On the number of prime factors of integers IV

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1. Introduction

In our papers I, II, and III,¹⁾ we were concerned with the number of distinct prime factors of integers. In this paper, we shall deal with the number of prime factors of integers, multiple factors being counted multiply, and the number of divisors of integers, or more generally, the number of representations of integers as a product of a fixed number of integers. The object of this paper is to prove the following theorems:

THEOREM A₁. *Let k be a positive integer. Let $\{f_1(\xi), \dots, f_k(\xi)\}$ be a set of polynomials in ξ , and $\{\mathfrak{P}_1, \dots, \mathfrak{P}_k\}$ a family of sets of rational prime numbers. Suppose that the following conditions are satisfied:*

(C₁) *For each $i=1, \dots, k$, the polynomial $f_i(\xi)$ is of positive degree with integral coefficients and with positive leading coefficient;*

(C₂) *For each pair of integers i, j with $1 \leq i < j \leq k$, at least one of the two requirements $(f_i, f_j)=1$ and $\mathfrak{P}_i \cap \mathfrak{P}_j = \emptyset$ holds;*

(C₃) *If we denote, for each $i=1, \dots, k$, by $\nu_i(p)$ the number of incongruent solutions of the congruence $f_i(\xi) \equiv 0 \pmod{p}$, then the series*

$$\sum_{p \in \mathfrak{P}_i} \frac{\nu_i(p)}{p}$$

is divergent.

Now let $\Omega_i(n)$ denote, for each $i=1, \dots, k$, the number of prime factors of a positive integer n which belong to the set $\mathfrak{P}_i$, multiple factors being counted multiply:

$$\Omega_i(n) = \sum_{p^{\alpha} | n, p \in \mathfrak{P}_i} 1^{(2)}.$$

Suppose that the canonical factorization of the polynomial $f_i(\xi)$ is, for each $i=1, \dots, k$, given by

$$(1.1) \quad f_i(\xi) = c_i \prod_{\rho=1}^{r_i} \{q_{i\rho}(\xi)\}^{a_{i\rho^{(3)}}},$$

and let $\nu_{i\rho}(p)$ denote, for each $i=1, \dots, k$ and for each $\rho=1, \dots, r_i$, the number of incongruent solutions of the congruence $q_{i\rho}(\xi) \equiv 0 \pmod{p}$. Put, for $n > 0$ ⁴⁾,

$$y_i(n) = \sum_{\rho=1}^{r_i} a_{i\rho} \sum_{p \leq n, p \in \mathfrak{P}_i} \frac{\nu_{i\rho}(p)}{p} \quad (i=1, \dots, k)$$

1) Tanaka [7], [8], and [9] will be quoted as I, II, and III respectively in this paper.

2) p^{α} runs through prime powers satisfying the indicated conditions.

3) We mean by the word 'canonical' that the irreducible factor $q_{i\rho}(\xi)$ has, for each $\rho=1, \dots, r_i$, integral coefficients with positive leading coefficient, and primitive in the sense that the greatest common divisor of its coefficients equals 1.

4) The following functions can be defined for any $n > 0$, not necessarily integral, but in this enunciation we use these functions only for integral n .

and

$$z_i(n) = \sum_{\rho=1}^{r_i} a_{i\rho}^2 \sum_{p \leq n, p \in \mathfrak{P}_i} \frac{\nu_{i\rho}(p)}{p} \quad (i=1, \dots, k).$$

Let n_0 be a positive integer such that $f_i(n) > 0$, $z_i(n) > 0$ ⁵⁾ ($i=1, \dots, k$) for integers $n \geq n_0$, and put, for $n \geq n_0$,

$$u_i(n) = \frac{\Omega_i\{f_i(n)\} - y_i(n)}{\sqrt{z_i(n)}} \quad (i=1, \dots, k).$$

Let E be a Jordan-measurable set, bounded or unbounded, in the space R^k of k dimensions. Let x be a variable $> n_0$, and let $A(x; E)$ denote the number of integers n with $n_0 \leq n \leq x$ for which the points $(u_1(n), \dots, u_k(n))$ of the space R^k belong to the set E . Then we have

$$\lim_{x \rightarrow \infty} \frac{A(x; E)}{x} = (2\pi)^{-k/2} \int_E \exp\left(-\frac{1}{2} \sum_{i=1}^k u_i^2\right) du_1 \cdots du_k.$$

THEOREM A₂. Let k be a positive integer. Let $\{f_1(\xi), \dots, f_k(\xi)\}$ be a set of polynomials in ξ . Let $s_1, \dots, s_k$ be integers > 1 , and $\{\mathfrak{P}_{i\sigma}\}$ ($i=1, \dots, k$; $\sigma=1, \dots, s_i$) a family of sets of rational prime numbers. Suppose that the following conditions are satisfied:

(C₁) For each $i=1, \dots, k$, the polynomial $f_i(\xi)$ is of positive degree with integral coefficients and with positive leading coefficient;

(C₂) For each $i=1, \dots, k$, the union $\mathfrak{P}_{i1} \cup \dots \cup \mathfrak{P}_{is_i}$ contains all prime numbers;

(C₃) If we denote, for each $i=1, \dots, k$, by $\mathfrak{Q}_i$ the set of primes which are contained in at least two of the sets $\mathfrak{P}_{i1}, \dots, \mathfrak{P}_{is_i}$, then, for each pair of integers i, j with $1 \leq i < j \leq k$, at least one of the two requirements $(f_i, f_j) = 1$ and $\mathfrak{Q}_i \cap \mathfrak{Q}_j = \emptyset$ holds;

(C₄) If we denote, for each $i=1, \dots, k$, by $\nu_i(p)$ the number of incongruent solutions of the congruence $f_i(\xi) \equiv 0 \pmod{p}$, then the series

$$\sum_{p \in \mathfrak{Q}_i} \frac{\nu_i(p)}{p}$$

is divergent.

Now let $\tau_i(n)$ denote, for each $i=1, \dots, k$, the number of representations of a positive integer n as the product $n = d_1 \cdots d_{s_i}$ such that d_σ is, for each $\sigma=1, \dots, s_i$, a positive integer composed only of primes belonging to the set $\mathfrak{P}_{i\sigma}$:

$$\tau_i(n) = \sum_{\substack{d_1 \cdots d_{s_i} = n \\ p | d_\sigma \Rightarrow p \in \mathfrak{P}_{i\sigma} (\sigma=1, \dots, s_i)}} 1.$$

Suppose that the canonical factorization of the polynomial $f_i(\xi)$ is, for each $i=1, \dots, k$, given by (1.1), and let $\nu_{i\rho}(p)$ have, for each $i=1, \dots, k$ and for each $\rho=1, \dots, r_i$, the same meaning as in Theorem A₁. Also let $\mathfrak{Q}_{i\sigma}$ denote, for each $i=1, \dots, k$ and for each $\sigma=2, \dots, s_i$, the set of primes which belong to just σ sets among the s_i sets $\mathfrak{P}_{i1}, \dots, \mathfrak{P}_{is_i}$ ⁶⁾. Put, for $n > 0$,

$$y_i(n) = \sum_{\rho=1}^{r_i} \sum_{\sigma=2}^{s_i} \log \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!} \sum_{p \leq n, p \in \mathfrak{Q}_{i\sigma}} \frac{\nu_{i\rho}(p)}{p} \quad (i=1, \dots, k)$$

5) By virtue of the condition (C₃), $z_i(n) \rightarrow \infty$ as $n \rightarrow \infty$.

6) $\mathfrak{Q}_i$ defined in (C₃) is the union $\mathfrak{Q}_{i2} \cup \dots \cup \mathfrak{Q}_{is_i}$.

and

$$z_i(n) = \sum_{\rho=1}^{r_i} \sum_{\sigma=2}^{s_i} \log^2 \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!} \sum_{p \leq n, p \in \Omega_{i\sigma}} \frac{\nu_{i\rho}(p)}{p} \quad (i=1, \dots, k).$$

Let n_0 be a positive integer such that $f_i(n) > 0$, $z_i(n) > 0$ ⁷⁾ ($i=1, \dots, k$) for integers $n \geq n_0$, and put, for $n \geq n_0$,

$$u_i(n) = \frac{\log \tau_i \{f_i(n)\} - y_i(n)}{\sqrt{z_i(n)}} \quad (i=1, \dots, k).$$

Let E be a Jordan-measurable set, bounded or unbounded, in the space R^k of k dimensions. Let x be a variable $> n_0$, and let $A(x; E)$ denote the number of integers n with $n_0 \leq n \leq x$ for which the points $(u_1(n), \dots, u_k(n))$ of the space R^k belong to the set E . Then we have

$$\lim_{x \rightarrow \infty} \frac{A(x; E)}{x} = (2\pi)^{-k/2} \int_E \exp\left(-\frac{1}{2} \sum_{i=1}^k u_i^2\right) du_1 \cdots du_k.$$

We shall prove Theorem A_1 in section 2, and prove Theorem A_2 in section 3. We deduce them from III, Theorem A which is concerned with the number of distinct prime factors of integers. In section 4, some special cases will be mentioned.

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2. The proof of the first theorem

We shall first prove a lemma concerning algebraic congruences with prime power modulus.

LEMMA 2.1.⁸⁾ Let $\varphi(\xi)$ be a squarefree polynomial of degree $d > 0$ with integral coefficients, and let D be its discriminant. Then, for any prime power p^a , the number of incongruent solutions of the congruence

$$(2.1) \quad \varphi(\xi) \equiv 0 \pmod{p^a}$$

does not exceed dD^2 .

PROOF. We regard $\varphi(\xi)$ and p as fixed, and denote the number of incongruent solutions of the congruence (2.1) by $N(a)$. Let $p^b \parallel D$ ⁹⁾. Let $\xi = n_1$ be a solution of the congruence (2.1), and let $p^m \parallel \varphi'(n_1)$ ¹⁰⁾. Then we can easily see that $m \leq b$ when $a > b$; in fact, if $a > b$ and $m > b$, i.e. $p^{b+1} \mid \varphi(n_1)$ and $p^{b+1} \mid \varphi'(n_1)$ at the same time, then we would have $p^{b+1} \mid D$ contrary to the meaning of the number b .

We shall, for a while, assume that $a > 2b$, and show that $N(a) = N(a+1)$ under this assumption. Again let n_1 be a solution of the congruence (2.1) with $p^m \parallel \varphi'(n_1)$ as above. Then $m \leq b$ as is proved, so that $a > 2m$. We now denote by $\mathfrak{M}_1$ the set of numbers of the form $n_1 + up^{a-m}$, where u runs through integers, and consider the Taylor's expansion

7) $z_i(n) \rightarrow \infty$ as $n \rightarrow \infty$ as in Theorem A_1 .

8) Cf. Nagell [6], p. 90, Theorem 54.

9) Since $\varphi(\xi)$ is assumed to be squarefree with integral coefficients, D is an integer $\neq 0$. $p^b \parallel D$ means, as usual, that $p^b \mid D$ and $p^{b+1} \nmid D$.

10) $\varphi'(\xi)$ is the derivative of $\varphi(\xi)$. We take n_1 such that $\varphi'(n_1) \neq 0$.

$$\varphi(n_1 + up^{a-m}) = \varphi(n_1) + up^{a-m}\varphi'(n_1) + u^2p^{2(a-m)}\frac{\varphi''(n_1)}{2} + \dots$$

Since $p^a \mid \varphi(n_1)$, $p^m \mid \varphi'(n_1)$, and $2(a-m) = a + (a-2m) > a$, this expansion shows that all the numbers of the set $\mathfrak{M}_1$ are the solutions of the congruence (2.1), and furthermore a number of the set $\mathfrak{M}_1$ can be a solution of the congruence

$$(2.2) \quad \varphi(\xi) \equiv 0 \pmod{p^{a+1}},$$

when and only when u satisfies the congruence

$$\varphi(n_1) + up^{a-m}\varphi'(n_1) \equiv 0 \pmod{p^{a+1}},$$

or equivalently the congruence

$$\frac{\varphi'(n_1)}{p^m}u \equiv -\frac{\varphi(n_1)}{p^a} \pmod{p}.$$

Since $p^m \nmid \varphi'(n_1)$, the coefficient of u in the last congruence is not divisible by p . Hence it has just one solution modulo p . Thus a number of the set $\mathfrak{M}_1$ can also be a solution of the congruence (2.2) when and only when u belongs to a definite class modulo p . It follows that in the set $\mathfrak{M}_1$ there are as many incongruent solutions of the congruence (2.2) as the incongruent solutions of the congruence (2.1).

