On triviality of the mod p Hopf invariant

By Nobuo SHIMADA and Tsuneyo YAMANOSHITA

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Introduction. Recently J. F. Adams [2, 3] gave an axiomatic treatment of stable secondary cohomology operations, and, applying this, proved the non-existence of elements of Hopf invariant one in $\pi_{2n-1}(S^n)$ ($n \geq 16$). Owing to that work, the secondary cohomology operations turn out to be a useful tool in the algebraic topology, along with the primary operations, provided with satisfactory practical machinery, which will thus serve to solve not a few interesting problems.

In the present paper, following the general scheme given by Adams, we shall extend his result on the mod 2 Hopf invariant to the mod p case, where p is an odd prime. It will be shown that the mod p Hopf invariant

$$H_p: \pi_{m+n-1}(S^m) \rightarrow Z_p \quad (n=2p^k(p-1))$$

is trivial for $k \geq 1$ (Theorem 5.3); the special case $k=1$ was already obtained by Toda [17]. Our result is deduced from the following theorem: For $k \geq 1$ the reduced power operation $\mathfrak{P}^{p^k}$ can be expressed in the form $\sum a_i \Phi_i$, where Φ_i are the basic secondary operations (§3) and a_i are suitable elements of the Steenrod algebra A over Z_p (Theorem 5.1 and 5.2). In choosing the basic secondary operations as generating system for secondary operations, we make use of Toda's result on exact sequences in the Steenrod algebra ([17] and Mukohda [9]) which may also be obtained by a computation of the cohomology of the Steenrod algebra [1]. Our problem, then, reduces to calculating the basic operations in the complex projective space P of infinitely-many dimensions, and this will be achieved in two ways, the one making use of functional operations (§6) and the other making use of a formula for the composite operation $\Phi_{-1, k} c(\mathfrak{P}^{p^k(p-1)})$ (§7).

The contents of this paper were previously announced in [14] and [18]. The same problem of the mod p Hopf invariant has been solved independently by A. Liulevicius [7].

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§1. Stable secondary cohomology operations and primary functional operations.

Let p be an odd prime. We shall denote by A the Steenrod algebra [6] over the prime field Z_p . As is well known, the algebra A is generated by the powers $\mathfrak{P}^i$ and the operator Δ . By Δ we mean the Bockstein operator ([6]) associated with the exact sequence of coefficients

$$0 \rightarrow Z_p \xrightarrow{(-1)^i p} Z_{p^2} \rightarrow Z_p \rightarrow 0$$

for i -dimensional cohomology classes. A has an additive base over Z_p consisting of the following "admissible" monomials [5]:

$$(1.1) \quad \mathfrak{P}(\Delta^{\varepsilon_0}, i_1, \Delta^{\varepsilon_1}, \dots, i_r, \Delta^{\varepsilon_r}) \text{ for } \varepsilon_j=0 \text{ or } 1, i_j \geq 0 \\ \text{and } i_j \geq pi_{j+1} + \varepsilon_j; j=1, \dots, r-1,$$

where $\mathfrak{P}(\Delta^{\varepsilon_0}, i_1, \Delta^{\varepsilon_1}, \dots, i_r, \Delta^{\varepsilon_r})$ stands for $\Delta^{\varepsilon_0} \mathfrak{P}^{i_1} \dots \mathfrak{P}^{i_r} \Delta^{\varepsilon_r}$. (ε_j takes values 0 or 1 and $\Delta^0 = id$. We omit Δ^0 in $\mathfrak{P}(\Delta^{\varepsilon_0}, i_1, \Delta^{\varepsilon_1}, \dots, i_r, \Delta^{\varepsilon_r})$ so that, for instance $\mathfrak{P}(r, s) = \mathfrak{P}^r \mathfrak{P}^s$. This abbreviation will be used throughout the paper.) Between the generators $\mathfrak{P}^i$ and Δ the following relations hold ([5] and [6]):

$$(1.2) \quad \mathfrak{P}(r, s) = \sum_{i \geq 0} (-1)^{r+i} \binom{(s-i)(p-1)-1}{r-ip} \mathfrak{P}(r+s-i, i) \\ \text{for } r < ps, \\ \mathfrak{P}(r, \Delta, s) = \sum_{i \geq 0} (-1)^{r+i} \binom{(s-i)(p-1)}{r-ip} \mathfrak{P}(\Delta, r+s-i, i) \\ + \sum_{i \geq 0} (-1)^{r+i-1} \binom{(s-i)(p-1)-1}{r-ip-1} \mathfrak{P}(r+s-i, \Delta, i) \\ \text{for } r \leq ps, \\ \Delta \Delta = 0.$$

In particular [5], we have

$$(1.3) \quad \mathfrak{P}(r, \Delta, s) = \binom{r+s-1}{r} \mathfrak{P}(\Delta, r+s) + \binom{r+s-1}{s} \mathfrak{P}(r+s, \Delta) \\ \text{for } r < p.$$

We shall recall the axioms and the fundamental properties of the stable secondary cohomology operations [2]. By an A -module we mean a graded left module over the graded algebra A . Let us write $H^*(X)$ for the singular cohomology ring $\sum_{q \geq 0} H^q(X, Z_p)$ of a space X and write $H^+(X)$ for $\sum_{q > 0} H^q(X, Z_p)$; $H^*(X)$ and $H^+(X)$ are A -modules.

Let C_0, C_1 be free A -modules of locally finite type such that $(C_i)_q = 0$ if $q < i$ ($i=0, 1$). Let (d, z) be a pair of an A -map $d: C_1 \rightarrow C_0$ of degree zero and a d -cycle z (i.e. a homogeneous element of $\text{Ker } d$). A stable secondary operation Φ associated with (d, z) , by definition, satisfies the following axioms.

Axiom (1). $\Phi(\varepsilon)$ is defined for each A -map $\varepsilon: C_0 \rightarrow H^+(X)$ of degree $m \geq 1$ such that $\varepsilon d = 0$.

For the next axiom, set $\deg z = n+1$, let $f: C_1 \rightarrow H^+(X)$ run over the A -maps of degree $m-1$, and let $Q^{m+n}(d, z; X)$ be the set of elements of the form $f(z)$.

Axiom (2). $\Phi(\varepsilon) \in H^{m+n}(X)/Q^{m+n}(d, z; X)$.

Axiom (3). $g^* \Phi(\varepsilon) = \Phi(g^* \varepsilon)$ for a map $g: Y \rightarrow X$.

Let (X, Y) be a pair of a space X and a subspace Y of X ; let $i: Y \rightarrow X$ be the injection, and let $\varepsilon: C_0 \rightarrow H^+(X)$ be an A -map of degree $m \geq 1$ such that $\varepsilon d = 0$ and $i^* \varepsilon = 0$. Then we can form the following diagram.

$$\begin{array}{ccccccc} H^+(Y) & \xleftarrow{i^*} & H^+(X) & \xleftarrow{j^*} & H^+(X, Y) & \xleftarrow{\delta} & H^+(Y) & \xleftarrow{i^*} & H^+(X) \\ & & \uparrow & & \uparrow \eta & & \uparrow \zeta & & \\ & & \xleftarrow{\varepsilon} & C_0 & \xleftarrow{(-1)^m d} & C_1 & & & \end{array}$$

Axiom (4). $i^*\Phi(\varepsilon) = \{\zeta z\} \bmod i^*Q^{m+n}(d, z; X)$.

For the next axiom, let $\sigma: H^+(X) \rightarrow H^+(SX)$ be the suspension isomorphism.

Axiom (5). $\sigma\Phi(\varepsilon) = \Phi(\sigma\varepsilon)$.

In particular, if C_0 is free on one given generator e_0 and $\varepsilon e_0 = u$, then we write $\Phi(u)$ for $\Phi(\varepsilon)$, and we say that $\Phi(u)$ is a stable secondary operation of one variable. A stable secondary operation of one variable is concretely defined from a relation in the Steenrod algebra as follows: Let $\sum_i a_i b_i = 0$ ($a_i, b_i \in A$) be such a relation, let C_1 be free on symbolic generators $[b_i]$, and let $d: C_1 \rightarrow C_0$ be an A -map of degree zero such that $d[b_i] = b_i e_0$. Then we have a d -cycle $z = \sum_i a_i [b_i]$, and a stable secondary operation Φ associated with (d, z) is called also associated with the relation $\sum_i a_i b_i = 0$. In this case, the domain of definition of Φ is $\bigcap_i \text{Ker } b_i \subset H^m(X)$ and the indeterminacy $Q^{m+n}(d, z; X) = \sum_i \text{Im } a_i$.

