2, \dots\}$ is a neighborhood basis of p for every point p of R .

THEOREM 10.2. In order that a topological space R be a metrizable space with $\dim R \leq n$ it is necessary and sufficient that there exists a sequence of coverings $\mathfrak{H}_i = \{H_\alpha; \alpha \in A_i\}, i=1, 2, \dots$, of R which satisfies the following conditions.

- (10.1) $\mathfrak{H}_i$ is a locally finite multiplicative covering for every i .
- (10.2) $\mathfrak{H}_{i+1}$ refines $\mathfrak{H}_i$ for every i .
- (10.3) *length* $\mathfrak{H}_i \leq n+1$ for every i .
- (10.4) For any point x of R and any neighborhood U of x there exists i with $\text{Star}(x, \mathfrak{H}_i) \subset U$.

PROOF. If R is a metric space with $\dim R \leq n$, there exists a sequence of locally finite coverings $\mathfrak{F}_i, i=1, 2, \dots$, of R with the conditions (3.1), (3.2), (3.3). Since $\mathfrak{F}_i$ plus all the non-empty intersections of a finite number of elements of $\mathfrak{F}_i$ is obviously a multiplicative covering with *length* $\leq n+1$, the necessity of the theorem is valid.

Let us assume the existence of a sequence satisfying the conditions of the proposition. We can assume without loss of generality that if α and β are

elements of A_i with $\alpha \neq \beta$, then $H_\alpha \neq H_\beta$. Since the conditions (10.2) and (10.4) yield the condition (7.1), R is metrizable by Lemma 7.1.

We shall prove $\dim R \leq n$ by the induction on n . When $n=0$, length $\mathfrak{H}_i \leq 1$ implies order $\mathfrak{H}_i \leq 1$. Moreover $\mathfrak{H}_i$ is a closed covering by (10.2). Hence we get $\dim R \leq 0$ by Theorem 3.2. Now make the induction assumption that if there is a sequence which satisfies the conditions (10.1), (10.2), (10.4) and (10.5) length $\mathfrak{H}_i \leq n$ for every i , then we get $\dim R \leq n-1$.

Now let $\mathfrak{H}_i$, $i=1, 2, \dots$, be a sequence satisfying the conditions of the theorem. For every i let us define a single-valued mapping $f_{i+1,i}: A_{i+1} \rightarrow A_i$ in such a way that $f_{i+1,i}(\alpha_{i+1}) = \alpha_i$ yields $H_{\alpha_{i+1}} \subset H_{\alpha_i}$. For i and j with $i > j$ let $f_{ij} = f_{j+1,j} \cdots f_{i-1,i-2} f_{i-1,i}$. Let A_0 be the aggregate of points $a = (\alpha_1, \alpha_2, \dots)$ of $\lim \{A_i, f_{ij}\}$, where every A_i is a topological space with discrete topology, such that $\bigcap_{i=1}^{\infty} H_{\alpha_i} \neq \emptyset$, and f a mapping of A_0 into R defined by $f(a) = \bigcap_{i=1}^{\infty} H_{\pi_i(a)}$, where π_i is the projection of A_0 into A_i . Then we can verify by the same argument as in the proof of (B) $\rightarrow$ (C) of §3 that i) f is onto and ii) $\mathfrak{F}_i = \{F_\alpha = f(\pi_i^{-1}(\alpha)); \alpha \in A_i\}$ is a locally finite closed covering of R such that (10.6) for any i and any $\alpha \in A_i$, $F_\alpha = \bigcup \{F_\beta; f_{i+1,i}(\beta) = \alpha\}$.

Let

$$(10.7) \quad F_i = \{x; x \in R, \text{order}(x, \mathfrak{F}_i) > 1\}, \quad i=1, 2, \dots;$$

then it can easily be seen that for every i F_i is a closed subset of R . Let x be an arbitrary point of F_i ; then there exist α_i and β_i of A_i with $\alpha_i \neq \beta_i$ such that $x \in F_{\alpha_i} \cap F_{\beta_i}$. Then by (10.6) there exist α_{i+1} and β_{i+1} with $f_{i+1,i}(\alpha_{i+1}) = \alpha_i$ and $f_{i+1,i}(\beta_{i+1}) = \beta_i$ such that $x \in F_{\alpha_{i+1}} \cap F_{\beta_{i+1}}$. Hence $F_i \subset F_{i+1}$.