We next consider the Taylor's expansion

$$\varphi'(n_1 + up^{a-m}) = \varphi'(n_1) + up^{a-m}\varphi''(n_1) + u^2p^{2(a-m)}\frac{\varphi'''(n_1)}{2} + \dots$$

Since $a > 2m$, i.e. $a-m > m$, this expansion shows that $p^m \mid \varphi'(n_1 + up^{a-m})$ for any number $n_1 + up^{a-m} \in \mathfrak{M}_1$. Now, if the set $\mathfrak{M}_1$ does not exhaust the solutions of the congruence (2.1), we take a solution $n_2 \notin \mathfrak{M}_1$, and, define $\mathfrak{M}_2$ to be the set consisting of the numbers of the form $n_2 + vp^{a-m'}$, where $p^{m'} \mid \varphi'(n_2)$ ¹¹⁾. Then $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \emptyset$; in fact, if we had $n_1 + up^{a-m} = n_2 + vp^{a-m'} \in \mathfrak{M}_1 \cap \mathfrak{M}_2$ for some u and v , then $m = m'$ by the above consideration, and we should have $n_1 + up^{a-m} = n_2 + vp^{a-m}$ or equivalently $n_2 = n_1 + (u-v)p^{a-m}$, which means $n_2 \in \mathfrak{M}_1$ contrary to the hypothesis. Next, if the union $\mathfrak{M}_1 \cup \mathfrak{M}_2$ does not exhaust the solutions of the congruence (2.1), we take a solution $n_3 \notin \mathfrak{M}_1 \cup \mathfrak{M}_2$, and we define the set $\mathfrak{M}_3$ similarly as the sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Then we can prove $(\mathfrak{M}_1 \cup \mathfrak{M}_2) \cap \mathfrak{M}_3 = \emptyset$ similarly as we proved $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \emptyset$. If the union $\mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$ does not exhaust the solutions of the congruence (2.1), we take a solution $n_4 \notin \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$, and define the set $\mathfrak{M}_4$. Proceeding in this way, we can obtain a finite number of sets $\mathfrak{M}_1, \dots, \mathfrak{M}_s$ such that they are mutually disjoint, and their union contains all the solutions of the congruence (2.1). Then, as is proved above concerning the set $\mathfrak{M}_1$, each of the sets $\mathfrak{M}_1, \dots, \mathfrak{M}_s$ contains the same number of incongruent solutions of the congruences (2.1) and (2.2). We can thus conclude $N(a) = N(a+1)$ under the assumption that $a > 2b$.

Hence it suffices to consider the case $a \leq 2b+1$. Let p^c be the highest power of p which divides the greatest common divisor of the coefficients of the polynomial $\varphi(\xi)$. Then $c \leq b$ since $p^c \mid D$. Now the case $a \leq c$ is trivial: $N(a) \leq p^a \leq p^c \leq p^b \leq D \leq dD^2$. When $a = c+1$, $N(a)$ is p^c times the number of incongruent solutions of the congruence

11) We take n_2 such that $\varphi'(n_2) \neq 0$.

$$\frac{\varphi(\xi)}{p^c} \equiv 0 \pmod{p},$$

which, being not identical, has at most d incongruent solutions. Thus $N(a) \leq dp^c \leq dp^b \leq dD \leq dD^2$. Finally, as for the case $c+2 \leq a \leq 2b+1$, it follows from the inequality $N(c+1) \leq dp^c$ now obtained and the trivial inequality $N(a) \leq p^{a-c-1}N(c+1)$ that $N(a) \leq dp^{a-1} \leq dp^{2b} \leq dD^2$. The proof of the lemma is now completed.

We shall next give a lemma which concerns with the relation between the number of distinct prime factors of integers and the number of prime factors of integers, multiple factors being counted multiply.

LEMMA 2.2. *Let $\varphi(\xi)$ be a polynomial of positive degree with integral coefficients and with positive leading coefficient, and suppose that its canonical factorization is given by*

$$(2.3) \quad \varphi(\xi) = c \prod_{\rho=1}^r \{q_\rho(\xi)\}^{a_\rho}.$$

Let n_1 be a positive integer such that $q_\rho(n) > 0$ ($\rho=1, \dots, r$) for integers $n \geq n_1$.

Let $\mathfrak{P}$ be a set of prime numbers, and let $\omega_{\mathfrak{P}}(n)$ denote the number of distinct prime factors of a positive integer n which belong to the set $\mathfrak{P}$:

$$\omega_{\mathfrak{P}}(n) = \sum_{p|n, p \in \mathfrak{P}} 1.$$

Let further $\Omega_{\mathfrak{P}}(n)$ denote the number of prime factors of a positive integer n which belong to the set $\mathfrak{P}$, multiple factors being counted multiply:

$$\Omega_{\mathfrak{P}}(n) = \sum_{p^a|n, p \in \mathfrak{P}} 1.$$

Then we have

$$\sum_{n_1 \leq n \leq x} \left| \Omega_{\mathfrak{P}}\{f(n)\} - \sum_{\rho=1}^r a_\rho \omega_{\mathfrak{P}}\{q_\rho(n)\} \right| = O(x).$$

PROOF. We may, for simplicity, assume that $n_1 = 1^{(12)}$. It is obvious that, for any positive integer n ,

$$\Omega_{\mathfrak{P}}\{f(n)\} = \sum_{\rho=1}^r a_\rho \Omega_{\mathfrak{P}}\{q_\rho(n)\},$$

so that

$$\sum_{n \leq x} \left| \Omega_{\mathfrak{P}}\{f(n)\} - \sum_{\rho=1}^r a_\rho \omega_{\mathfrak{P}}\{q_\rho(n)\} \right|^{(13)} = \sum_{\rho=1}^r a_\rho \sum_{n \leq x} [\Omega_{\mathfrak{P}}\{q_\rho(n)\} - \omega_{\mathfrak{P}}\{q_\rho(n)\}].$$

Hence we want to prove that

$$\sum_{n \leq x} [\Omega_{\mathfrak{P}}\{q_\rho(n)\} - \omega_{\mathfrak{P}}\{q_\rho(n)\}] = O(x) \quad (\rho=1, \dots, r).$$

In the proof which follows, we fix the subscript ρ to one of the values $1, \dots, r$, and write simply $q(n)$ for $q_\rho(n)$. We put

$$S = \sum_{n \leq x} [\Omega_{\mathfrak{P}}\{q(n)\} - \omega_{\mathfrak{P}}\{q(n)\}].$$

¹²⁾ We have only to consider $f(\xi+n_1)$ instead of $f(\xi)$ if necessary.

¹³⁾ Each expression between the symbol $|$ is, in fact, non-negative.

Now

$$\omega_{\mathfrak{P}}\{q(n)\} = \sum_{p|q(n), p \in \mathfrak{P}} 1,$$

and, if we denote, for convenience, by $h(n, p)$ the exponent of the power of p such that $p^{h(n, p)} || n$, then we can write

$$\Omega_{\mathfrak{P}}\{q(n)\} = \sum_{p|q(n), p \in \mathfrak{P}} h\{q(n), p\},$$

so that

$$\Omega_{\mathfrak{P}}\{q(n)\} - \omega_{\mathfrak{P}}\{q(n)\} = \sum_{p|q(n), p \in \mathfrak{P}} [h\{q(n), p\} - 1],$$

which implies

$$\Omega_{\mathfrak{P}}\{q(n)\} - \omega_{\mathfrak{P}}\{q(n)\} \leq \sum_{p|q(n)} [h\{q(n), p\} - 1],$$

omitting the restriction that the variable prime p must be contained in the set $\mathfrak{P}$. Hence

$$S \leq \sum_{n \leq x} \sum_{p|q(n)} [h\{q(n), p\} - 1].$$

Separating the sum on the right-hand side into two partial summations, we rewrite the inequality as

$$(2.4) \quad S \leq \Sigma_1 + \Sigma_2$$

with

$$\Sigma_1 = \sum_{n \leq x} \sum_{p \leq x, p|q(n)} [h\{q(n), p\} - 1]$$

and

$$\Sigma_2 = \sum_{n \leq x} \sum_{p > x, p|q(n)} [h\{q(n), p\} - 1].$$

We shall consider the sums Σ_1 and Σ_2 separately.

We first deal with Σ_1 . On putting

$$H(x, p) = \max_{n \leq x} h\{q(n), p\},$$

we can write, for any n and p such that $n \leq x$ and $p^2 | n$,

$$h\{q(n), p\} - 1 = \sum_{a=2}^{h\{q(n), p\}} 1 = \sum_{\substack{a=2 \\ p^a | q(n)}}^{H(x, p)} 1;$$

In fact, the set of the numbers $a=2, \dots, h\{q(n), p\}$ is equal to the set of the numbers $a=2, \dots, H(x, p)$ for which $p^a | q(n)$. Hence we can rewrite the double sum defining Σ_1 as

$$\Sigma_1 = \sum_{n \leq x} \sum_{p \leq x, p^2 | q(n)} \sum_{a=2}^{H(x, p)} 1 = \sum_{p \leq x} \sum_{a=2}^{H(x, p)} \sum_{n \leq x, p^a | q(n)} 1,$$

inverting the order of summation. Now the innermost sum of the last triple summation counts the number of solutions n , $1 \leq n \leq x$, of the congruence $q(\xi) \equiv 0 \pmod{p^a}$. Here we can apply Lemma 2.1 to $q(\xi)$, since $q(\xi)$ is irreducible,

and a fortiori squarefree. Denoting by d and D the degree and discriminant of $q(\xi)$ respectively, we obtain, by Lemma 2.1,

$$(2.5) \quad \sum_{n \leq x, p^a | q(n)} 1 \leq dD^2 \left(\frac{x}{p^a} + 1 \right).$$

In order to estimate the sum Σ_1 , we must also estimate $H(x, p)$. When x is sufficiently large, $q(n) < x^{d+1}$ for $n \leq x$, which implies that $p^{h(q(n), p)} < x^{d+1}$ for $n \leq x$, i. e. $p^{H(x, p)} < x^{d+1}$. We deduce that

$$(2.6) \quad H(x, p) < \frac{(d+1) \log x}{\log p} \leq \frac{2d \log x}{\log 2} < 3d \log x$$

for sufficiently large x . Now, by (2.5) and (2.6),

$$\begin{aligned} \Sigma_1 &\leq dD^2 \sum_{p \leq x} \sum_{a=2}^{3d \log x} \left(\frac{x}{p^a} + 1 \right) \\ &\leq dD^2 x \sum_{p \leq x} \sum_{a=2}^{3d \log x} \frac{1}{p^a} + 3d^2 D^2 \pi(x) \log x, \end{aligned}$$

where $\pi(x)$ denotes, as usual, the number of primes not exceeding x . Here the sum

$$\sum_{p \leq x} \sum_{a=2}^{3d \log x} \frac{1}{p^a}$$

is evidently bounded, and, as is well known,

$$\pi(x) = O\left(\frac{x}{\log x}\right).$$

Hence

$$(2.7) \quad \Sigma_1 = O(x).$$

We turn to the sum Σ_2 . We fix a positive integer $n \leq x$, and assume that there are s primes p such that $p > x$ and $p | q(n)$, say $p_1, \dots, p_s$. Then

$$p_1^{h(q(n), p_1)} \dots p_s^{h(q(n), p_s)} | q(n),$$

which implies

$$x^{h(q(n), p_1) + \dots + h(q(n), p_s)} < q(n).$$

Again we consider sufficiently large x . Then $q(n) < x^{d+1}$, and hence

$$x^{h(q(n), p_1) + \dots + h(q(n), p_s)} < x^{d+1}$$

which gives

$$h(q(n), p_1) + \dots + h(q(n), p_s) < d+1,$$

i. e.

$$\sum_{p > x, p | q(n)} h(q(n), p) \leq d.$$

Thus this inequality holds for any positive integer n such that $n \leq x$ when x is sufficiently large, and from this we at once deduce that

$$(2.8) \quad \Sigma_2 = O(x)$$

From (2.4), (2.7), and (2.8) we obtain

$$S=O(x),$$

and the lemma is proved.