The following theorem of Adams [2] is basic for the present paper.

THEOREM 1.1. (1) *Given (d, z) as above, there is at least one stable secondary cohomology operation Φ associated with (d, z) .*

(2) *If Φ, Φ' are two operations associated with the same (d, z) then there is an element e in $(C_0/dC_1)_n$ such that*

$$\Phi(\varepsilon) - \Phi'(\varepsilon) = \{\varepsilon e\}.$$

(3) *Suppose given d , element z_i in $\text{Ker } d$, and operations Φ_i associated with the pairs (d, z_i) . Suppose $z = \sum_i a_i z_i$ ($a_i \in A$). Then there is an operation Φ associated with (d, z) such that*

$$\sum_i a_i \Phi_i(\varepsilon) = \{\Phi(\varepsilon)\} \bmod \sum_i a_i Q^{m+n_i}(d, z_i; X).$$

(4) *Suppose given a diagram*

$$\begin{array}{ccc} & m_1 & \\ C_1 & \longrightarrow & C_1' \\ d \downarrow & & \downarrow d' \\ C_0 & \longrightarrow & C_0' \\ & m_0 & \end{array}$$

in which d, d' are as above, and m_0, m_1 are A -maps of degree zero. Let Φ be an operation associated with a pair (d, z) . Then there is an operation Φ' associated with (d', mz) such that

$$\Phi(\varepsilon' m_0) = \{\Phi'(\varepsilon')\}$$

for each $\varepsilon': C_0' \rightarrow H^+(X, Z_p)$ of the sort considered above.

As is well known, the functional primary cohomology operations are closely related to the secondary operations [12, 15]. For later use we shall state the axioms and fundamental properties of the stable functional primary operation.

With the above notations, let C_0, C_1 be free A -modules of locally finite type such that $(C_i)_q = 0$ if $q < i$ ($i = 0, 1$). Let (d, v) be a pair of an A -map $d: C_1 \rightarrow C_0$ of degree zero and a homogeneous element v of C_1 . We call φ a stable functional primary cohomology operation associated with (d, v) , if it satisfies the following axioms.

F1. $\varphi(f, \varepsilon)$ is defined for each pair of a map $f: Y \rightarrow X$ and an A -map $\varepsilon: C_0 \rightarrow H^+(X)$ of degree $m \geq 1$ such that $f^*\varepsilon = 0$ and $\varepsilon d = 0$.

In particular, if $C_0 = A$ and C_1 is free on one generator $[a]$ and $d[a] = a \in A$, then we write $u = \varepsilon(1)$ and we have $au = 0$ and $f^*u = 0$. In this case we write $a_f(u)$ for $\varphi(f, \varepsilon)$ as usual, where φ is associated with $(d, [a])$.

For the next axiom, set $\deg(v) = v$, let $\lambda: C_0 \rightarrow H^*(Y)$, $\mu: C_1 \rightarrow H^*(X)$ run over the A -maps of degree $m-1$, and let $L^{m+v-1}(d, v; f)$ be the set of elements of the form $\lambda dv + f^*\mu v$.

F2. $\varphi(f, \varepsilon) \in H^{m+v-1}(Y)/L^{m+v-1}(d, v; f)$.

For the next axiom, consider the following homotopy commutative diagram.

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \bar{g} \uparrow & f' & \uparrow g \\ Y' & \xrightarrow{\quad} & X' \end{array}$$

F3. $\varphi(f', g^*\varepsilon) = \bar{g}^*\varphi(f, \varepsilon)$.

For the next axiom, let (X, Y) be a pair, and let $\varepsilon: C_0 \rightarrow H^+(Y)$ be a map of degree $m \geq 1$ such that $\varepsilon d = 0$ and $i^*\varepsilon = 0$. We can form the same diagram as in the above Axiom (4).

F4. $\varphi(i, \varepsilon) = \{\zeta(v)\} \bmod L^{m+v-1}(d, v; i)$.

F5. $\sigma\varphi(f, \varepsilon) = \varphi(Sf, \sigma\varepsilon)$.

THEOREM 1.2. (1) *Given a pair (d, v) as above, there is a uniquely determined stable functional operation φ associated with it (in the sense of the axioms above).*

(2) *Suppose given d , elements v_i in C_1 and operations φ_i associated with the pairs (d, v_i) respectively. Suppose $v = \sum_i a_i v_i (a_i \in A)$. Then, we have*

$$\{\varphi(f, \varepsilon)\} = \{\sum_i a_i \varphi_i(f, \varepsilon)\} \bmod \sum_i a_i L^{m+v_i-1}(d, v_i; f)$$

for the operation φ associated with (d, v) .

(3) *Suppose given a diagram*

$$\begin{array}{ccc} C_1 & \xrightarrow{m_1} & C_1' \\ d \downarrow & m_0 & \downarrow d' \\ C_0 & \xrightarrow{\quad} & C_0' \end{array}$$

in which d, d' are as above, and m_0, m_1 are A -maps of degree zero. Let φ be the operation associated with a pair (d, v) , and let φ' be the operation associated with $(d', m_1 v)$. Then we have

$$\varphi(f, \varepsilon' m_0) = \{\varphi'(f, \varepsilon')\}$$

where $\varepsilon': C_0' \rightarrow H^+(X)$ is of the sort considered above.

This theorem is easily deduced from the axioms above.

One may generalize the two formulas of Peterson and Stein [12] as follows.

THEOREM 1.3. (1) *Suppose given d as above, element $z = \sum_i a_i e_i (a_i \in A)$ in $\text{Ker } d$, where the elements e_i form an A -base of C_1 . Let Φ be any stable secondary cohomology operation associated with (d, z) , and let φ_i be the functional operations associated with (d, e_i) . Then*

$$f^*\Phi(\varepsilon) = \sum_i a_i \varphi_i(f, \varepsilon) \bmod \sum_i a_i L^{m+v_i-1}(d, e_i; f)$$

for each $\varepsilon: C_0 \rightarrow H^+(X)$ of degree $m \geq 1$ and each $f: Y \rightarrow X$ such that $\varepsilon d = 0$ and $f^* \varepsilon = 0$.

(2) Suppose given d as above, element z in $\text{Ker } d$. Let $\bar{C}$ be a free A -module generated by one generator $\bar{e}$, let $\bar{d}: \bar{C} \rightarrow C_1$ be an A -map of degree zero such that $\bar{d}(\bar{e}) = z$ and let $\bar{\varphi}$ be the functional operation associated with $(\bar{d}, \bar{e})$. Then there exists a secondary operation Φ associated with (d, z) such that

$$\Phi(f^* \varepsilon) = -\bar{\varphi}(f, \varepsilon d) \bmod L^{m+\nu-1}(\bar{d}, \bar{e}; f)$$

for each $\varepsilon: C_0 \rightarrow H^*(X)$ of degree $m \geq 1$ and each $f: Y \rightarrow X$ such that $f^* \varepsilon d = 0$.

The first formula of this theorem is nothing but a rewriting of the above Axiom (4) for secondary operations by using the axiom F4 and Theorem 1.2.