Moreover we can conclude from the above argument that if we set

$$(10.8) \quad \mathfrak{H}_j' = \{H_{\alpha_j} \cap H_{\beta_j}; f_{ji}(\alpha_j) \neq f_{ji}(\beta_j), F_{f_{ji}(\alpha_j)} \cap F_{f_{ji}(\beta_j)} \neq \emptyset\}, \quad j=i, i+1, \dots, \text{ and}$$

$$(10.9) \quad \mathfrak{H}_j'' = \mathfrak{H}_j' \text{ plus all the non-empty intersections of a finite number of elements of } \mathfrak{H}_j', \quad j=i, i+1, \dots,$$

then $F_i \cap \mathfrak{H}_j''$, $j=i, i+1, \dots$, is a sequence of locally finite multiplicative coverings of F_i which satisfies the following conditions.

$$(10.10) \quad \overline{F_i \cap \mathfrak{H}_{j+1}''} = F_i \cap \mathfrak{H}_{j+1}'' \text{ refines } F_i \cap \mathfrak{H}_j'' \text{ for } j=i, i+1, \dots.$$

$$(10.11) \quad \text{length } F_i \cap \mathfrak{H}_j'' \leq n \text{ for } j=i, i+1, \dots.$$

$$(10.12) \quad \text{For any point } x \text{ of } F_i \text{ and any (relatively) open neighborhood } U \text{ of } x \text{ there exists } j(\geq i) \text{ with } \text{Star}(x, \overline{F_i \cap \mathfrak{H}_j''}) \subset U.$$

Therefore we get

$$(10.13) \quad \dim F_i \leq n-1, \quad i=1, 2, \dots,$$

by the induction assumption. Then by the sum theorem we get

$$(10.14) \quad \dim \bigcup_{i=1}^{\infty} F_i \leq n-1.$$

On the other hand $(R - F_i) \cap \mathfrak{F}_i$ is a locally finite closed covering of $R - F_i$ whose order ≤ 1 . Hence

$$(10.15) \quad \mathfrak{L}_i = (R - \bigcup_{j=1}^{\infty} F_j) \cap \mathfrak{F}_i$$

is a locally finite closed covering of $R - \bigcup_{j=1}^{\infty} F_j$ whose order ≤ 1 .

Thus a sequence $\mathfrak{L}_i$, $i=1, 2, \dots$, satisfies the following conditions.

$$(10.16) \quad \mathfrak{L}_{i+1} \text{ refines } \mathfrak{L}_i \text{ for every } i.$$

$$(10.17) \quad \text{order } \mathfrak{L}_i \leq 1 \text{ for every } i.$$

$$(10.18) \quad \text{For any } x \in R - \bigcup_{j=1}^{\infty} F_j \text{ and any neighborhood } U \text{ of } x \text{ there exists } i \text{ such that } \text{Star}(x, \mathfrak{L}_i) \subset U.$$

Therefore by Theorem 3.2 we get

$$(10.19) \quad \dim(R - \bigcup_{j=1}^{\infty} F_j) \leq 0.$$

Combining (10.19) with (10.14) we get by Theorem 3.2 $\dim R \leq n$, which completes the induction. Thus the theorem is completely proved.

§ 11. Characterization of dimension of non-metrizable spaces by means of length of coverings

THEOREM 11.1. *For a normal space R the following conditions are equivalent.*

- (11.1) $\dim R \leq n$.
- (11.2) Every finite open covering can be refined by a finite multiplicative open covering whose length $\leq n+1$.
- (11.3) Every locally finite open covering can be refined by a locally finite multiplicative open covering whose length $\leq n+1$.
- (11.4) Every finite open covering can be refined by a finite multiplicative closed covering whose length $\leq n+1$.
- (11.5) Every locally finite open covering can be refined by a locally finite multiplicative closed covering whose length $\leq n+1$.

PROOF. (11.1) $\rightarrow$ (11.2): Let $\dim R \leq n$ and $\mathcal{G}$ an arbitrary finite open covering of R . Then there exists a finite open covering $\mathcal{U}$ of R with order $\mathcal{U} \leq n+1$ such that $\mathcal{U}$ refines $\mathcal{G}$. Let $\mathfrak{H}$ be $\mathcal{U}$ plus all the non-empty intersections of a finite number of elements of $\mathcal{U}$ and then it is obvious that $\mathfrak{H}$ is a finite multiplicative open covering of R with length $\mathfrak{H} \leq n+1$ such that $\mathfrak{H}$ refines $\mathcal{G}$.

The implications (11.1) $\rightarrow$ (11.3), (11.1) $\rightarrow$ (11.4) and (11.1) $\rightarrow$ (11.5) are proved by the same way.

(11.2) $\rightarrow$ (11.1) is proved essentially by P. Alexandroff and A. Kolmogoroff [1].

(11.3) $\rightarrow$ (11.1): Let $\mathcal{G}_1$ be an arbitrary locally finite open covering of R and $\mathfrak{H}_1$ be a locally finite multiplicative open covering of R with length $\mathfrak{H}_1 \leq n+1$ and $\mathcal{G}_1 > \mathfrak{H}_1$. Then by J. Dieudonné [2] there exists a locally finite open covering $\mathcal{G}_2$ of R with $\mathfrak{H}_1 > \mathcal{G}_2^*$. Let $\mathfrak{H}_2$ be a locally finite multiplicative open covering of R with length $\mathfrak{H}_2 \leq n+1$ and $\mathcal{G}_2 > \mathfrak{H}_2$. Since $\mathcal{G}_1 > \mathfrak{H}_1 > \mathcal{G}_2^* > \mathcal{G}_2 > \mathfrak{H}_2$, we get $\mathcal{G}_1 > \mathfrak{H}_1 > \mathfrak{H}_2^*$. Repeating this argument we get a sequence of locally finite multiplicative open coverings $\mathfrak{H}_i$, $i=1, 2, \dots$, of R which satisfies the following conditions.