We shall next give a lemma which is similar to III, Theorem A.

LEMMA 2.3. *Let l be a positive integer. Let $\{f_1^*(\xi), \dots, f_l^*(\xi)\}$ be a set of polynomials in ξ , and $\{\mathfrak{P}_1^*, \dots, \mathfrak{P}_l^*\}$ a family of sets of prime numbers. Suppose that the following conditions are satisfied:*

(C₁) *For each $j=1, \dots, l$, the polynomial $f_j^*(\xi)$ is of positive degree with integral coefficients and with positive leading coefficient;*

(C₂) *For each pair of integers j, k with $1 \leq j < k \leq l$, at least one of the two requirements $(f_j^*, f_k^*)=1$ and $\mathfrak{P}_j^* \cap \mathfrak{P}_k^* = \emptyset$ holds;*

(C₃) *If we denote, for each $j=1, \dots, l$, by $\nu_j^*(p)$ the number of incongruent solutions of the congruence $f_j^*(\xi) \equiv 0 \pmod{p}$, then the series*

$$\sum_{p \in \mathfrak{P}_j^*} \frac{\nu_j^*(p)}{p}$$

is divergent.

Now let $\omega_j^*(n)$ denote, for each $j=1, \dots, l$, the number of distinct prime factors of a positive integer n which belong to the set $\mathfrak{P}_j^*$:

$$\omega_j^*(n) = \sum_{p|n, p \in \mathfrak{P}_j^*} 1.$$

Put, for $x > 0$,

$$y_j^*(x) = \sum_{p \leq x, p \in \mathfrak{P}_j^*} \frac{\nu_j^*(p)}{p} \quad (j=1, \dots, l).$$

Let x_0 be a positive number such that $f_j^*(x) > 0$, $y_j^*(x) > 0$ ($j=1, \dots, l$) for $x \geq x_0$.

Put, for $x > x_0$ and integers $n \geq x_0$,

$$u_j^*(n, x) = \frac{\omega_j^*\{f_j^*(n)\} - y_j^*(x)}{\sqrt{y_j^*(x)}} \quad (j=1, \dots, l)$$

For $x > x_0$ and for any convex set E^* , bounded or unbounded, in the space R^l of l dimensions, let $A^*(x; E^*)$ denote the number of integers n with $x_0 \leq n \leq x$ for which the points $U^*(n, x) = (u_1^*(n, x), \dots, u_l^*(n, x))$ of the space R^l belong to the set E^* . Then we have

$$\lim_{x \rightarrow \infty} \frac{A^*(x; E^*)}{x} = (2\pi)^{-l/2} \int_{E^*} \exp\left(-\frac{1}{2} \sum_{j=1}^l u_j^{*2}\right) du_1^* \dots du_l^*,$$

the convergence being uniform with respect to the set E^* provided it is convex.

This lemma differs in substance from III, Theorem A only in the respects that the definition of $u_j^*(n, x)$ is slightly different from that of $u_i(n)$, and that we assert the uniformness of the resulting convergence with respect to the set E^* . We changed the notations as follows to avoid the confusion: We used the letters l and j in the places where we used the letters k and i in III, Theorem A respectively, and we asterisked the letters $f, \mathfrak{P}, \nu, \omega, y, u, E$, and A .

We assumed the convexity of the set E^* to assure the uniform convergence with respect to the set E^* , which means the following:

(i) For any given $\varepsilon > 0$, we can find an $x_1 = x_1(\varepsilon) > x_0$ such that, for $x > x_1$ and for any convex set E^* ,

$$\left| \frac{A^*(x; E^*)}{x} - (2\pi)^{-l/2} \int_{E^*} \exp\left(-\frac{1}{2} \sum_{j=1}^l u_j^{*2}\right) du_1^* \cdots du_l^* \right| < \varepsilon,$$

and the number x_1 can be chosen independently of the shape and position of the set E^* provided it is convex. Or in other words:

(ii) Suppose there is given an arbitrary family of sets $\{E_x^*\}$ such that, to each real number $x > x_0$, there corresponds a convex set E_x^* of the space R^l . Then we have

$$(2.9) \quad \lim_{x \rightarrow \infty} \left\{ \frac{A(x; E_x^*)}{x} - (2\pi)^{-l/2} \int_{E_x^*} \exp\left(-\frac{1}{2} \sum_{j=1}^l u_j^{*2}\right) du_1^* \cdots du_l^* \right\} = 0.$$

PROOF. During this proof, the letters l and j will be rechanged to k and i respectively, and the asterisks will be omitted, in order that the notations in this proof should be congruous with those in the lemmas in III.

Our proof now starts from III, p. 116, Lemma 2.11. By similar arguments as in II, pp. 184–187, the proofs of Lemmas 6, 7, 8, and 9, we can deduce that, when the set E is an interval, say I : $\alpha_i \leq u_i < \beta_i (i=1, \dots, k)^{14)}$,

$$(2.10) \quad \lim_{x \rightarrow \infty} \frac{A(x; I)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i.$$

This corresponds to II, Lemma 9. Note that, in the deduction of this result from III, Lemma 2.11, we use III, p. 106, Lemma 2.4 and p. 119, Lemma 2.12.¹⁵⁾

If we do not require the uniformness of the convergence, we can, of course, obtain our lemma by the same reasoning as in II, pp. 187–189, but, for securing the uniformness, we must enter into a somewhat more complicated consideration.

We put, for brevity,

$$\Phi(E) = (2\pi)^{-k/2} \int_E \exp\left(-\frac{1}{2} \sum_{i=1}^k u_i^2\right) du_1 \cdots du_k.$$

in this proof.

Now let g be a positive number, and let I_g be the interval $-g \leq u_i < g$ ($i=1, \dots, k$) in the space R^k . Let h be an integer > 1 , and we divide the interval I_g by the hyperplanes $u_i = -g + (2g/h)j$ ($i=1, \dots, k; j=1, \dots, h-1$) into h^k small intervals, each closed to the left and open to the right. Let these small intervals be called $I_{g,h}^{(t)}$ ($t=1, \dots, h^k$) in some order. Now, E being a convex set, bounded or unbounded, in the space R^k , we denote by $S_{g,h}$ the union of the intervals $I_{g,h}^{(t)}$ which are entirely contained in the set E , and by $S'_{g,h}$ the union of the intervals $I_{g,h}^{(t)}$ such that $I_{g,h}^{(t)} \cap E$ is not empty. Then, since E is convex, it is easily seen that the number of intervals $I_{g,h}^{(t)}$ constituting the difference $S'_{g,h} - S_{g,h}$ does not exceed the number of intervals $I_{g,h}^{(t)}$ whose boundaries come in contact with the boundary of the interval I_g . Hence, if we denote by $\text{vol}(S'_{g,h} - S_{g,h})$ the volume of the set $S'_{g,h} - S_{g,h}$, then

14) The half closedness of the interval I has scarcely any influence on the proof of (2.10).

15) Cf. III, p. 119, lines just before Lemma 2.12.

$$\text{vol}(S'_{g,h} - S_{g,h}) \leq \left(\frac{2g}{h}\right)^k \{h^k - (h-2)^k\} < \left(\frac{2g}{h}\right)^k \cdot 2kh^{k-1} = \frac{2k(2g)^k}{h},$$

from which we obtain

$$(2.11) \quad \Phi(S'_{g,h}) - \Phi(S_{g,h}) < \frac{2k(2g)^k}{h}.$$

Now let ε be an arbitrarily given positive number. Then we can take $g=g(\varepsilon)$ so large that

$$\Phi(I_g^c) < \varepsilon,^{16)}$$

which implies that

$$(2.12) \quad \Phi(E \cap I_g^c) < \varepsilon.$$

Also, applying (2.10) to the interval I_g , we get

$$\lim_{x \rightarrow \infty} \frac{A(x; I_g)}{x} = \Phi(I_g),$$

and hence

$$\lim_{x \rightarrow \infty} \frac{A(x; I_g^c)}{x} = \Phi(I_g^c) < \varepsilon.$$

Hence we can take a number $x_1 = x_1(\varepsilon, g) = x_1(\varepsilon) > x_0$ such that, when $x > x_1$,

$$\frac{A(x; I_g^c)}{x} < \varepsilon,$$

which implies that

$$(2.13) \quad \frac{A(x; E \cap I_g^c)}{x} < \varepsilon.$$

Here we put

$$h = h(\varepsilon, g) = h(\varepsilon) = \left\lceil \frac{2k(2g)^k}{\varepsilon} \right\rceil + 1,$$

then, by (2.11),

$$(2.14) \quad \Phi(S'_{g,h}) - \Phi(S_{g,h}) < \varepsilon.$$

Also, applying (2.10) to the intervals $I_{g,h}^{(t)}$, we get

$$\lim_{x \rightarrow \infty} \frac{A(x; I_{g,h}^{(t)})}{x} = \Phi(I_{g,h}^{(t)}),$$

so that we can take a number $x_2 = x_2(\varepsilon, g, h) = x_2(\varepsilon) > x_1$ such that, when $x > x_2$,

$$\left| \frac{A(x; I_{g,h}^{(t)})}{x} - \Phi(I_{g,h}^{(t)}) \right| < \frac{\varepsilon}{h^k}$$

hold for $t=1, \dots, h^k$ simultaneously. It obviously follows from this that, when $x > x_2$,

16) I_g^c means the complement of I_g .

$$(2.15) \quad \left| \frac{A(x; S_{g,h})}{x} - \Phi(S_{g,h}) \right| < \varepsilon$$

and

$$(2.16) \quad \left| \frac{A(x; S'_{g,h})}{x} - \Phi(S'_{g,h}) \right| < \varepsilon .$$

Now, since obviously

$$\begin{aligned} A(x; E) &= A(x; E \cap I_g) + A(x; E \cap I_g^c) , \\ \Phi(E) &= \Phi(E \cap I_g) + \Phi(E \cap I_g^c) , \end{aligned}$$

and also

$$\begin{aligned} A(x; S_{g,h}) &\leq A(x; E \cap I_g) \leq A(x; S'_{g,h}) , \\ \Phi(S_{g,h}) &\leq \Phi(E \cap I_g) \leq \Phi(S'_{g,h}) , \end{aligned}$$

it follows from (2.12), (2.13), (2.14), (2.15), and (2.16) that

$$\left| \frac{A(x; E)}{x} - \Phi(E) \right| < 4\varepsilon$$

when $x > x_2$, and, since the number $x_2 = x_2(\varepsilon)$ can be chosen independently of the shape and position of the set E , provided it is convex, we see that our lemma is proved.

THE PROOF OF THEOREM A₁. The notations in the enunciation of Theorem A₁ will be preserved in this proof. The letter i will always represent one of the integers $1, \dots, k$, and we shall omit the proviso ' $i=1, \dots, k$ ' except when it is desirable to emphasize that we are considering the k values of i simultaneously.¹⁷⁾

We divide the proof in four steps.