In order to prove the second formula, we use the universal examples. We begin with some preliminaries. Let C_0 be free on generators e_j of degree μ_j . Let $\mu = \text{Min } \mu_j$. Let B_0 be a generalized Eilenberg-MacLane space with fundamental classes y_j of degree $m + \mu_j$ (i.e. B_0 is a cartesian product of spaces $K(Z_p, m + \mu_j)$). We can define an A -map $\theta_0: C_0 \rightarrow H^+(B_0)$ of degree m by $\theta_0(e_j) = y_j$, then θ_0 is an isomorphism in the stable range (i.e. the range of degrees $< 2(m + \mu)$). Similarly define B_1 , θ_1 and $\bar{B}$, $\bar{\theta}$ corresponding to C_1 and $\bar{C}$ (m being fixed). We can define two maps $\psi_0: B_0 \rightarrow B_1$ and $\psi_1: B_1 \rightarrow \bar{B}$ such that

$$(1.4) \quad \psi_0^* \theta_1 = \theta_0 d, \quad \psi_1^* \bar{\theta} = \theta_1 \bar{d}$$

Let $(\Omega B_1, E_0, B_0)$ be the fibre space over base B_0 with fibre ΩB_1 induced by the map ψ_0 from the path-space fibering $(\Omega B_1, LB_1, B_1)$ introduced by Serre [13]. Let $(\Omega \bar{B}, E_1, B_1)$ be similarly defined by the map ψ_1 and the fibering $(\Omega \bar{B}, L\bar{B}, \bar{B})$. Let $\sigma_1: H^*(B_1) \rightarrow H^*(\Omega B_1)$ and $\bar{\sigma}: H^*(\bar{B}) \rightarrow H^*(\Omega \bar{B})$ be the suspension homomorphisms of degree -1 . We can form the following diagram

$$\begin{array}{ccccccc} H^+(\Omega B_1) & \xleftarrow{i_0^*} & H^+(E_0) & \xleftarrow{p_0^*} & H^+(B_0) & \xleftarrow{\tau_0} & H^+(\Omega B_1) \\ & & \uparrow \varepsilon_0 & & \uparrow \theta_0 & & \uparrow \zeta_0 \\ & & & & C_0 & \xleftarrow{(-1)^m d} & C_1 \\ & & & & & & \downarrow \theta_1 \\ & & & & & & H^+(B_1) \end{array}$$

$\begin{array}{c} \swarrow (-1)^m \sigma_1 \\ \searrow \end{array}$

where $i_0: \Omega B_1 \rightarrow E_0$ is the injection, $p_0: E_0 \rightarrow B_0$ is the projection and τ_0 is the transgression. We have a similar diagram corresponding to the fibre space $(\Omega \bar{B}, E_1, B_1)$. In particular, we may set

$$(1.5) \quad \zeta_0 = (-1)^m \sigma_1 \theta_1, \quad \zeta_1 = (-1)^m \bar{\sigma} \bar{\theta}.$$

Now take $z \in \text{Ker } d$ and $\bar{e} \in \bar{C}$ as in the theorem, then, from (1.4) and (1.5), we have

$$\begin{aligned} (1.6) \quad \zeta_0(z) &= \zeta_0 \bar{d}(\bar{e}) = (-1)^m \sigma_1 \theta_1 \bar{d}(\bar{e}) = (-1)^m \sigma_1 \psi_1^* \bar{\theta}(\bar{e}) \\ &= (-1)^m (\Omega \psi_1)^* \bar{\sigma} \bar{\theta}(\bar{e}) = (\Omega \psi_1)^* \zeta_1(\bar{e}). \end{aligned}$$

Let $g: X \rightarrow B_0$ be a map such that

$$(1.7) \quad g^*\theta_0 = \varepsilon.$$

Then we can form the following (homotopy commutative) diagram (cf. Nomura [11])

$$\begin{array}{ccccccc} & & Y & \xrightarrow{f} & X & & \\ & & \downarrow \bar{g} & & \downarrow g & & \\ \Omega B_1 & \xrightarrow{i_0} & E_0 & \xrightarrow{p_0} & B_0 & \xrightarrow{\psi_0} & B_1 \\ \parallel & \Omega^{-1}\psi_1 \downarrow & \downarrow \bar{h} & i_1 \downarrow & \downarrow h & p_1 \parallel & \psi_1 \downarrow \\ \Omega B_1 & \xrightarrow{\quad} & \Omega \bar{B} & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & B_1 \xrightarrow{\quad} \bar{B} \end{array}$$

where Ω^{-1} means the composition of the loop functor and of the inversion of loops, so that we may regard $(\Omega^{-1}\psi_1)^*$ as $-(\Omega\psi_1)^*$ in the stable range.

Let $\bar{\varphi}$ be the functional operation associated with $(\bar{d}, \bar{e})$. Then the axiom 4 implies

$$(1.8) \quad \bar{\varphi}(i_1, p_1^*\theta_1) = \{\zeta_1(\bar{e})\} \bmod L^{m+\nu-1}(\bar{d}, \bar{e}; i_1).$$

Making use of the axiom F3, (1.4), (1.6) and (1.8), we have

$$\begin{aligned} (1.9) \quad i_0^*\bar{\varphi}(p_0, \theta_0 d) &= i_0^*\bar{h}^*\bar{\varphi}(i_1, p_1^*\theta_1) \\ &= i_0^*\bar{h}^*\{\zeta_1(\bar{e})\} = \{(\Omega^{-1}\psi_1)^*\zeta_1(\bar{e})\} = -\{\zeta_0(z)\} \\ &\quad \bmod i_0^*L^{m+\nu-1}(\bar{d}, \bar{e}; p_0) \end{aligned}$$

where the subgroup $i_0^*L^{m+\nu-1}(\bar{d}, \bar{e}; p_0)$ is equal to $i_0^*Q^{m+\nu-1}(d, z; E_0)$. Therefore, from the Axiom (4), we can assert that there exists a secondary operation Φ associated with the pair (d, z) such that

$$(1.10) \quad \Phi(p_0^*\theta_0) = -\bar{\varphi}(p_0, \theta_0 d) \bmod L^{m+\nu-1}(\bar{d}, \bar{e}; p_0)$$

It follows immediately that

$$\begin{aligned} (1.11) \quad \Phi(f^*\varepsilon) &= \Phi(f^*g^*\theta_0) = \Phi(\bar{g}^*p_0^*\theta_0) = \bar{g}^*\Phi(p_0^*\theta_0) \\ &= -\bar{g}^*\bar{\varphi}(p_0, \theta_0 d) = -\bar{\varphi}(f, \varepsilon d) \bmod L^{m+\nu-1}(\bar{d}, \bar{e}; f). \end{aligned}$$

It follows from Theorem 1.1. (2) that

$$\Phi'(f^*\varepsilon) = \Phi(f^*\varepsilon) \bmod L^{m+\nu-1}(\bar{d}, \bar{e}; f)$$

for another operation Φ' associated with the pair (d, z) . Thus we have the required result.

For later use we mention of a special case of Th. 1.3, (2).

THEOREM 1.4. *Let $\sum_i a_i b_i = 0$ be a relation in A , where b_i are of positive degree. Let $f: Y \rightarrow X$ be a map. Let $u \in H^m(X)$ be a class such that $a_i b_i u = 0$ and $f^* b_i u = 0$ for any i . Then, for any stable secondary operation Φ associated with the relation $\sum_i a_i b_i = 0$, we have*

$$\Phi(f^*u) = -\sum_i a_i \cdot f(b_i u) \bmod \text{Im } f^* + \sum_i \text{Im } a_i,$$

where $a_i \cdot$ denotes the functional operation.

§2. The Steenrod algebra and the conjugation operation.

We begin with introducing the following left and right ideals of A . For $k \geq 1$, put

$$\begin{aligned} M(k) &= A\mathfrak{P}^{p^k} + \cdots + A\mathfrak{P}^1 + A\Delta, \\ M'(k) &= A\mathfrak{P}^{p^k} + \cdots + A\mathfrak{P}^1, \\ M_*(k) &= \mathfrak{P}^{p^k}A + \cdots + \mathfrak{P}^1A + \Delta A, \\ M_*'(k) &= \mathfrak{P}^{p^k}A + \cdots + \mathfrak{P}^1A, \end{aligned}$$

and $M(-1)=A\Delta$, $M_*(-1)=\Delta A$, $M'(-1)=M_*'(-1)=0$,
and for $k < -1$, put

$$M(k)=M'(k)=M_*(k)=M_*'(k)=0.$$

Furthermore, define a homomorphism $L_k(k \geq -1)$ of A into $A/M_*(k-1)$ as follows :

$$\begin{aligned} L_k(a) &= \mathfrak{P}^{p^k} a \mod M_*(k-1) \quad (k \geq 0) \\ L_{-1}(a) &= \Delta a \quad \text{for } a \in A. \end{aligned}$$

It is well-known that the kernel of L_{-1} is equal to ΔA .

We cite the following results of Toda ([17] and Mukohda [9]).

PROPOSITION 2.1. For $k \geq 1$, the kernel of L_k is given by

$$M_*(k-2) + \mathfrak{P}^{2p^{k-1}} A + (2\mathfrak{P}^{p^k+p^{k-1}} - \mathfrak{P}^{p^k}\mathfrak{P}^{p^{k-1}}) A + \mathfrak{P}^{p^k(p-1)} A.$$

PROPOSITION 2.2. The kernel of L_0 is given by

$$(2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta) A + \mathfrak{P}^{p-1} A.$$

Next, we have the following

LEMMA 2.3. For $k \geq 0$, $0 < j < p$,

$$\mathfrak{P}^{jp^k}\mathfrak{P}^{p^{k+1}l} \equiv \mathfrak{P}^{p^{k+1}l+jp^k} \mod M'(k-1) \cap M_*'(k-1).$$

Proof. We will proceed by induction on k . The assertion is trivial for $k=0$. Assume that the lemma is true for $k \leq n-1$.