$$(11.6) \quad \mathcal{G}_1 > \mathfrak{H}_1 > \mathfrak{H}_2^* > \mathfrak{H}_2 > \mathfrak{H}_3^* > \dots$$

$$(11.7) \quad \text{length } \mathfrak{H}_i \leq n+1 \text{ for every } i.$$

Then we can construct an open covering $\mathcal{U}$ whose order $\leq n+1$ such that $\mathcal{U}$ refines $\mathcal{G}_1$ by the same argument employed in J. Nagata [24, Theorem 4], which shows $\dim R \leq n$.

(11.5) $\rightarrow$ (11.1): Let $\mathcal{G} = \{G_\lambda; \lambda \in A\}$ be an arbitrary locally finite open covering of R and $\mathfrak{H} = \{H_\alpha; \alpha \in A\}$ a locally finite multiplicative closed covering of R with length $\mathfrak{H} \leq n+1$ and $\mathcal{G} > \mathfrak{H}$ such that whenever $\alpha \neq \beta$ ($\alpha, \beta \in A$), $H_\alpha \neq H_\beta$. Let $\mathfrak{H}_i = \{H_\alpha; \alpha \in A_i\}$, $i=1, 2, \dots, n+1$, be the aggregate of elements of $\mathfrak{H}$ whose rank $= i$. Assume that $A = \{1, 2, \dots, \lambda, \dots\}$ is well-ordered. Let $A_{i\lambda} = \{\alpha; \alpha \in A_i, H_\alpha \not\subset G_\mu \text{ for any } \mu < \lambda \text{ and } H_\alpha \subset G_\lambda\}$, $i=1, \dots, n+1$, $\lambda \in A$. Let $H_{i\lambda} = \bigcup \{H_\alpha; \alpha \in A_{i\lambda}\}$, $i=1, \dots, n+1$, $\lambda \in A$. Then $H_{i\lambda}$ is a closed set with $H_{i\lambda} \subset G_\lambda$ for every $\lambda \in A$. Since order $\mathfrak{H}_1 \leq 1$, order $(H_{i\lambda}; \lambda \in A) \leq 1$. Hence there exists by Morita [15, Theorem 1.3] a collection of open sets $\mathcal{U}_1 = \{U_{i\lambda}; \lambda \in A\}$ with order $\mathcal{U}_1 \leq 1$ such that

$H_{1\lambda} \subset U_{1\lambda} \subset G_\lambda$ for every $\lambda \in A$.

It is to be noted that for any $\alpha, \beta \in A_2$ with $\alpha \neq \beta$ either $H_\alpha \cap H_\beta \in \mathfrak{H}_1$ or $H_\alpha \cap H_\beta = \emptyset$ holds. Hence $\{H_{2\lambda} - \bigcup\{U_{1\mu}; \mu \in A\}; \lambda \in A\}$ is a collection of mutually disjoint closed sets of R such that $H_{2\lambda} - \bigcup\{U_{1\mu}; \mu \in A\} \subset G_\lambda$ for every $\lambda \in A$. Therefore by Morita [15, Theorem 1.3] there exists a collection of open sets $\mathfrak{U}_2 = \{U_{2\lambda}; \lambda \in A\}$ with order $\mathfrak{U}_2 \leq 1$ such that $H_{2\lambda} - \bigcup\{U_{1\mu}; \mu \in A\} \subset U_{2\lambda} \subset G_\lambda$ for every $\lambda \in A$.

Repeating such processes we get collections of open sets $\mathfrak{U}_i = \{U_{i\lambda}; \lambda \in A\}$, $i = 1, 2, \dots, n+1$, with order $\mathfrak{U}_i \leq 1$ such that $H_{i\lambda} - \bigcup\{U_{j\mu}; j = 1, \dots, i-1, \mu \in A\} \subset U_{i\lambda} \subset G_\lambda$ for every $\lambda \in A$. Since $\{H_{i\lambda}; i = 1, \dots, n+1, \lambda \in A\}$ covers R , $\{U_{i\lambda}; i = 1, \dots, n+1, \lambda \in A\}$ covers R . Therefore $\mathfrak{U} = \{U_\lambda = \bigcup\{U_{i\lambda}; i = 1, \dots, n+1\}; \lambda \in A\}$ is an open covering of R such that $U_\lambda \subset G_\lambda$ for every $\lambda \in A$. Moreover it is evident that order $\mathfrak{U} \leq n+1$. Thus an arbitrary locally finite open covering $\mathfrak{G}$ can be refined by an open covering whose order $\leq n+1$, which shows $\dim R \leq n$.

The implication (11.4) $\rightarrow$ (11.1) is proved by the same way as above and the theorem is completely proved.

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