1. We denote by $\omega_i(n)$ the number of distinct prime factors of a positive integer n which belong to the set $\mathfrak{P}_i$:

$$\omega_i(n) = \sum_{p|n, p \in \mathfrak{P}_i} 1 .$$

We put, $x > 0$,

$$(2.17) \quad y_{i\rho}(x) = \sum_{p \leq x, p \in \mathfrak{P}_i} \frac{\nu_{i\rho}(p)}{p} \quad (\rho=1, \dots, r_i),$$

and then put

$$(2.18) \quad y_i(x) = \sum_{\rho=1}^{r_i} a_{i\rho} y_{i\rho}(x)$$

and

$$(2.19) \quad z_i(x) = \sum_{\rho=1}^{r_i} a_{i\rho}^2 y_{i\rho}(x) .$$

Thus defined $y_i(x)$ and $z_i(x)$ coincide with $y_i(n)$ and $z_i(n)$ in the enunciation of the theorem. Now, since $\nu_{i1}(p) + \dots + \nu_{ir_i}(p) \geq \nu_i(p)$ for any prime p ,

17) We shall obey this convention also in section 3, THE PROOF OF THEOREM A₂, and in section 4.

$$\sum_{\rho=1}^{r_i} y_{i\rho}(x) \geq \sum_{p \leq x, p \in \mathfrak{P}_i} \frac{\nu_i(p)}{p}.$$

Also the condition (C₃) means that

$$\sum_{p \leq x, p \in \mathfrak{P}_i} \frac{\nu_i(p)}{p} \rightarrow \infty \text{ as } x \rightarrow \infty.$$

Hence it follows that, for each $i=1, \dots, k$, among the functions $y_{i\rho}(x)$ ($\rho=1, \dots, r_i$), there are at least one which tends to infinity with x . We suppose that, for each $i=1, \dots, k$, there are t_i such functions, say $y_{i\rho_{i\tau}}(x)$ ($\tau=1, \dots, t_i$), and that the remaining $r_i - t_i$ functions are bounded. We put

$$(2.20) \quad \tilde{y}_i(x) = \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} y_{i\rho_{i\tau}}(x)$$

and

$$(2.21) \quad \tilde{z}_i(x) = \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}}^2 y_{i\rho_{i\tau}}(x).$$

Then we have

$$(2.22) \quad y_i(x) - \tilde{y}_i(x) = O(1)$$

and

$$(2.23) \quad z_i(x) - \tilde{z}_i(x) = O(1).$$

Let x_0 be a positive number such that $q_{i\rho}(x) > 0$ ($i=1, \dots, k; \rho=1, \dots, r_i$) and $y_{i\rho_{i\tau}}(x) > 0$ ($i=1, \dots, k; \tau=1, \dots, t_i$) for $x \geq x_0$, and we put, for $x > x_0$ and integers $n \geq x_0$,

$$(2.24) \quad u_{i\tau}(n, x) = \frac{\omega_i \{q_{i\rho_{i\tau}}(n)\} - y_{i\rho_{i\tau}}(x)}{\sqrt{y_{i\rho_{i\tau}}(x)}} \quad (\tau=1, \dots, t_i).$$

Then, for each $i=1, \dots, k$, we can associate with any pair n, x with $x > x_0$ and $n \geq x_0$ a point $U_i(n, x) = (u_{i1}(n, x), \dots, u_{it_i}(n, x))$ of the space R^{t_i} of t_i dimensions, and, if we further write

$$U^*(n, x) = (U_1(n, x), \dots, U_k(n, x)),$$

then $U^*(n, x)$ can be regarded as a point of the product space

$$R^T = R^{t_1} \times \dots \times R^{t_k}$$

of $T = t_1 + \dots + t_k$ dimensions. Also let $\alpha_i < \beta_i$ ($i=1, \dots, k$) be real numbers, and we define, for $i=1, \dots, k$ and for each value of x such that $x > x_0$, the set $E_{i,x}$ in the space R^{t_i} as consisting of the points $(u_{i1}, \dots, u_{it_i})$ which satisfy the inequalities

$$(2.25) \quad \alpha_i < \frac{\sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} u_{i\tau} \sqrt{y_{i\rho_{i\tau}}(x)}}{\sqrt{\tilde{z}_i(x)}} < \beta_i.$$

We further define the set E_x^* in the space R^T as the product

$$E_x^* = E_{1,x} \times \dots \times E_{k,x}.$$

Then the set E_x^* is certainly convex as is seen from the definition.

We denote, for $x > x_0$, by $A^*(x; E_x^*)$ the number of integers n with $x_0 \leq n \leq x$ for which the points $U^*(n, x)$ belong to the set E_x^* , that is, the points $U_i(n, x)$ belong to the sets $E_{i,x}$ for $i=1, \dots, k$ respectively. Here we want to apply Lemma 2.3 to $A^*(x; E_x^*)$ in such a fashion that the correspondence between the quantities in the enunciation of Lemma 2.3 and those now under consideration is given as follows:

$$\left. \begin{aligned} l &= T \\ f_j^*(\xi) &= q_{i\rho_{i\tau}}(\xi) \\ \mathfrak{P}_j^* &= \mathfrak{P}_i \\ \nu_j^*(p) &= \nu_{i\rho_{i\tau}}(p) \\ \omega_j^*(n) &= \omega_i(n) \\ y_j^*(x) &= y_{i\rho_{i\tau}}(x) \\ u_j^*(n, x) &= u_{i\tau}(n, x) \end{aligned} \right\} \text{ with } j = t_1 + \dots + t_{i-1} + \tau, \tau = 1, \dots, t_i.$$

In fact, the conditions (C_1) , (C_2) , and (C_2) of Lemma 2.3 are surely satisfied by virtue of the conditions (C_1) and (C_2) of the theorem and the unboundedness of the functions $y_{i\rho_{i\tau}}(x)$, and, if we write down the result thus obtained in applying Lemma 2.3 in the form (2.9), we obtain

$$(2.26) \quad \lim_{x \rightarrow \infty} \left\{ \frac{A^*(x; E_x^*)}{x} - (2\pi)^{-T/2} \int_{E_x^*} \exp\left(-\frac{1}{2} \sum_{i=1}^k \sum_{\tau=1}^{t_i} u_{i\tau}^2\right) du_{11} \dots du_{kt_k} \right\} = 0,$$

where the integral is T dimensional with the integrating variables $u_{i\tau}$ ($i=1, \dots, k$; $\tau=1, \dots, t_i$). Since the set E_x^* is the product of the sets $E_{1,x}, \dots, E_{k,x}$, this integral can also be decomposed into the product of k integrals:

$$\begin{aligned} (2\pi)^{-T/2} \int_{E_x^*} \exp\left(-\frac{1}{2} \sum_{i=1}^k \sum_{\tau=1}^{t_i} u_{i\tau}^2\right) du_{11} \dots du_{kt_k} \\ = \prod_{i=1}^k (2\pi)^{-t_i/2} \int_{E_{i,x}} \exp\left(-\frac{1}{2} \sum_{\tau=1}^{t_i} u_{i\tau}^2\right) du_{i1} \dots du_{it_i}, \end{aligned}$$

and, on noting the definition of the set $E_{i,x}$, we further obtain

$$(2\pi)^{-t_i/2} \int_{E_{i,x}} \exp\left(-\frac{1}{2} \sum_{\tau=1}^{t_i} u_{i\tau}^2\right) du_{i1} \dots du_{it_i} = \frac{1}{\sqrt{2\pi}} \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i$$

by easy calculations of repeated integrals.¹⁸⁾ Hence

$$(2\pi)^{-T/2} \int_{E_x^*} \exp\left(-\frac{1}{2} \sum_{i=1}^k \sum_{\tau=1}^{t_i} u_{i\tau}^2\right) du_{11} \dots du_{kt_k} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i,$$

and this equality shows that the integral on the left-hand side is, in fact, independent of x . Hence (2.26) can be rewritten as

$$(2.27) \quad \lim_{x \rightarrow \infty} \frac{A^*(x; E_x^*)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i.$$

On the other hand, by (2.20) and (2.24),

18) Or by the theory of many dimensional normal distributions.

$$(2.28) \quad \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} u_{i\tau}(n, x) \sqrt{y_{i\rho_{i\tau}}(x)} = \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} \omega_i\{q_{i\rho_{i\tau}}(n)\} - \tilde{y}_i(x),$$

and comparing this with (2.25) defining the set $E_{i,x}$, we see that $A^*(x; E_x^*)$ can be defined as the number of integers n with $x_0 \leq n \leq x$ for which the inequalities

$$(2.29) \quad \tilde{y}_i(x) + \alpha_i \sqrt{\tilde{z}_i(x)} < \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} \omega_i\{q_{i\rho_{i\tau}}(n)\} < \tilde{y}_i(x) + \beta_i \sqrt{\tilde{z}_i(x)} \quad (i=1, \dots, k)$$

hold simultaneously. For later convenience, we now rewrite $A^*(x; E_x^*)$ as $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ indicating that it depends not only on x but also on $\alpha_i, \beta_i (i=1, \dots, k)$. (2.27) becomes

$$(2.30) \quad \lim_{x \rightarrow \infty} \frac{\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i.$$

2. We denote by $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ the number of integers n with $x_0 \leq n \leq x$ for which the inequalities

$$(2.31) \quad \tilde{y}_i(x) + \alpha_i \sqrt{\tilde{z}_i(x)} < \Omega_i\{f_i(n)\} < \tilde{y}_i(x) + \beta_i \sqrt{\tilde{z}_i(x)} \quad (i=1, \dots, k)$$

hold simultaneously, and, in this second step, we want to show that, in (2.30), $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ can be replaced by $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$, the right-hand side being left as it is.

For this purpose we estimate the sum

$$\sum_{x_0 \leq n \leq x} \omega_i\{q_{i\rho}(n)\} \text{ with } \rho \neq \rho_{i\tau} (\tau=1, \dots, t_i).$$

We put

$$\sum_{x_0 \leq n \leq x} \omega_i\{q_{i\rho}(n)\} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{x_0 \leq n \leq x} \sum_{\substack{p | q_{i\rho}(n) \\ p \leq x, p \in \mathfrak{P}_i}} 1$$

and

$$\Sigma_2 = \sum_{x_0 \leq n \leq x} \sum_{\substack{p | q_{i\rho}(n) \\ p > x, p \in \mathfrak{P}_i}} 1$$

Now, changing the order of summation defining Σ_1 , and noting that $\nu_{i\rho}(p)$ means the number of incongruent solutions of the congruence $q_{i\rho}(\xi) \equiv 0 \pmod{p}$, we have

$$\Sigma_1 = \sum_{p \leq x, p \in \mathfrak{P}_i} \sum_{\substack{x_0 \leq n \leq x \\ q_{i\rho}(n) \equiv 0 \pmod{p}}} 1 \leq \sum_{p \leq x, p \in \mathfrak{P}_i} \nu_{i\rho}(p) \left(\frac{x}{p} + 1 \right),$$

and hence, by (2.17),

$$\Sigma_1 \leq xy_{i\rho}(x) + \sum_{p \leq x} \nu_{i\rho}(p).$$

Also, since we are considering the values of the subscript ρ such that $\rho \neq \rho_{i\tau} (\tau=1, \dots, t_i)$, we have

$$y_{i\rho}(x) = O(1).$$

Hence, noting that $\nu_{i\rho}(p)$ does not exceed the degree of $q_{i\rho}(\xi)$, we obtain

$$\Sigma_1 = O(x).$$

As for the sum Σ_2 , we obtain

$$\Sigma_2 = O(x)$$

by similar reasoning as we treated Σ_2 in the proof of III, p. 119, Lemma 2.12. It now follows from thus obtained estimations for the sums Σ_1 and Σ_2 that

$$(2.32) \quad \sum_{x_0 \leq n \leq x} \omega_i\{q_{i\rho}(n)\} = O(x) \text{ when } \rho \neq \rho_{i\tau} (\tau=1, \dots, t_i),$$

and hence

$$\sum_{x_0 \leq n \leq x} \left| \sum_{\rho=1}^{r_i} a_{i\rho} \omega_i\{q_{i\rho}(n)\} - \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} \omega_i\{q_{i\rho_{i\tau}}(n)\} \right| = O(x).$$