Let us verify it for $k=n$. By Adem's relation, we have

$$\mathfrak{P}(jp^n, p^{n+1}l) = \mathfrak{P}(p^{n+1}l + jp^n) + \sum_{i=1}^{jp^n-1} *\mathfrak{P}(p^{n+1}l + jp^n - i, i).$$

On the other hand, by the hypothesis of induction it is easy to see that $\mathfrak{P}(p^{n+1}l + jp^n - i, i)$ is contained in not only $M'(n-1)$ but also $M_*'(n-1)$ for $1 \leq i \leq jp^{n-1}$. q.e.d.

From now on we shall consider the conjugation operation c [8] which is an anti-automorphism of the Steenrod algebra A with the following properties :

$$\begin{aligned} (2.1) \quad c(ab) &= (-1)^{rs} c(b)c(a), \text{ for } a \in A_r, b \in A_s, \\ c(\Delta) + \Delta &= 0, \sum_{i \geq 0} \mathfrak{P}^i c(\mathfrak{P}^{i-t}) = 0 \text{ for } t > 0, c^2 = \text{identity}. \end{aligned}$$

LEMMA 2.4. (1) For $k \geq 0$,

$$c(\mathfrak{P}^{p^k}) \equiv -\mathfrak{P}^{p^k} \mod M'(k-2) \cap M_*'(k-2),$$

(2) For $k \geq 0$ and $j < p$

$$c(\mathfrak{P}^{jp^k}) \equiv (-1)^j \mathfrak{P}^{jp^k} \mod M'(k-1) \cap M_*'(k-1).$$

Proof. We shall prove this lemma by induction on k . From (2.1) it is easily seen that $c(\mathfrak{P}^j) = (-1)^j \mathfrak{P}^j$ for $j < p$. Thus the lemma is true for $k=0$. Assume that the lemma is true for $k \leq r-1$. We shall prove it for $k=r$.

First, using Lemma 2.3, we can observe that

$$\begin{aligned} (2.2) \quad c(\mathfrak{P}(p^{k+1}l + jp^k)) &\equiv c(\mathfrak{P}(jp^r, p^{k+1}l)) \\ &\equiv c(\mathfrak{P}^{p^{k+1}l}) c(\mathfrak{P}^{jp^r}) \\ &\equiv (-1)^j c(\mathfrak{P}^{p^{k+1}l}) \mathfrak{P}^{jp^k} \mod M'(k-1), \end{aligned}$$

for $k \leq r-1$. From (2.1) and (2.2), we obtain

$$(2.3) \quad c(\mathfrak{P}^{p^r}) + \mathfrak{P}^{p^r} + \sum_{i=1}^{p-1} \mathfrak{P}^{i p^{r-1}} c(\mathfrak{P}^{(p-i)p^{r-1}}) \equiv 0 \pmod{M'(r-2)}.$$

On the other hand, from the Adem relation we have

$$(2.4) \quad \mathfrak{P}^{i p^{r-1}} \mathfrak{P}^{j p^{r-1}} \equiv \binom{i+j}{i} \mathfrak{P}^{(i+j)p^{r-1}} \pmod{M'(r-2)}$$

for $0 \leq i, j < p$.

From the assumption of induction, (2.3) and (2.4), it follows that

$$(2.5) \quad \begin{aligned} c(\mathfrak{P}^{p^r}) + \mathfrak{P}^{p^r} &\equiv \sum (-1)^i \mathfrak{P}^{i p^{r-1}} \mathfrak{P}^{(p-i)p^{r-1}} \\ &\equiv 0 \pmod{M'(r-2)}. \end{aligned}$$

On the other hand, from the assumption of induction it follows immediately that $c(M'(k)) = M'_*(k)$ for $k \leq r-1$. Thus the part (1) of the lemma has been proved.

Next, we shall prove the part (2) of the lemma by induction on j . Assume that the assertion is verified for $j < l$. Then from (2.1) we have

$$\begin{aligned} -c(\mathfrak{P}^{l p^k}) &\equiv \sum_{i=1}^{l-1} \mathfrak{P}^{i p^k} c(\mathfrak{P}^{(l-i)p^k}) + \mathfrak{P}^{l p^k} \pmod{M'(k-1)} \\ &\equiv (-1)^l \sum_{i=1}^{l-1} (-1)^i \mathfrak{P}^{i p^k} \mathfrak{P}^{(l-i)p^k} + \mathfrak{P}^{l p^k} \pmod{M'(k-1)} \\ &\equiv (-1)^l \left\{ \sum_{i=1}^{l-1} (-1)^i \binom{l}{i} \right\} \mathfrak{P}^{l p^k} + \mathfrak{P}^{l p^k} \pmod{M'(k-1)} \\ &\equiv -(-1)^l \mathfrak{P}^{l p^k} \pmod{M'(k-1)}. \end{aligned}$$

By taking the conjugates on both sides, we obtain the part (2) of the lemma. q.e.d.

Furthermore we have

LEMMA 2.5. For $k \geq 1$,

$$c(\mathfrak{P}^{l k + p^{k-1}}) \equiv \mathfrak{P}^{p^k} \mathfrak{P}^{p^{k-1}} \pmod{M'(k-2)}.$$

Proof. Similarly as in Proof of Lemma 2.4, we have

$$\begin{aligned} c(\mathfrak{P}^{p^k + p^{k-1}}) + \mathfrak{P}^{p^{k-1}} c(\mathfrak{P}^{p^k}) + \sum_{j=2}^{p-1} \mathfrak{P}^{(p-j+1)p^{k-1}} c(\mathfrak{P}^{j p^{k-1}}) \\ + \mathfrak{P}^{p^k} c(\mathfrak{P}^{p^{k-1}}) + \mathfrak{P}^{p^k + p^{k-1}} \equiv 0 \pmod{M'(k-2)}. \end{aligned}$$

The second term is equal to $-\mathfrak{P}^{p^k + p^{k-1}} \pmod{M'(k-2)}$ by Lemmas 2.4 and 2.3. The third term is contained in $M'(k-2)$ by Lemma 2.4 and (2.4). As to the fourth term, $\mathfrak{P}^{p^k} c(\mathfrak{P}^{p^{k-1}}) \equiv -\mathfrak{P}^{p^k} \mathfrak{P}^{p^{k-1}} \pmod{M'(k-2)}$. The proof is thus completed.

Remark 2.6. In case $p=2$, set $M(k) = ASq^{2^k} + \dots + ASq^1$, $M_*(k) = Sq^{2^k} A + \dots + Sq^1 A$ and $M(k) = M_*(k) = 0$ for $k < 0$. Then we have similar relations as in the above lemmas:

$$\begin{aligned} \text{For } k \geq 0, \quad c(Sq^{2^k}) &\equiv Sq^{2^k} \pmod{M(k-2) \cap M_*(k-2)}, \\ \text{and for } k \geq 1, \quad c(Sq^{2^k + 2^{k-1}}) &\equiv Sq^{2^k} Sq^{2^{k-1}} \pmod{M(k-2)}. \end{aligned}$$

§ 3. The stable secondary cohomology operations defined for

$$\bigcap_{i=0}^k \text{Ker } \mathfrak{P}^{p^i} \cap \text{Ker } \Delta.$$

Let $C_i^k (k \geq 0)$ be an A -free module with the symbolic base consisting of

$[c\mathfrak{P}^{p^i}]$ ($i=0, \dots, k$) and $[c\Delta]$. Let C_1^{-1} be an A -free module with the single base $[c\Delta]$. Then we obtain the following inclusion relation

$$C_1^k \supset C_1^{k-1} \supset \dots \supset C_1^0 \supset C_1^{-1}.$$

Furthermore, define a graduation for the A -free base as follows: $\deg. \text{ of } [c\mathfrak{P}^{p^i}] = 2p^i(p-1)$, $\deg. \text{ of } [c\Delta] = 1$. This graduation can be extended to all C_1^k naturally.