Now Lemma 2.2 gives

$$\sum_{x_0 \leq n \leq x} \left| \Omega_i\{f_i(n)\} - \sum_{\rho=1}^{r_i} a_{i\rho} \omega_i\{q_{i\rho}(n)\} \right| = O(x),$$

Combining these estimations, we have

$$(2.33) \quad \sum_{x_0 \leq n \leq x} \left| \Omega_i\{f_i(n)\} - \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} \omega_i\{q_{i\rho_{i\tau}}(n)\} \right| = O(x).$$

Since $\tilde{z}_i(x) \rightarrow \infty$ with x , it can easily be concluded from (2.33) that, given an arbitrary $\varepsilon > 0$, we can take a number $x_1 = x_1(\varepsilon) > x_0$ such that, when $x > x_1$, the number of integers n with $x_0 \leq n \leq x$, for which at least one of the inequalities

$$\left| \Omega_i\{f_i(n)\} - \sum_{\tau=1}^{t_i} a_{i\rho_{i\tau}} \omega_i\{q_{i\rho_{i\tau}}(n)\} \right| > \varepsilon \sqrt{\tilde{z}_i(x)} \quad (i=1, \dots, k)$$

holds, is less than εx . Hence, comparing (2.29) and (2.31) defining $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ and $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ respectively, we see that, when $x > x_1$,

$$\begin{aligned} & \tilde{A}(x; \alpha_1 + \varepsilon, \beta_1 - \varepsilon, \dots, \alpha_k + \varepsilon, \beta_k - \varepsilon) - \varepsilon x \\ & \leq \tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k) \\ & \leq \tilde{A}(x; \alpha_1 - \varepsilon, \beta_1 + \varepsilon, \dots, \alpha_k - \varepsilon, \beta_k + \varepsilon) + \varepsilon x. \end{aligned}$$

Here we apply (2.30), and obtain

$$\begin{aligned} (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i + \varepsilon}^{\beta_i - \varepsilon} \exp\left(-\frac{u_i^2}{2}\right) du_i - \varepsilon & \leq \liminf_{x \rightarrow \infty} \frac{\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} \\ & \leq \limsup_{x \rightarrow \infty} \frac{\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} \leq (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i - \varepsilon}^{\beta_i + \varepsilon} \exp\left(-\frac{u_i^2}{2}\right) du_i + \varepsilon, \end{aligned}$$

which, on making $\varepsilon \rightarrow 0$, gives

$$(2.34) \quad \lim_{x \rightarrow \infty} \frac{\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i.$$

3. We denote by $A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ the number of integers n with $x_0 \leq n \leq x$ for which the inequalities

$$(2.35) \quad y_i(n) + \alpha_i \sqrt{z_i(n)} < \Omega_i\{f_i(n)\} < y_i(n) + \beta_i \sqrt{z_i(n)} \quad (i=1, \dots, k)$$

hold simultaneously.

Since $\nu_{i\rho}(p)$ does not exceed the degree of $q_{i\rho}(\xi)$, it follows from (2.17), (2.18), and (2.19) that, when $\sqrt{x} < n \leq x$,

$$y_i(x) - y_i(n) = O\{y_i(x) - y_i(\sqrt{x})\} = O\left(\sum_{\sqrt{x} < p \leq x} \frac{1}{p}\right) = O(1),$$

and similarly

$$z_i(x) - z_i(n) = O(1).$$

Hence, by (2.22) and (2.23), when $\sqrt{x} < n \leq x$,

$$\tilde{y}_i(x) - y_i(n) = O(1)$$

and

$$\tilde{z}_i(x) - z_i(n) = O(1).$$

Hence it follows that, given an arbitrary $\varepsilon > 0$, we can take a positive number $x_2 = x_2(\varepsilon) > x_1$ such that, when $x > x_2$ and $\sqrt{x} < n \leq x$,

$$\tilde{y}_i(x) + (\alpha_i - \varepsilon)\sqrt{\tilde{z}_i(x)} < y_i(n) + \alpha_i\sqrt{z_i(n)} < \tilde{y}_i(x) + (\alpha_i + \varepsilon)\sqrt{\tilde{z}_i(x)} \quad (i=1, \dots, k)$$

and

$$\tilde{y}_i(x) + (\beta_i - \varepsilon)\sqrt{\tilde{z}_i(x)} < y_i(n) + \beta_i\sqrt{z_i(n)} < \tilde{y}_i(x) + (\beta_i + \varepsilon)\sqrt{\tilde{z}_i(x)} \quad (i=1, \dots, k)$$

hold simultaneously. Hence, comparing (2.31) and (2.35) defining $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ and $A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ respectively, we see that, when $x > x_2$,

$$\begin{aligned} \tilde{A}(x; \alpha_1 + \varepsilon, \beta_1 - \varepsilon, \dots, \alpha_k + \varepsilon, \beta_k - \varepsilon) - \sqrt{x} \\ \leq A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k) \\ \leq \tilde{A}(x; \alpha_1 - \varepsilon, \beta_1 + \varepsilon, \dots, \alpha_k - \varepsilon, \beta_k + \varepsilon) + \sqrt{x}, \end{aligned}$$

on which we apply (2.34), and obtain

$$\begin{aligned} (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i + \varepsilon}^{\beta_i - \varepsilon} \exp\left(-\frac{u_i^2}{2}\right) du_i \leq \liminf_{x \rightarrow \infty} \frac{A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} \\ \leq \limsup_{x \rightarrow \infty} \frac{A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} \leq (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i - \varepsilon}^{\beta_i + \varepsilon} \exp\left(-\frac{u_i^2}{2}\right) du_i. \end{aligned}$$

On making $\varepsilon \rightarrow 0$, this gives

$$(2.36) \quad \lim_{x \rightarrow \infty} \frac{A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u_i^2}{2}\right) du_i.$$

4. (2.36) is a special case of Theorem A₁ when the set E is an interval, and, for accomplishing the proof of Theorem A₁, it only remains to follow the lines in II, pp. 187–189, THE PROOF OF THEOREM A.

3. The proof of the second theorem

We shall first prove a lemma which will play a role corresponding to that played by Lemma 2.2 in the proof of Theorem A₁. Then we can reach Theorem A₂ in a similar way as above.

LEMMA 3.1. *Let $\varphi(\xi)$ be, as in Lemma 2.2, a polynomial of positive degree with integral coefficients and with positive leading coefficient, and suppose that its canonical factorization is given by (2.3). Let the notations n_1 , $\mathfrak{P}$, and $\omega_{\mathfrak{P}}(n)$ have the same meanings as in Lemma 2.2.*

Let s be an integer >1 , and let $\tau_{\mathfrak{P}}^{(s)}(n)$ denote the number of representations of a positive integer n as the product $n=d_1 \cdots d_s d_{s+1}$ such that $d_1, \dots, d_s$ are positive integers composed only of primes belonging to the set $\mathfrak{P}$, and d_{s+1} is a positive integer composed only of primes not belonging to the set $\mathfrak{P}$:

$$\tau_{\mathfrak{P}}^{(s)}(n) = \sum_{\substack{d_1 \cdots d_s d_{s+1} = n \\ p|d_\sigma \Rightarrow p \in \mathfrak{P} \ (\sigma=1, \dots, s) \\ p|d_{s+1} \Rightarrow p \notin \mathfrak{P}}} 1.$$

Then

$$\sum_{n_1 \leq n \leq x} \left| \log \tau_{\mathfrak{P}}^{(s)}\{\varphi(n)\} - \sum_{\rho=1}^r \omega_{\mathfrak{P}}\{q_\rho(n)\} \log \frac{(a_\rho + s - 1)!}{a_\rho! (s-1)!} \right| = O(x).$$

PROOF. We assume that $n_1=1$ as in the proof of Lemma 2.2. We first consider the sum

$$T = \sum_{n \leq x} \left| \log \tau_{\mathfrak{P}}^{(s)}\{\varphi(n)\} - \sum_{\rho=1}^r \log \tau_{\mathfrak{P}}^{(s)}\{q_\rho^{a_\rho}(n)\} \right|.$$

Let $h(n, p)$ have the same meaning as in the proof of Lemma 2.2: $p^{h(n, p)} || n$. Then we can easily see from the elementary theory of combinations that

$$(3.1) \quad \tau_{\mathfrak{P}}^{(s)}(n) = \prod_{p|n, p \in \mathfrak{P}} \frac{\{h(n, p) + s - 1\}!}{h(n, p)! (s-1)!}.$$

Also, as is easily seen,

$$(3.2) \quad \log \frac{\{h(n, p) + s - 1\}!}{h(n, p)! (s-1)!} \leq sh(n, p)^{(19)}.$$

It now follows from I, p. 2, Lemma 2.1 that there are at most a finite number of primes p such that at least two of the congruences $q_\rho(\xi) \equiv 0 \pmod{p}$ ($\rho=1, \dots, r$) have at least a common solution. Let $\mathfrak{Q}$ be the finite set of these primes. Then, for $p \notin \mathfrak{Q}$,

$$h\{\varphi(n), p\} = \begin{cases} h\{q_\rho^{a_\rho}(n), p\} & \text{when } \rho \text{ takes an adequate value out of } 1, \dots, r, \\ 0 & \text{when } \rho \text{ takes other values of } 1, \dots, r. \end{cases}$$

Hence, by (3.1),

$$\begin{aligned} & \log \tau_{\mathfrak{P}}^{(s)}\{\varphi(n)\} - \sum_{\rho=1}^r \log \tau_{\mathfrak{P}}^{(s)}\{q_\rho^{a_\rho}(n)\} \\ &= \sum_{p|\varphi(n), p \in \mathfrak{P}} \log \frac{\{h\{\varphi(n), p\} + s - 1\}!}{h\{\varphi(n), p\}! (s-1)!} - \sum_{\rho=1}^r \sum_{p|q_\rho(n), p \in \mathfrak{P}} \log \frac{\{h\{q_\rho^{a_\rho}(n), p\} + s - 1\}!}{h\{q_\rho^{a_\rho}(n), p\}! (s-1)!} \\ &= \sum_{p|\varphi(n), p \in \mathfrak{P} \cap \mathfrak{Q}} \log \frac{\{h\{\varphi(n), p\} + s - 1\}!}{h\{\varphi(n), p\}! (s-1)!} - \sum_{\rho=1}^r \sum_{p|q_\rho(n), p \in \mathfrak{P} \cap \mathfrak{Q}} \log \frac{\{h\{q_\rho^{a_\rho}(n), p\} + s - 1\}!}{h\{q_\rho^{a_\rho}(n), p\}! (s-1)!}. \end{aligned}$$

Hence, by (3.2),

19) For h a positive integer,

$$\frac{(h+s-1)!}{h!(s-1)!} = \frac{h+1}{1} \cdot \frac{h+2}{2} \cdots \frac{h+s-1}{s-1} \leq (h+1)^{s-1} < (h+1)^s,$$

so that

$$\log \frac{(h+s-1)!}{h!(s-1)!} < s \log (h+1) < sh.$$

$$\begin{aligned}
& \left| \log \tau_{\mathfrak{P}}^{(s)}\{\varphi(n)\} - \sum_{\rho=1}^r \log \tau_{\mathfrak{P}}^{(s)}\{q_{\rho}^{a_{\rho}}(n)\} \right| \\
& \leq s \sum_{p|q(n), p \in \mathfrak{P} \cap \Omega} h\{\varphi(n), p\} + s \sum_{\rho=1}^r \sum_{p|q_{\rho}(n), p \in \mathfrak{P} \cap \Omega} h\{q_{\rho}^{a_{\rho}}(n), p\} \\
& = 2s \sum_{\rho=1}^r a_{\rho} \sum_{p|q_{\rho}(n), p \in \mathfrak{P} \cap \Omega} h\{q_{\rho}(n), p\},
\end{aligned}$$

since, by (2.3),

$$h\{\varphi(n), p\} = \sum_{\rho=1}^r a_{\rho} h\{q_{\rho}(n), p\}.$$

Thus

$$T \leq 2s \sum_{\rho=1}^r a_{\rho} \sum_{n \leq x} \sum_{p|q_{\rho}(n), p \in \mathfrak{P} \cap \Omega} h\{q_{\rho}(n), p\}.$$

Here we put

$$U_{\rho} = \sum_{n \leq x} \sum_{p|q_{\rho}(n), p \in \mathfrak{P} \cap \Omega} h\{q_{\rho}(n), p\} \quad (\rho=1, \dots, r),$$

then

$$(3.3) \quad T \leq 2s \sum_{\rho=1}^r a_{\rho} U_{\rho}.$$

For a while, we fix the subscript ρ to one of the values $1, \dots, r$, and write simply $q(n)$ and U for $q_{\rho}(n)$ and U_{ρ} respectively:

$$U = \sum_{n \leq x} \sum_{p|q(n), p \in \mathfrak{P} \cap \Omega} h\{q(n), p\}.$$

We can now estimate the sum U in a similar way as we treated Σ_1 in the proof of Lemma 2.2. Putting it in brief,

$$\begin{aligned}
U &= \sum_{n \leq x} \sum_{p|q(n), p \in \mathfrak{P} \cap \Omega} \sum_{\substack{a=1 \\ p^a|q(n)}}^{H(x, p)} 1 \\
&= \sum_{p \in \mathfrak{P} \cap \Omega} \sum_{a=1}^{H(x, p)} \sum_{n \leq x, p^a|q(n)} 1 \\
&\leq dD^2 \sum_{p \in \mathfrak{P} \cap \Omega} \sum_{a=1}^{H(x, p)} \left(\frac{x}{p^a} + 1 \right) \\
&\leq dD^2 x \sum_{p \in \mathfrak{P} \cap \Omega} \sum_{a=1}^{3d \log x} \frac{1}{p^a} + 3d^2 D^2 \log x \cdot \sum_{p \in \mathfrak{P} \cap \Omega} 1,
\end{aligned}$$

noting that the set $\mathfrak{P} \cap \Omega$ is finite. Hence

$$U = O(x),$$

or, writing again with the subscript ρ ,

$$U_{\rho} = O(x) \quad (\rho=1, \dots, r).$$

Hence, by (3.3),

$$(3.4) \quad T = O(x).$$

Noting the definition of T , we have

$$(3.5) \quad \sum_{n \leq x} \left| \log \tau_{\mathfrak{P}}^{(s)} \{ \varphi(n) \} - \sum_{\rho=1}^r \omega_{\mathfrak{P}} \{ q_{\rho}(n) \} \log \frac{(a_{\rho} + s - 1)!}{a_{\rho}! (s-1)!} \right| \\ \leq T + \sum_{\rho=1}^r \sum_{n \leq x} \left| \log \tau_{\mathfrak{P}}^{(s)} \{ q_{\rho}^{a_{\rho}}(n) \} - \omega_{\mathfrak{P}} \{ q_{\rho}(n) \} \log \frac{(a_{\rho} + s - 1)!}{a_{\rho}! (s-1)!} \right|.$$

Hence it suffices, in the rest of the proof, to deal with the inner sums of the second member on the right-hand side of (3.5). Again we fix the subscript ρ to one of the values $1, \dots, r$, and write simply a and $q(n)$ for a_{ρ} and $q_{\rho}(n)$ respectively. We put

$$V = \sum_{n \leq x} \left| \log \tau_{\mathfrak{P}}^{(s)} \{ q^a(n) \} - \omega_{\mathfrak{P}} \{ q(n) \} \log \frac{(a + s - 1)!}{a! (s-1)!} \right|.$$

Now we easily see that, when $h\{q(n), p\} > 0$, i. e. $p \mid q(n)$,

$$\frac{(a + s - 1)!}{a! (s-1)!} \leq \frac{[ah\{q(n), p\} + s - 1]!}{[ah\{q(n), p\}]! (s-1)!} \leq \frac{1}{(s-1)!} \left\{ \frac{(a + s - 1)!}{a!} \right\}^{h\{q(n), p\} 20}.$$

Now let p run through the prime factors of $q(n)$, and multiply thus obtained inequalities. Then, noting that

$$\omega_{\mathfrak{P}} \{ q(n) \} = \sum_{p \mid q(n), p \in \mathfrak{P}} 1, \\ \Omega_{\mathfrak{P}} \{ q(n) \} = \sum_{p \mid q(n), p \in \mathfrak{P}} h\{q(n), p\},$$

and

$$\tau_{\mathfrak{P}}^{(s)} \{ q^a(n) \} = \prod_{p \mid q(n), p \in \mathfrak{P}} \frac{[ah\{q(n), p\} + s - 1]!}{[ah\{q(n), p\}]! (s-1)!},$$

we obtain

$$\left\{ \frac{(a + s - 1)!}{a! (s-1)!} \right\}^{\omega_{\mathfrak{P}} \{ q(n) \}} \leq \tau_{\mathfrak{P}}^{(s)} \{ q^a(n) \} \leq \frac{1}{\{(s-1)!\}^{\omega_{\mathfrak{P}} \{ q(n) \}}} \left\{ \frac{(a + s - 1)!}{a!} \right\}^{\Omega_{\mathfrak{P}} \{ q(n) \}}.$$

Hence

$$0 \leq \log \tau_{\mathfrak{P}}^{(s)} \{ q^a(n) \} - \omega_{\mathfrak{P}} \{ q(n) \} \log \frac{(a + s - 1)!}{a! (s-1)!} \\ \leq [\Omega_{\mathfrak{P}} \{ q(n) \} - \omega_{\mathfrak{P}} \{ q(n) \}] \log \frac{(a + s - 1)!}{a!},$$

i. e.

$$0 \leq V \leq S \log \frac{(a + s - 1)!}{a!},$$

20) Multiplying the trivial inequalities

$$a + 1 \leq ah + 1 \leq (a + 1)^h, \\ a + 2 \leq ah + 2 \leq (a + 2)^h, \\ \dots \dots \dots \\ a + s - 1 \leq ah + s - 1 \leq (a + s - 1)^h,$$

we obtain

$$\frac{(a + s - 1)!}{a!} \leq \frac{(ah + s - 1)!}{(ah)!} \leq \left\{ \frac{(a + s - 1)!}{a!} \right\}^h.$$

where S has the same meaning as in the proof of Lemma 2.2. Since $S=O(x)$ as was proved there, we have

$$V=O(x),$$

or rewriting with the subscript ρ ,

$$(3.6) \quad \sum_{n \leq x} \left| \log \tau_{\mathfrak{P}}^{(s)}\{q_{\rho}^{a_{\rho}}(n)\} - \omega_{\mathfrak{P}}\{q_{\rho}(n)\} \log \frac{(a_{\rho}+s-1)!}{a_{\rho}! (s-1)!} \right| = O(x) \quad (\rho=1, \dots, r).$$

Finally the lemma follows from (3.4), (3.5), and (3.6).

THE PROOF OF THEOREM A_2 . Using Lemma 3.1 instead of Lemma 2.2, we can proceed on the same lines as in the proof of Theorem A_1 , so we shall explain the proof shortly, avoiding detailed repetitions, but indicating explicitly the respects in which the proof differs from that of Theorem A_1 . The notations in the enunciation of Theorem A_2 will be preserved. Our explanation will be divided into four steps.

1. We denote, for each $i=1, \dots, k$ and for each $\sigma=2, \dots, s_i$, by $\omega_{i\sigma}(n)$ the number of distinct prime factors of a positive integer n which belong to the set $\Omega_{i\sigma}$:

$$\omega_{i\sigma}(n) = \sum_{p|n, p \in \Omega_{i\sigma}} 1.$$

We denote, for each $i=1, \dots, k$ and for each $\sigma=2, \dots, s_i$, by $\tau_{i\sigma}(n)$ the number of representations of a positive integer n as the product $n=d_1 \cdots d_{\sigma} d_{\sigma+1}$ such that $d_1, \dots, d_{\sigma}$ are positive integers composed only of primes belonging to the set $\Omega_{i\sigma}$, and $d_{\sigma+1}$ is a positive integer composed only of primes not belonging to the set $\Omega_{i\sigma}$:

$$\tau_{i\sigma}(n) = \prod_{p|n, p \in \Omega_{i\sigma}} \frac{\{h(n, p) + \sigma - 1\}!}{h(n, p)! (\sigma - 1)!},$$

where $h(n, p)$ has the same meaning as in the proof of Lemma 2.2 and 3.1. Comparing the definitions of $\tau_i(n)$ in the enunciation of the theorem and of $\tau_{i\sigma}(n)$ now given, and noting the condition (C_2) , we easily see that, for any positive integer n ,

$$(3.7) \quad \tau_i(n) = \prod_{\sigma=2}^{s_i} \tau_{i\sigma}(n).$$

We next put, for $x > 0$,

$$y_{i\rho\sigma}(x) = \sum_{p \leq x, p \in \Omega_{i\sigma}} \frac{\nu_{i\rho}(p)}{p} \quad (\rho=1, \dots, r_i; \sigma=2, \dots, s_i),$$

and furthermore

$$y_i(x) = \sum_{\rho=1}^{r_i} \sum_{\sigma=2}^{s_i} y_{i\rho\sigma}(x) \log \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!}$$

and

$$z_i(x) = \sum_{\rho=1}^{r_i} \sum_{\sigma=2}^{s_i} y_{i\rho\sigma}(x) \log^2 \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!}.$$

These definitions correspond to (2.17), (2.18), and (2.19), and thus defined $y_i(x)$ and $z_i(x)$ coincide with $y_i(n)$ and $z_i(n)$ in the enunciation of the theorem. As in

the proof of Theorem A₁, it follows from I, Lemma 2.1 that, for each $i=1, \dots, k$, there is among the functions $y_{i\rho\sigma}(x)$ ($\rho=1, \dots, r_i; \sigma=2, \dots, s_i$) at least one which tends to infinity with x . Suppose that, for each $i=1, \dots, k$, there are, among the $r_i(s_i-1)$ pairs of integers ρ, σ ($\rho=1, \dots, r_i; \sigma=2, \dots, s_i$), t_i pairs, say $\rho_{i\tau}, \sigma_{i\tau}$ ($\tau=1, \dots, t_i$), such that the functions $y_{i\rho_{i\tau}\sigma_{i\tau}}(x)$ ($\tau=1, \dots, t_i$) tend to infinity with x , and that the remaining $r_i(s_i-1)-t_i$ functions are bounded. We put

$$\tilde{y}_i(x) = \sum_{\tau=1}^{t_i} y_{i\rho_{i\tau}\sigma_{i\tau}}(x) \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!}$$

and

$$\tilde{z}_i(x) = \sum_{\tau=1}^{t_i} y_{i\rho_{i\tau}\sigma_{i\tau}}(x) \log^2 \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!}.$$

These definitions correspond to (2.20) and (2.21), and the formulas of the same form as (2.22) and (2.23) hold, in which however $y_i(x)$, $\tilde{y}_i(x)$, $z_i(x)$, and $\tilde{z}_i(x)$ have modified meanings. These formulas will be denoted with (2.22') and (2.23'). In the sequel also, we shall denote with (2.26'), (2.27'), etc. the formulas of the same form as (2.26), (2.27), etc., in which the meanings of some notations are modified. We replace (2.24) by

$$u_{i\tau}(n, x) = \frac{\omega_{i\sigma_{i\tau}}\{q_{i\rho_{i\tau}}(n)\} - y_{i\rho_{i\tau}\sigma_{i\tau}}(x)}{\sqrt{y_{i\rho_{i\tau}\sigma_{i\tau}}(x)}} \quad (\tau=1, \dots, k),$$

and define the point $U^*(n, x)$ in the space R^T of $T=t_1+\dots+t_k$ dimensions as in the proof of Theorem A₁. We replace the inequalities (2.25) defining the set $E_{i,x}$ by

$$\alpha_i < \frac{\sum_{\tau=1}^{t_i} u_{i\tau} \sqrt{y_{i\rho_{i\tau}\sigma_{i\tau}}(x)} \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!}}{\sqrt{\tilde{z}_i(x)}} < \beta_i,$$

and define the product set E_x^* in the space R^T .