Now we fix one k ($-1 \leq k$) and substitute by C_1^k and A the modules C_1 and C_0 considered in §1, and we define A -map d of C_1 into C_0 by $d[c\mathfrak{P}^{p^i}] = c(\mathfrak{P}^{p^i})$ ($i=0, 1, \dots, k$), $d[c\Delta] = c(\Delta)$. Choose a homogeneous d -cycle z of C_1^k , then (d, z) yields a stable secondary cohomology operation Φ . The domain of this Φ is given by the first formula of Lemma 3.1.

LEMMA 3.1. *We have*

$$\bigcap_{i=0}^k \text{Ker } c(\mathfrak{P}^{p^i}) \cap \text{Ker } c(\Delta) = \bigcap_{i=0}^k \text{Ker } \mathfrak{P}^{p^i} \cap \text{Ker } \Delta,$$

$$\text{Im } c(\mathfrak{P}^1) + \dots + \text{Im } c(\mathfrak{P}^{p^k}) + \text{Im } c(\Delta) = \text{Im } \mathfrak{P}^1 + \dots + \text{Im } \mathfrak{P}^{p^k} + \text{Im } \Delta.$$

This is obviously deduced from Lemma 2.4.

Next, we shall specify some d -cycles in the following way.

(3.1) For $k \geq 1$,

$$Z_{i,k} = c(\mathfrak{P}^{p^i})[c\mathfrak{P}^{p^k}] + \sigma_{i,k} \quad (0 \leq i \leq k-2)$$

$$Z_{-1,k} = c(\Delta)[c\mathfrak{P}^{p^k}] - c(\Delta\mathfrak{P}^{p^{k-1}})[c\mathfrak{P}^1] - c(\mathfrak{P}^{p^k})[c\Delta],$$

$$U_k = c(\mathfrak{P}^{2p^{k-1}})[c\mathfrak{P}^{p^k}] + \sigma'_k,$$

$$V_k = c(2\mathfrak{P}^{p^k+p^{k-1}} - \mathfrak{P}(p^k, p^{k-1}))[c\mathfrak{P}^{p^k}] + \sigma''_k,$$

$$W_k = c(\mathfrak{P}^{p^k(p-1)})[c\mathfrak{P}^{p^k}] + \sigma'''_k,$$

(3.2) for $k=0$,

$$V_0 = c(2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)[c\mathfrak{P}^1] - 2c(\mathfrak{P}^2)[c\Delta]$$

$$W_0 = c(\mathfrak{P}^{p-1})[c\mathfrak{P}^1],$$

(3.3) for $k=-1$

$$Z_{-1,-1} = c(\Delta)[c\Delta],$$

where $\sigma_{i,k}$, σ'_k , σ''_k and σ'''_k are elements of C_1^{k-p} . By a suitable choice of these elements, the elements in (3.1), (3.2), (3.3) can be regarded as d -cycles. This is assured by Proposition 2.1 and 2.2. For example,

$$\begin{aligned} \alpha W_k &= c(\mathfrak{P}^{p^k(p-1)})c(\mathfrak{P}^{p^k}) + d\sigma'''_k \\ &= c(\mathfrak{P}^{p^k}\mathfrak{P}^{p^k(p-1)}) + d\sigma'''_k \\ &= c(\mathfrak{P}^{p^k}\mathfrak{P}^{p^k(p-1)} + c^{-1}(d\sigma'''_k)). \end{aligned}$$

By Proposition 2.1 $\mathfrak{P}^{p^k}\mathfrak{P}^{p^k(p-1)}$ is contained in $M_*(k-1)$. On the other hand, we have $dC_1^{k-1} = M(k-1)$, i.e. $c^{-1}(dC_1^{k-1}) = M_*(k-1)$.

In the sequel, $\sigma_{i,k}$, σ'_k , σ''_k and σ'''_k will be kept fixed. Then the stable secondary operation associated with $Z_{i,k}$ ($0 \leq i \leq k-2$ or $i=-1, k=-1$), U_j ($j \geq 1$), V_k , W_k ($k \geq 0$) are uniquely determined, as A/dC_1^k is zero in degrees i , $0 < i < 2p^k(p-1)$ (Cf. Theorem 1.1 (2)). These operations associated will be denoted by $\Phi_{i,k}$, $\Phi_j^{(0)}$, $\Phi_k^{(1)}$, $\Phi_k^{(2)}$ respectively, and will be called basic operations.

The following proposition justifies the word “basic”.

PROPOSITION 3.2. *The kernel of $d : C_1^k \rightarrow A$ is just*

$$\sum_{j \leq k} AZ_{i,j} + \sum_{j \leq k} AU_j + \sum_{j \leq k} AV_j + \sum_{j \leq k} AW_j$$

Proof. We proceed by induction on k . Let $k = -1$. Let z be a d -cycle of $C_1^{-1} = A[c\Delta]$, which we denote by $c(\alpha)[c\Delta]$. Then we have

$$dz = c(\alpha)c(\Delta) = c(\Delta\alpha) = 0.$$

That is, $\Delta\alpha = 0$.

As the kernel of L_{-1} is equal to ΔA , there exists an element $\alpha_{-1, -1}$ of A such that $\alpha = \Delta\alpha_{-1, -1}$. Therefore we obtain $z = c(\alpha)[c\Delta] = c(\alpha_{-1, -1})c(\Delta)[c\Delta] = c(\alpha_{-1, -1})z_{-1, -1}$.

Let us assume that our assertion is verified for $r(<k)$. Now, we take a d -cycle z of C_1^k , and write it in the following form

$$z = c(\alpha_k)[c\mathfrak{P}^k] + \sum_{i=0}^{k-1} c(\alpha_i)[c\mathfrak{P}^{p^i}] + c(\alpha_{-1})[c\Delta].$$

Then we have

$$\begin{aligned} dz &= c(\alpha_k)c(\mathfrak{P}^k) + \sum_{i=0}^{k-1} c(\alpha_i)c(\mathfrak{P}^{p^i}) + c(\alpha_{-1})c(\Delta) \\ &= c(\mathfrak{P}^k\alpha_k + \sum_{i=0}^{k-1} \mathfrak{P}^{p^i}\alpha_i + \Delta\alpha_{-1}) \\ &= 0. \end{aligned}$$

$$\text{Thus } L_k\alpha_k = \mathfrak{P}^k\alpha_k = -\sum_{i=0}^{k-1} \mathfrak{P}^{p^i}\alpha_i - \Delta\alpha_{-1} \equiv 0.$$

By Proposition 2.1 and 2.2 there exist elements $\alpha_{i,k}, \alpha'_k, \alpha''_k, \alpha'''_k$ of A such that

$$\begin{aligned} \alpha_k &= \sum_{i=0}^{k-2} \mathfrak{P}^{p^i}\alpha_{i,k} + \Delta\alpha_{-1,k} + \mathfrak{P}^{2p^{k-1}}\alpha'_k + (2\mathfrak{P}^{p^k+p^{k-1}} - \mathfrak{P}(p^k, p^{k-1}))\alpha''_k \\ &\quad + \mathfrak{P}^{p^{k(p-1)}}\alpha'''_k \quad (k \geq 1), \\ \alpha_k &= (2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)\alpha''_0 + \mathfrak{P}^{p-1}\alpha'''_0 \quad (k=0). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} z &= \left\{ \sum_{i=0}^{k-1} c(\alpha_{i,k})c(\mathfrak{P}^{p^i}) + c(\alpha_{-1,k})c(\Delta) + c(\alpha'_k)c(\mathfrak{P}^{2p^{k-1}}) \right. \\ &\quad \left. + c(\alpha''_k)c(2\mathfrak{P}^{p^k+p^{k-1}} - \mathfrak{P}(p^k, p^{k-1})) + c(\alpha'''_k)c(\mathfrak{P}^{p^{k(p-1)}}) \right\} [c\mathfrak{P}^k] \\ &\quad + \sum_{i=0}^{k-1} c(\alpha_i)[c\mathfrak{P}^{p^i}] + c(\alpha_{-1})c(\Delta) \quad (k \geq 1), \\ z &= \{c(\alpha''_0)c(2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta) + c(\alpha'''_0)c(\mathfrak{P}^{p-1})\} [c\mathfrak{P}^1] + c(\alpha_{-1})[c\Delta] \quad (k=0). \end{aligned}$$

Then the cycles

$$\begin{aligned} z - \sum_{i=-1}^{k-1} c(\alpha_{i,k})Z_{i,k} - c(\alpha'_k)U_k - c(\alpha''_k)V_k - c(\alpha'''_k)W_k &\quad (\text{for } k \geq 1), \\ z - c(\alpha''_0)V_0 - c(\alpha'''_0)W_0 &\quad (\text{for } k=0), \end{aligned}$$

are contained in C_1^{k-1} . By the assumption of induction, the proof is completed.