We define $A^*(x; E_x^*)$ as in the proof of Theorem A₁, and apply Lemma 2.3 to $A^*(x; E_x^*)$ with the following notational correspondence:

$$\left. \begin{aligned} l &= T \\ f_j^*(\xi) &= q_{i\rho_{i\tau}}(\xi) \\ \mathfrak{P}_j^* &= \Omega_{i\sigma_{i\tau}} \\ \nu_j^*(p) &= \nu_{i\rho_{i\tau}}(p) \\ \omega_j^*(n) &= \omega_{i\sigma_{i\tau}}(n) \\ y_j^*(x) &= y_{i\rho_{i\tau}\sigma_{i\tau}}(x) \\ u_j^*(n, x) &= u_{i\tau}(n, x) \end{aligned} \right\} \text{ with } j = t_1 + \dots + t_{i-1} + \tau, \tau = 1, \dots, t_i.$$

Thus we obtain (2.26'), from which we derive (2.27'), i. e. (2.30'), similarly as in the proof of Theorem A₁. The only distinction is that (2.28) and (2.29) are replaced by

$$\begin{aligned} \sum_{\tau=1}^{t_i} u_{i\tau}(n, x) \sqrt{y_{i\rho_{i\tau}\sigma_{i\tau}}(x) \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!}} \\ = \sum_{\tau=1}^{t_i} \omega_{i\sigma_{i\tau}} \{q_{i\rho_{i\tau}}(n)\} \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!} - \tilde{y}_i(x) \end{aligned}$$

and

$$\tilde{y}_i(x) + \alpha_i \sqrt{\tilde{z}_i(x)} < \sum_{\tau=1}^{t_i} \omega_{i\sigma_{i\tau}} \{q_{i\rho_{i\tau}}(n)\} \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!} < \tilde{y}_i(x) + \beta_i \sqrt{\tilde{z}_i(x)}$$

respectively.

2. The inequalities (2.31) defining $\tilde{A}(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ is replaced by

$$\tilde{y}_i(x) + \alpha_i \sqrt{\tilde{z}_i(x)} < \log \tau_i \{f_i(n)\} < \tilde{y}_i(x) + \beta_i \sqrt{\tilde{z}_i(x)}.$$

We can prove that

$$\sum_{x_0 \leq n \leq x} \omega_{i\sigma} \{q_{i\rho}(n)\} = O(x) \text{ when } (\rho, \sigma) \neq (\rho_{i\tau}, \sigma_{i\tau}) \ (\tau=1, \dots, t_i)$$

in a similar way as we proved (2.32). Hence

$$(3.8) \quad \sum_{x_0 \leq n \leq x} \left| \sum_{\rho=1}^{r_i} \sum_{\sigma=2}^{s_i} \omega_{i\sigma} \{q_{i\rho}(n)\} \log \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!} - \sum_{\tau=1}^{t_i} \omega_{i\sigma_{i\tau}} \{q_{i\rho_{i\tau}}(n)\} \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!} \right| = O(x).$$

Here we use Lemma 3.1 in place of Lemma 2.2, and obtain

$$\sum_{x_0 \leq n \leq x} \left| \log \tau_{i\sigma} \{f_i(n)\} - \sum_{\rho=1}^{r_i} \omega_{i\sigma} \{q_{i\rho}(n)\} \log \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!} \right| = O(x) \quad (\sigma=2, \dots, s_i),$$

which implies

$$(3.9) \quad \sum_{x_0 \leq n \leq x} \left| \sum_{\sigma=2}^{s_i} \log \tau_{i\sigma} \{f_i(n)\} - \sum_{\rho=1}^{r_i} \sum_{\sigma=2}^{s_i} \omega_{i\sigma} \{q_{i\rho}(n)\} \log \frac{(a_{i\rho} + \sigma - 1)!}{a_{i\rho}! (\sigma - 1)!} \right| = O(x).$$

Also, by (3.7),

$$\log \tau_i \{f_i(n)\} = \sum_{\sigma=2}^{s_i} \log \tau_{i\sigma} \{f_i(n)\}.$$

Hence, by (3.8) and (3.9),

$$\sum_{x_0 \leq n \leq x} \left| \log \tau_i \{f_i(n)\} - \sum_{\tau=1}^{t_i} \omega_{i\sigma_{i\tau}} \{q_{i\rho_{i\tau}}(n)\} \log \frac{(a_{i\rho_{i\tau}} + \sigma_{i\tau} - 1)!}{a_{i\rho_{i\tau}}! (\sigma_{i\tau} - 1)!} \right| = O(x).$$

This corresponds to (2.33), and we can obtain (2.34') similarly as in the proof of Theorem A₁.

3. The inequalities (2.35) defining $A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ is replaced by

$$y_i(n) + \alpha_i \sqrt{z_i(n)} < \log \tau_i \{f_i(n)\} < y_i(n) + \beta_i \sqrt{z_i(n)}.$$

We can obtain (2.36') similarly as in the proof of Theorem A₁.

4. As in the 4th step of the proof of Theorem A₁, we can reach Theorem A₂ by the same argument as in II, pp. 187–189, THE PROOF OF THEOREM A.

4. Some special cases

We shall first refer to the special case of Theorem A₁ when all the sets $\mathfrak{P}_1, \dots, \mathfrak{P}_k$ are equal to the set of all prime numbers.

THEOREM 1. *Let k be a positive integer. Let $\{f_1(\xi), \dots, f_k(\xi)\}$ be a set of polynomials satisfying the following conditions:*

(C₁) *Each of the polynomials $f_1(\xi), \dots, f_k(\xi)$ is of positive degree with integral coefficients and with positive leading coefficient;*

(C₂) *The polynomials $f_1(\xi), \dots, f_k(\xi)$ are relatively prime in pairs.*

Now let $\Omega(n)$ denote the number of prime factors of a positive integer n , multiple factors being counted multiply:

$$\Omega(n) = \sum_{p^a | n} 1.$$

Suppose that the canonical factorization of $f_i(\xi)$ is given by (1.1), and put

$$\begin{aligned} R_i^{(1)} &= a_{i1} + \dots + a_{ir_i}, \\ R_i^{(2)} &= a_{i1}^2 + \dots + a_{ir_i}^2. \end{aligned}$$

Let n_0 be an integer ≥ 3 such that $f_i(n) > 0$ ($i=1, \dots, k$) for integers $n \geq n_0$, and put, for $n \geq n_0$,

$$u_i(n) = \frac{\Omega\{f_i(n)\} - R_i^{(1)} \log \log n}{\sqrt{R_i^{(2)} \log \log n}}.$$

Let E be a Jordan-measurable set, bounded or unbounded, in the space R^k of k dimensions. Let x be a variable $> n_0$, and let $A(x; E)$ denote the number of integers n with $n_0 \leq n \leq x$ for which the points $(u_1(n), \dots, u_k(n))$ of the space R^k belong to the set E . Then we have

$$\lim_{x \rightarrow \infty} \frac{A(x; E)}{x} = (2\pi)^{-k/2} \int_E \exp\left(-\frac{1}{2} \sum_{i=1}^k u_i^2\right) du_1 \dots du_k.$$

We shall next refer to the special case of Theorem A₂ when $s_1 = \dots = s_k = 2$, and all the sets $\mathfrak{P}_{i1}, \mathfrak{P}_{i2}$ ($i=1, \dots, k$) are equal to the set of all prime numbers.

THEOREM 2. *Let $\tau(n)$ denote the number of positive divisors of a positive integer n :*

$$\tau(n) = \sum_{d|n} 1.$$

Then Theorem 1 remains true when we replace the definitions of $R_i^{(1)}, R_i^{(2)}$, and $u_i(n)$ by

$$\begin{aligned} R_i^{(1)} &= \sum_{\rho=1}^{r_i} \log(a_{i\rho} + 1), \\ R_i^{(2)} &= \sum_{\rho=1}^{r_i} \log^2(a_{i\rho} + 1), \end{aligned}$$

and

$$u_i(n) = \frac{\log \tau\{f_i(n)\} - R_i^{(1)} \log \log n}{\sqrt{R_i^{(2)} \log \log n}}$$

respectively.

We shall further refer to the case when somewhat generally $s_1 = \dots = s_k = s > 1$ and all the sets $\mathfrak{P}_{i\sigma} (i=1, \dots, k; \sigma=1, \dots, s)$ are equal to the set all prime numbers.

THEOREM 3. Let s be an integer > 1 , and let $\tau^{(s)}(n)$ denote the number of representations of a positive integer n as a product of s positive integers:

$$\tau^{(s)}(n) = \sum_{d_1 \dots d_s = n} 1.$$

Then Theorem 1 remains true when we replace the definitions of $R_i^{(1)}$, $R_i^{(2)}$, and $u_i(n)$ by

$$R_i^{(1)} = \sum_{\rho=1}^{r_i} \log \frac{(a_{i\rho} + s - 1)!}{a_{i\rho}! (s-1)!},$$

$$R_i^{(2)} = \sum_{\rho=1}^{r_i} \log^2 \frac{(a_{i\rho} + s - 1)!}{a_{i\rho}! (s-1)!},$$

and

$$u_i(n) = \frac{\log \tau^{(s)}\{f_i(n)\} - R_i^{(1)} \log \log n}{\sqrt{R_i^{(2)}} \log \log n}$$

respectively.

These three theorems correspond to I, Theorem A.

Strictly speaking, Theorem 1 is not a special case of Theorem A₁, but the former can easily be derived from the latter, and Theorems 2 and 3 are similarly related to Theorem A₂. In deriving Theorem 1 from Theorem A₁, we had better start rather from (2.34) than from Theorem A₁ itself. By virtue of I, p. 2, Lemma 2.3,

$$y_i(n) = R_i^{(1)} \log \log n + O(1)$$

and

$$z_i(n) = R_i^{(2)} \log \log n + O(1).$$

Hence, by similar reasoning as in the 3rd step of the proof of Theorem A₁, we can obtain from (2.34) the case of Theorem 1 when the set E is an interval. Then we have only to follow the lines in II, pp. 187-189.

We now mention some special cases of Theorem 1.