§ 4. Calculation of the basic secondary operations in the complex projective space of infinitely-many dimensions.

Let us denote by P the complex projective space of infinitely many dimensions and by y a generator of $H^2(P, Z_p)$. Then $H^*(P, Z_p)$ is regarded as the polynomial algebra with a single generator y . We shall consider the effects of our basic stable secondary cohomology operations in $H^*(P, Z_p)$.

LEMMA 4.1. *We have for $s > 0$*

$$\Delta y^s = 0$$

$$\mathfrak{P}^r y^s = \binom{s}{r} y^{s+r(p-1)}$$

Since the proof is easy by induction on s , so it is omitted.

LEMMA 4.2. *If y^s is contained in $\bigcap_{i=0}^k \text{Ker } \mathfrak{P}^{p^i} \cap \text{Ker } \Delta$, then s is a multiple of p^{k+1} .*

This follows immediately from the computation of binomial coefficient modulo p (Cf. [6]).

Thus, by Lemma 3.1 the basic operations $\Phi_{i,k} (-1 \leq i \leq k-2)$, $\Phi_k^{(j)} (j=0, 1, 2)$ are all defined for cohomology classes of the form $y^{p^{k+1}n}$. These operations except for $\Phi_{-1,k} (k \geq 1)$ and $\Phi_0^{(1)}$ are of odd degrees i.e. they map subgroups of $H^n(P, Z_p)$ into $H^{n+i}(P, Z_p)/Q$ for certain subgroup Q where i is odd. Therefore these operations operate trivially in $H^*(P, Z_p)$.

Now, we shall show that $\Phi_{-1,k}(y^{p^{k+1}n})$ ($k \geq 1$) and $\Phi_0^{(1)}(y^{p^n})$ may be also calculated with null indeterminacy. $\Phi_0^{(1)}$ is evaluated modulo:

$$(4.1) \quad \text{Im } c(2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta) + \text{Im } c(\mathfrak{P}^2).$$

Obviously, the first term in (4.1) is zero as $H^{2j+1}(P, Z_p) = 0$. With respect to the second term, we can assert that it is zero by $c(\mathfrak{P}^2)y^{p^n} = \mathfrak{P}^2y^{p^n} = 0$. Next, recall $\Phi_{-1,k} (k \geq 1)$ is evaluated modulo

$$(4.2) \quad \text{Im } c(\Delta) + \text{Im } c(\Delta\mathfrak{P}^{p^k-1}) + \text{Im } c(\mathfrak{P}^{p^k}).$$

Both the first and the second term are zero as $H^{2j+1}(P, Z_p) = 0$. As for the third term we obtain $\text{Im } c(\mathfrak{P}^{p^k}) = 0$ by $c(\mathfrak{P}^{p^k})y^{p^{k+1}n} = -\mathfrak{P}^{p^k}y^{p^{k+1}n} = 0$.

Later, in § 6 and § 7 we shall prove the following theorems.

THEOREM 4.3₀. $\Phi_0^{(1)}y^{p^n} = -ny^{p^n+2(p-1)} \pmod{\text{zero}}$.

THEOREM 4.3_k. $\Phi_{-1,k}y^{p^{k+1}n} = -ny^{p^{k+1}n+p(p-1)} \pmod{\text{zero}}$.

§ 5. The mod p Hopf invariant.

If we assume Theorem 4.3₀ and Theorem 4.3_k, then we can obtain the following theorems.

THEOREM 5.1. *For $u \in \text{Ker } \mathfrak{P}^1 \cap \text{Ker } \Delta$, we have*

$$\mathfrak{P}^p u = (-\Delta\Phi_0^{(2)} - \mathfrak{P}^{p-2}\Phi_0^{(1)})u.$$

THEOREM 5.2. $\mathfrak{P}^{p^{k+1}} (k \geq 1)$ may be expressed by stable secondary cohomology operations as follows:

$$\begin{aligned} \mathfrak{P}^{p^2} u &\equiv (c(\Delta)\Phi_1^{(2)} + c(\Delta\mathfrak{P}^{p(p-1)-2})\Phi_1^{(0)} - c(\mathfrak{P}^{p(p-1)})\Phi_{-1,1})u \\ &\pmod{(A\Phi_0^{(1)} + A\Phi_0^{(2)} + A\Phi_{-1,-1})u \quad (k=1),} \end{aligned}$$

$$\begin{aligned} \mathfrak{P}^{pk-1}u &\equiv (c(\Delta)\Phi_k^{(2)} - c(\Delta\mathfrak{P}^{p^k(p-1)-1})\Phi_{0,k} - c(\mathfrak{P}^{p^k(p-1)}\Phi_{-1,k})u \\ &\quad \text{mod } (\sum_{j < k} A\Phi_{i,j} + \sum_{j \leq k} A\Phi_j^{(i)})u \quad (k \geq 2), \end{aligned}$$

where $u \in \bigcap_{i=0}^k \text{Ker } \mathfrak{P}^{p^i} \cap \text{Ker } \Delta$.

Proof of Theorem 5.1. By the Adem relation we have

$$\mathfrak{P}^{p-1}\Delta - (2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)\mathfrak{P}^{p-2} = 0.$$

Consequently

$$c(\Delta)c(\mathfrak{P}^{p-1})[c\mathfrak{P}^1] + c(\mathfrak{P}^{p-2})c(2\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)[c\mathfrak{P}^1] = 0.$$

This implies

$$0 = -\Delta W_0 - \mathfrak{P}^{p-2}V_0,$$

that is,

$$(5.1) \quad \mu_0\mathfrak{P}^p = -\Delta\Phi_0^{(2)} - \mathfrak{P}^{p-2}\Phi_0^{(1)}.$$

If we let it operate on y^p , then by Theorem 4.3₀ we have

$$\begin{aligned} \mu_0\mathfrak{P}^p y^p &= \mu_0 y^{p^2} \\ &= -\mathfrak{P}^{p-2}\Phi_0^{(1)}y^p = \mathfrak{P}^{p-2}y^{p+2(p-1)} = y^{p^2} \quad \text{mod zero.} \end{aligned}$$

Therefore we obtain $\mu_0 = 1$.

Proof of Theorem 5.2. We shall try to express $\mathfrak{P}^{p^2}$ by the stable secondary operations. By the Adem relation we have

$$(5.2) \quad c(\Delta\mathfrak{P}^{p(p-1)}) + c(\mathfrak{P}^2\Delta\mathfrak{P}^{p(p-1)-2}) - c(\Delta\mathfrak{P}^{p(p-1)}) = 0.$$

Therefore it holds that

$$\{c(\Delta)c(\mathfrak{P}^{p(p-1)}) + c(\Delta\mathfrak{P}^{p(p-1)-2})c(\mathfrak{P}^2) - c(\mathfrak{P}^{p(p-1)})c(\Delta)\}[c\mathfrak{P}^p] = 0.$$

This implies

$$c(\Delta)W_1 + c(\Delta\mathfrak{P}^{p(p-1)-2})U_1 - c(\mathfrak{P}^{p(p-1)})Z_{-1,1} \equiv 0 \quad \text{mod } AV_0 + AW_0 + AZ_{-1,-1}.$$

In other words, this asserts that the following equality holds.

$$(5.3) \quad \begin{aligned} \mu_1\mathfrak{P}^{p^2} &\equiv c(\Delta)\Phi_1^{(2)} + c(\Delta\mathfrak{P}^{p(p-1)-2})\Phi_1^{(0)} - c(\mathfrak{P}^{p(p-1)})\Phi_{-1,1} \\ &\quad \text{mod } A\Phi_0^{(1)} + \Phi_0^{(2)} + A\Phi_{-1,-1}. \end{aligned}$$

Let this operate on $y^{p^2(p-1)}$, then by Theorem 4.3₀, Theorem 4.3_k ($k=1$) and Lemma 2.4 we have

$$\begin{aligned} \mu_1\mathfrak{P}^{p^2}y^{p^2(p-1)} &= -\mu_1y^{2p^2(p-1)} \\ &= -c(\mathfrak{P}^{p(p-1)})\Phi_{-1,1}y^{p^2(p-1)} = -c(\mathfrak{P}^{p(p-1)})y^{p(p-1)(p+1)} \\ &= -y^{2p^2(p-1)} \quad \text{mod zero.} \end{aligned}$$

Therefore we obtain $\mu_1 = 1$.

Let Theorem 4.3_j be proved for $j < k$ (≥ 2) and

$$(5.4) \quad \Phi_{-1,k}y^{p^{k+1}(p-1)} = y^{p^k(p-1)(p+1)}$$

be proved, and notice that we have then for $j \leq k-1$

$$(5.5) \quad \Phi_{-1,j}y^{p^{k+1}n} = 0, \quad \Phi_0^{(1)}y^{p^{k+1}n} = 0$$

without any indeterminacy.

We shall utilize the Adem relation

$$(5.6) \quad c(\mathfrak{P}^{pk(p-1)}\Delta) - c(\mathfrak{P}^1\Delta\mathfrak{P}^{p^k(p-1)-1}) - c(\Delta\mathfrak{P}^{p^k(p-1)}) = 0$$

corresponding to (5.2). In a similar way as above, we may, using Theorem 1.1, put for $k \geq 2$

$$(5.7) \quad \mu_k \mathfrak{P}^{p^{k+1}} \equiv c(\Delta) \Phi_k^{(2)} - c(\Delta \mathfrak{P}^{p^k(p-1)}) \Phi_{0,k} - c(\mathfrak{P}^{p^k(p-1)}) \Phi_{-1,k} \pmod{\sum_{j < k} A \Phi_{i,j} + \sum_{j < k} A \Phi_j^{(i)}}.$$

Let this operate on $y^{p^{k+1}(p-1)}$, then by (5.5), (5.4) and Lemma 2.7 we have

$$\begin{aligned} \mu_k \mathfrak{P}^{p^{k+1}} y^{p^{k+1}(p-1)} &= -\mu_k y^{2p^{k+1}(p-1)} \\ &= -c(\mathfrak{P}^{p^k(p-1)}) \Phi_{-1,k} y^{p^{k+1}(p-1)} = -c(\mathfrak{P}^{p^k(p-1)}) y^{p^k(p-1)(p+1)} \\ &= -y^{2p^{k+1}(p-1)} \pmod{\text{zero}}. \end{aligned}$$

Consequently we obtain $\mu_k = 1$. (q.e.d.)

Let $K = S^m \cup_f e^{m+n}$ ($n = 2t(p-1)$) be a cell complex in which e^{m+n} is attached to S^m by a mapping $f: S^{m+n-1} \rightarrow S^m$. Then the following statements are equivalent:

(a) The Steenrod reduced power

$$\mathfrak{P}^t: H^m(K, Z_p) \rightarrow H^{m+n}(K, Z_p) \quad (n = 2t(p-1))$$

is zero for any mapping f and any t ($p^{k+1} > t \geq p^k$)

(b) For $t = p^k$, the above reduced power is zero.

(c) The mod p Hopf invariant (homomorphism) [17]

$$H_p: \pi_{m+n-1}(S^m) \rightarrow Z_p \quad (n = 2p^k(p-1))$$

is trivial.

It is a classical result that the conclusions are false for $k=0$. Here, we shall show the following theorem.

THEOREM 5.3. *The mod p Hopf invariant (homomorphism)*

$$H_p: \pi_{m+2p^k(p-1)-1}(S^m) \rightarrow Z_p$$

is trivial for $k \geq 1$.

Proof. Let u be a non-zero element of $H^m(K, Z_p)$. Then we have $\Delta u = \mathfrak{P}^{p^i} u = 0$ ($i < k$). We see by Theorem 5.1 and Theorem 5.2

$$\mathfrak{P}^{p^k} u = \sum a_i \Phi_i u, \pmod{a_i Q_i(u)},$$

where Φ_i are the secondary cohomology operations and a_i are the elements of the Steenrod algebra with positive degrees. Obviously, we have by the reason on dimension that $\Phi_i u = 0$, mod zero. That is, $\mathfrak{P}^{p^k} u = 0$, mod zero.

Remark 5.4. If $p=2$, we have analogues to Theorem 5.1 and Theorem 5.2 for $k \geq 3$.

$$\text{Sq}^{2^{k+1}} = c(\text{Sq}^1) \Phi_{k,k} + c(\text{Sq}^{2^{k-1}}) \Phi_{1,k} + c(\text{Sq}^{2^k}) \Phi_{0,k} + \sum_{j \leq k-1} a_{i,j} \Phi_{i,j},$$

where $\Phi_{i,j}$ are the stable secondary cohomology operations defined in Remark 7.4 and $a_{i,j}$ are elements of the Steenrod algebra over Z_2 . From this, we can obtain Adams' theorem on the Hopf invariant [2, 3].

§ 6. First proof.

In this section we shall give a proof of Theorem 4.3_k by making use of Theorem 1.4. In applying Theorem 1.4, we use the complex projective space P of infinitely-many dimensions as the space Y , a certain fibre space $\tilde{K}_k$ (for each $k \geq 0$) as the space X , and the composition $f_k = g_k \circ d$ of the

diagonal map $d: P \rightarrow P^p$ (cartesian product of p copies of P) and of a suitable map $g_k: P^p \rightarrow \tilde{K}_k$ as the map $f: Y \rightarrow X$.

We shall begin with some preparations.

Let p be an odd prime, and let n be a positive integer. For each $k \geq 0$, set $N_k = 2p^{k+1}n$. Then the space $\tilde{K}_k$ will be defined as a fibre space over the base $B_k = K(Z, N_k)$ with fibre F_k , a generalized Eilenberg-MacLane space of the following type:

$$(6.1) \quad \begin{aligned} F_0 &= K(Z, N_0 + 2(p-1)) \times K(Z, N_0 + 4(p-1)), \\ F_k &= K(Z, N_k + 2p^k(p-1)) \times \prod_{j=0}^{k-1} K(Z_p, N_k + 2p^j(p-1) - 1) \\ &\quad (\text{for } k \geq 1). \end{aligned}$$

Let $\zeta_j \in H^{\mathbf{N}_0 + 2j(p-1)}(F_0, Z)$ ($j=1, 2$), $\eta_k \in H^{\mathbf{N}_k + 2p^k(p-1)}(F_k, Z)$ ($k \geq 1$),

$$\eta_{k,j} \in H^{\mathbf{N}_k + 2p^j(p-1) - 1}(F_k, Z_p) \quad (k \geq 1, 0 \leq j \leq k-1) \quad \text{and}$$

$\iota_k \in H^{\mathbf{N}_k}(B_k, Z)$ ($k \geq 0$) be fundamental classes of the respective spaces.

Denote by τ the transgression homomorphism. Then the fibre space $(F_k, \tilde{K}_k, B_k)$ is to be defined so that

$$(6.2) \quad \begin{aligned} \tau \zeta_j &= \Delta^* \mathfrak{P}^j(\iota_0)_p \quad \text{for } k=0, \\ \tau \eta_k &= \Delta^* \mathfrak{P}^{i^k}(\iota_k)_p, \\ \tau \eta_{k,j} &= \mathfrak{P}^{i^j}(\iota_k)_p \quad (0 \leq j \leq k-1) \text{ for } k \geq 1, \end{aligned}$$

where Δ^* denotes the Bockstein operator associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{(-1)^i p} Z \longrightarrow Z_p \longrightarrow 0$$

and $(\ )_p$ denotes the reduction mod p of coefficients.

The following lemma is well known.

Lemma 6.1. *Let $f: Y \rightarrow X$ be a map, and let $v \in H^m(X, Z)$ $\eta \in H^m(Y, Z)$ be such classes that $f^*v = p\eta$. Then we have*

$$\Delta_{\sharp}(v)_p = \{\eta_p\} \mod \text{Im } f^* + \text{Im } \Delta.$$

For the following lemmas, let $i: F_k \rightarrow \tilde{K}_k$ be the injection, $\rho: \tilde{K}_k \rightarrow B_k$ the projection, and set $\bar{\iota}_k = \rho^* \iota_k$.