THEOREM 4. Let $\alpha < \beta$, and $A(x) = A(x; \alpha, \beta)$ denote the number of integers n with $3 \leq n \leq x$ for which

$$\log \log n + \alpha \sqrt{\log \log n} < \Omega(n) < \log \log n + \beta \sqrt{\log \log n}.$$

Then

$$\lim_{x \rightarrow \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(-\frac{u^2}{2}\right) du.$$

THEOREM 5. Let $\alpha_i < \beta_i (i=1, \dots, k)$, and let $A(x) = A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ denote the number of integers n with $3 \leq n \leq x$ for which

$$\log \log n + \alpha_i \sqrt{\log \log n} < \Omega(n+i-1) < \log \log n + \beta_i \sqrt{\log \log n} \quad (i=1, \dots, k)$$

simultaneously. Then

$$\lim_{x \rightarrow \infty} \frac{A(x)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u^2}{2}\right) du.$$

THEOREM 6. Let $\alpha_i < \beta_i (i=1, \dots, k)$, and let $A(x) = A(x; \alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ denote the number of integers n with $3 \leq n \leq x$ for which

$$\log \log n + \alpha_i \sqrt{\log \log n} < \Omega(n^2 + i) < \log \log n + \beta_i \sqrt{\log \log n} \quad (i=1, \dots, k)$$

simultaneously. Then

$$\lim_{x \rightarrow \infty} \frac{A(x)}{x} = (2\pi)^{-k/2} \prod_{i=1}^k \int_{\alpha_i}^{\beta_i} \exp\left(-\frac{u^2}{2}\right) du.$$

Let $P(a_1, \dots, a_k)$ be an arbitrary permutation of the numbers $1, \dots, k$. Then we have

THEOREM 7. Let $B(x) = B(x; P)$ denote the number of positive integers $n \leq x$ for which

$$\Omega(n + a_1 - 1) < \Omega(n + a_2 - 1) < \dots < \Omega(n + a_k - 1).$$

Then

$$\lim_{x \rightarrow \infty} \frac{B(x)}{x} = \frac{1}{k!}.$$

THEOREM 8. Let $B(x) = B(x; P)$ denote the number of positive integers $n \leq x$ for which

$$\Omega(n^2 + a_1) < \Omega(n^2 + a_2) < \dots < \Omega(n^2 + a_k).$$

Then

$$\lim_{x \rightarrow \infty} \frac{B(x)}{x} = \frac{1}{k!}.$$

As special cases of Theorem 2, we have

THEOREM 9. Theorem 4 remains true when we replace $\Omega(n)$ by $\log \tau(n)/\log 2$.

THEOREM 10. Theorem 5 remains true when we replace $\Omega(n+i-1)$ by $\log \tau(n+i-1)/\log 2$.

THEOREM 11. Theorem 6 remains true when we replace $\Omega(n^2+i)$ by $\log \tau(n^2+i)/\log 2$.

THEOREM 12. Theorem 7 remains true when we replace $\Omega(n+a_i-1)$ by $\tau(n+a_i-1)$.

THEOREM 13. Theorem 8 remains true when we replace $\Omega(n^2+a_i)$ by $\tau(n^2+a_i)$.

Let s be an integer > 1 . Then, as special cases of Theorem 3, we have

THEOREM 14. Theorem 4 remains true when we replace $\Omega(n)$ by $\log \tau^{(s)}(n)/\log s$.

THEOREM 15. Theorem 5 remains true when we replace $\Omega(n+i-1)$ by $\log \tau^{(s)}(n+i-1)/\log s$.

THEOREM 16. Theorem 6 remains true when we replace $\Omega(n^2+i)$ by $\log \tau^{(s)}(n^2+i)/\log s$.

THEOREM 17. Theorem 7 remains true when we replace $\Omega(n+a_i-1)$ by $\tau^{(s)}(n+a_i-1)$.

THEOREM 18. Theorem 8 remains true when we replace $\Omega(n^2+a_i)$ by $\tau^{(s)}(n^2+a_i)$.

Here we briefly mention the fact that Theorems 4 and 9 can easily be derived from I, p. 17, Theorem 1: Let $\alpha < \beta$, and let $A(x) = A(x; \alpha, \beta)$ denote the number of integers n with $3 \leq n \leq x$ for which

$$\log \log n + \alpha \sqrt{\log \log n} < \omega(n) < \log \log n + \beta \sqrt{\log \log n}.$$

Then

$$\lim_{x \rightarrow \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(-\frac{u^2}{2}\right) du.$$

We can easily prove that

$$(4.1) \quad \sum_{n \leq x} \{\Omega(n) - \omega(n)\} = O(x).$$

and

$$(4.2) \quad 2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)}$$

as is shown in Hardy-Wright [2], 22.11 and 22.13. Then it is easy to derive Theorems 4 and 9 from I, Theorems 1 by the aid of (4.1) and (4.2). This derivation can be performed by a similar method as in the 2nd step of THE PROOF OF THEOREM A₁, and, in fact, this may be said to be a prototype of THE PROOF OF THEOREM A₁ and THE PROOF OF THEOREM A₂. By (4.1) and (4.2), we have

$$(4.3) \quad \sum_{n \leq x} \left\{ \frac{\log \tau(n)}{\log 2} - \omega(n) \right\} = O(x),$$

and (4.1) and (4.3) may also be considered as prototypes of Lemmas 2.2 and 3.1 respectively.

Theorems 9, 10, and 12 overlap with the results obtained in Erdős [1], Kac [3], [4], and LeVeque [5] concerning $\tau(n)$.

We turn to the case when $f_1(\xi) = \cdots = f_k(\xi) = \xi$ of Theorems A₁ and A₂. As a special case of Theorem A₁, we obtain

THEOREM 19. Let k be a positive integer. Let $\{\mathfrak{P}_1, \dots, \mathfrak{P}_k\}$ be a family of sets of prime numbers satisfying the following conditions:

- (C₁) The sets $\mathfrak{P}_1, \dots, \mathfrak{P}_k$ are mutually disjoint;
- (C₂) The series

$$\sum_{p \in \mathfrak{P}_i} \frac{1}{p}$$

are all divergent.

Now let $\Omega_i(n)$ denote the number of prime factors of a positive integer n which belong to the set $\mathfrak{P}_i$, multiple factors being counted multiply:

$$\Omega_i(n) = \sum_{p^a | n, p \in \mathfrak{P}_i} 1.$$

Put also

$$y_i(n) = \sum_{p \leq n, p \in \mathfrak{P}_i} \frac{1}{p},$$

and let n_0 be a positive integer such that $y_i(n) > 0$ ($i=1, \dots, k$) for $n \geq n_0$. Put, for $n \geq n_0$,

$$u_i(n) = \frac{\Omega_i(n) - y_i(n)}{\sqrt{y_i(n)}}.$$

Let E be a Jordan-measurable set, bounded or unbounded, in the space R^k of k dimensions. Let x be a variable $> n_0$, and let $A(x; E)$ denote the number of integers n with $n_0 \leq n \leq x$ for which the points $(u_1(n), \dots, u_k(n))$ belong to the set E . Then we have

$$\lim_{x \rightarrow \infty} \frac{A(x; E)}{x} = (2\pi)^{-k/2} \int_E \exp\left(-\frac{1}{2} \sum_{i=1}^k u_i^2\right) du_1 \cdots du_k.$$

As a special case of Theorem A₂, we obtain

THEOREM 20. Let $\{\mathfrak{P}_1, \dots, \mathfrak{P}_k\}$ have the same meaning as in Theorem 19 satisfying the same conditions (C₁) and (C₂). Let $\tau_i(n)$ denote the number of positive divisors of n which are composed only of primes belonging to the set $\mathfrak{P}_i$:

$$\tau_i(n) = \sum_{\substack{d|n \\ p|d \Rightarrow p \in \mathfrak{P}_i}} 1.$$

Then Theorem 19 remains true when we replace the definition of $u_i(n)$ by

$$u_i(n) = \frac{\log \tau_i(n) / \log 2 - y_i(n)}{\sqrt{y_i(n)}}.$$

These two theorems correspond to II, Theorem A, which is concerned with the number of distinct prime factors of integers. We can also prove theorems which correspond to II, pp. 189–190, Theorems 1–4, and to III, pp. 126–127, Theorems 1–6.

We next mention the special case of Theorem A₂ when $k=1$ and $f_1(\xi)=\xi$.

THEOREM 21. Let s be an integer > 1 . Let $\{\mathfrak{P}_1, \dots, \mathfrak{P}_s\}$ be a family of sets of prime numbers satisfying the following conditions:

(C₁) The union $\mathfrak{P}_1 \cup \dots \cup \mathfrak{P}_s$ contains all prime numbers;

(C₂) If we denote by Ω the set of primes which belong to at least two of the sets $\mathfrak{P}_1, \dots, \mathfrak{P}_s$, then the series

$$\sum_{p \in \Omega} \frac{1}{p}$$

is divergent.

Now let $\tau^*(n)$ denote the number of representations of a positive integer n as the product $n = d_1 \cdots d_s$ such that d_σ is, for each $\sigma=1, \dots, s$, a positive integer composed only of primes belonging to the set $\mathfrak{P}_\sigma$:

$$\tau^*(n) = \sum_{\substack{d_1 \cdots d_s = n \\ p|d_\sigma \Rightarrow p \in \mathfrak{P}_\sigma (\sigma=1, \dots, s)}} 1.$$

Let Ω_σ denote, for each $\sigma=2, \dots, s$, the set of primes which belong to just σ

sets among the s sets $\mathfrak{P}_1, \dots, \mathfrak{P}_s$,²¹⁾ and put, for $n > 0$,

$$y(n) = \sum_{\sigma=2}^s \log \sigma \cdot \sum_{p \leq n, p \in \mathfrak{Q}_\sigma} \frac{1}{p}$$

and

$$z(n) = \sum_{\sigma=1}^s \log^2 \sigma \cdot \sum_{p \leq n, p \in \mathfrak{Q}_\sigma} \frac{1}{p}.$$

Let $\alpha < \beta$, and let $A(x) = A(x; \alpha, \beta)$ denote the number of integers $n \leq x$ for which

$$y(n) + \alpha \sqrt{z(n)} < \log \tau^*(n) < y(n) + \beta \sqrt{z(n)}.$$

Then

$$\lim_{x \rightarrow \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(-\frac{u^2}{2}\right) du.$$

For example we have

THEOREM 22. *Let all the prime numbers be numbered in the order of their magnitudes: $p_1=2, p_2=3, p_3=5, \dots$. Let $\tau^*(n)$ denote the number of representations of a positive integer n as the product $n=d_1 d_2 d_3$ such that d_σ is, for each $\sigma=1, 2, 3$, composed only of primes p_j with the subscript j divisible by σ .²²⁾ Let $\alpha < \beta$, and let $A(x) = A(x; \alpha, \beta)$ denote the number of integers n with $3 \leq n \leq x$ for which*

$$\begin{aligned} & \left(\frac{1}{2} \log 2 + \frac{1}{6} \log 3 \right) \log \log n + \alpha \sqrt{\left(\frac{1}{2} \log^2 2 + \frac{1}{6} \log^2 3 \right) \log \log n} < \tau^*(n) \\ & < \left(\frac{1}{2} \log 2 + \frac{1}{6} \log 3 \right) \log \log n + \beta \sqrt{\left(\frac{1}{2} \log^2 2 + \frac{1}{6} \log^2 3 \right) \log \log n}. \end{aligned}$$

Then

$$\lim_{x \rightarrow \infty} \frac{A(x)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(-\frac{u^2}{2}\right) du.$$

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References

- [1] P. Erdős, On a problem of Chowla and some related problems, Proc. Cambridge Philos. Soc., 32 (1932), pp. 530-540.
- [2] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, 3rd ed., Oxford, 1954.
- [3] M. Kac, Note on the distribution of values of the arithmetic function $d(m)$, Bull. Amer. Math. Soc., 47 (1941), pp. 815-817.
- [4] M. Kac, Probability methods in some problems of analysis and number theory, Bull. Amer. Math. Soc., 55 (1949), pp. 641-655.
- [5] W. J. LeVeque, On the size of certain number-theoretic functions, Trans. Amer. Math. Soc., 66 (1949), pp. 440-463.
- [6] T. Nagell, Introduction to number theory, New York, 1951.

21) $\mathfrak{Q} = \mathfrak{Q}_2 \cup \dots \cup \mathfrak{Q}_s$.

22) We impose no restriction on the prime factors of d_1 .

- [7] M. Tanaka, On the number of prime factors of integers, Jap. J. Math., 25 (1955), pp. 1-20. This paper will be referred to as I in this paper.
- [8] M. Tanaka, On the number of prime factors of integers II, J. Math. Soc. Japan, 9 (1957), pp. 171-191. This paper will be referred to as II in this paper.
- [9] M. Tanaka, On the number of prime factors of integers III, Jap. J. Math., 27 (1957), pp. 103-127. This paper will be referred to as III in this paper.