Lemma 6.2₀. *There exist classes $v_{0,j} \in H^{\mathbf{N}_0 + 2j(p-1)}(\tilde{K}_0, Z)$ ($j=1, 2$) such that*

$$(1) \quad i^* v_{0,j} = p \zeta_j,$$

$$(2) \quad (v_{0,j})_p = \mathfrak{P}^j(\bar{\iota}_0)_p \quad (j=1, 2).$$

Lemma 6.2_k. *For each $k \geq 1$, there exists a class $v_k \in H^{\mathbf{N}_k + 2p^k(p-1)}(\tilde{K}_k, Z)$ such that*

$$(1) \quad i^* v_k = p \eta_k,$$

$$(2) \quad (v_k)_p = \mathfrak{P}^{i^k}(\bar{\iota}_k)_p.$$

Proof. We shall treat only the case $k \geq 1$. The case $k=0$ will be analogously proved. From the exact sequence of cohomology groups in the stable range of dimensions:

$$\longrightarrow H^+(B_k, Z) \xrightarrow{\rho^*} H^+(\tilde{K}_k, Z) \xrightarrow{i^*} H^+(F_k, Z) \xrightarrow{\tau} H^+(B_k, Z) \longrightarrow,$$

it follows that there exists such a class v_k as satisfying the first condition of Lemma 6.2_k. Then $(v_k)_p \in \text{Im } \rho^*$. But $\rho^* H^{N_k+2p^k(p-1)}(B_k, Z_p)$ is generated by a single element $\mathfrak{P}^{p^k}(\iota_k)_p$. Therefore we can set

$$(6.3) \quad (v_k)_p = \nu \mathfrak{P}^{p^k}(\iota_k)_p, \quad \nu \in Z_p.$$

From the condition (1) of the lemma, it follows that

$$(6.4) \quad \Delta_i(v_k)_p = \{(\eta_k)_p\} \pmod{\text{Im } i^* + \text{Im } \Delta}.$$

On the other hand we know

$$(6.5) \quad \Delta_i(\mathfrak{P}^{p^k}(\iota_k)_p) = \{(\eta_k)_p\} \pmod{\text{Im } i^* + \text{Im } \Delta}.$$

It follows that $\nu=1$ in (6.3), and this proves the condition (2).

For the next lemmas, let $y_j \in H^2(P^p, Z)$ ($j=1, \dots, p$) be those generators of the cohomology ring $H^*(P^p, Z)$ which correspond to a fixed generator $y \in H^*(P, Z)$ by the diagonal map $d: P \rightarrow P^p$.

Lemma 6.3₀. *There exists a map $g_0: P^p \rightarrow \tilde{K}_0$ such that*

$$\begin{aligned} (1) \quad & g_0^* \tau_0 = y_1^n \dots y_p^n \\ (2) \quad & g_0^* v_{0,1} = n \sum y_1^{n+p-1} y_2^n \dots y_p^n \\ & g_0^* v_{0,2} = n(n-1)/2 \cdot \sum y_1^{n+2(p-1)} y_2^n \dots y_p^n \\ & \quad + n^2 \sum y_1^{n+p-1} y_2^{n+p-1} y_3^n \dots y_p^n. \end{aligned}$$

Lemma 6.3_k. *For each $k \geq 1$, there exists a map $g_k: P^p \rightarrow \tilde{K}_k$ such that*

$$\begin{aligned} (1) \quad & g_k^* \tau_k = y_1^{p^k n} \dots y_p^{p^k n}, \\ (2) \quad & g_k^* v_k = n \sum y_1^{p^k(n+p-1)} y_2^{p^k n} \dots y_p^{p^k n}. \end{aligned}$$

In these lemmas the right-hand sides of the respective formulas are the symmetric polynomials of $y_1, \dots, y_p$ having the indicated monomial as a representative, and all cohomology classes involved are of integral coefficients.

Proof. Since Lemmas 6.3₀ and 6.3_k are similarly proved, we treat only the case $k \geq 1$. As is easily seen, there exists a map $g': P^p \rightarrow \tilde{K}_k$ such that the condition (1) is fulfilled. From Lemma 6.2_k, (2) it follows that

$$(6.6) \quad g'^* v_k = n \sum y_1^{p^k(n+p-1)} y_2^{p^k n} \dots y_p^{p^k n} + pw,$$

where w is a suitable element of $H^{N_k+2p^k(p-1)}(P^p, Z)$.

Let $h: P^p \rightarrow K(Z, N_k+2p^k(p-1)) \subset F_k$ be a map such that

$$(6.7) \quad h^* \eta_k = -w.$$

Set $g_k = h \cdot g'$, where $\cdot$ means the multiplication of the principal fibre space $(F_k, \tilde{K}_k, B_k)$. Then the map g_k satisfies the condition (2).

Now we will return to the proof of Theorem 4.3_k. Set $f_k = g_k \circ d$, the composition of the diagonal map $d: P \rightarrow P^p$ and of the map $g_k: P^p \rightarrow \tilde{K}_k$ appeared in Lemma 6.3_k. Then we have

$$(6.8) \quad f_k^* \tau_k = y^{p^k+1} n \quad (k \geq 0),$$

and

$$\begin{aligned} (6.9) \quad & f_0^* v_{0,1} = pny^{p n + (p-1)}, \\ & f_0^* v_{0,2} = pn(pn-1)/2 y^{p n + 2(p-1)}, \end{aligned}$$

$$f_k^* v_k = p n y^{p^{k+1}n + l^k(p-1)} \quad (k \geq 1)$$

By Lemma 6.1, we have

$$(6.10) \quad \begin{aligned} \Delta_{f_0}(v_{0,1})_p &= \{(n y^{pn+(p-1)})_p\} \bmod \text{Im } f_0^*, \\ \Delta_{f_0}(v_{0,2})_p &= \{(n(\dot{p}n-1)/2 y^{pn+2(p-1)})_p\} \bmod \text{Im } f_0^*, \\ \Delta_{f_k}(v_k)_p &= \{(n y^{p^{k+1}n + l^k(p-1)})_p\} \bmod \text{Im } f_k^*. \end{aligned}$$

Recalling that the operation $\Phi_0^{(1)}$ corresponds to the relation

$$(\Delta \mathfrak{P}^1 - 2\mathfrak{P}^1 \Delta) \mathfrak{P}^1 - 2\mathfrak{P}^2 \Delta = 0$$

and that $\Phi_{-1,k}$ corresponds to the relation

$$c(\Delta) c(\mathfrak{P}^{p^k}) - c(\Delta \mathfrak{P}^{l^k-1}) c(\mathfrak{P}^1) - c(\mathfrak{P}^{l^k}) c(\Delta) = 0,$$

and applying Theorem 1.4, we have

$$(6.11) \quad \begin{aligned} \Phi_0^{(1)}(y^{pn})_p &= \Phi_0^{(1)} f_0^*(\bar{\tau}_0)_p = -(\Delta \mathfrak{P}^1 - 2\mathfrak{P}^1 \Delta)_{f_0} \mathfrak{P}^1(\bar{\tau}_0)_p \\ &= 2\mathfrak{P}^1 \Delta_{f_0}(\mathfrak{P}^1(\bar{\tau}_0)_p) - 2\Delta_{f_0} \mathfrak{P}^2(\bar{\tau}_0)_p \\ &= 2\mathfrak{P}^1 \Delta_{f_0}(v_{0,1})_p - 2\Delta_{f_0}(v_{0,2})_p \\ &= (-n y^{pn+2(p-1)})_p \bmod \text{Im } f_0^* \end{aligned}$$

and

$$(6.12) \quad \begin{aligned} \Phi_{-1,k}(y^{p^{k+1}n})_p &= \Phi_{-1,k} f_k^*(\bar{\tau}_k)_p = -c(\Delta)_{f_k} (c(\mathfrak{P}^{l^k})(\bar{\tau}_k)_p) \\ &= -\Delta_{f_k}(\mathfrak{P}^{p^k}(\bar{\tau}_k)_p) = -\Delta_{f_k}(v_k)_p \\ &= (-n y^{p^{k+1}n + l^k(p-1)})_p \bmod \text{Im } f_k^*. \end{aligned}$$

It is easy to see $\text{Im } f_0^* \cap H^{N_0+4(p-1)}(P, Z_p) = 0$. We shall prove

$$\text{Im } f_k^* \cap H^{N_k+2l^k(p-1)}(P, Z_p) = 0$$

by induction on k .

Lemma 6.4. *In the range of degrees less than $2N_k(k \geq 1)$ the graded A -module $H^+(\tilde{K}_k, Z_p)$ is generated by the classes $((\bar{\tau}_k)_p)$ and $\Phi_{Z*}((\bar{\tau}_k)_p)$ (Z_* run over all the basic d -cycles belonging to C_{k-1}).*

This lemma is easily proved. Now assume that Theorem 4.3_k is true for smaller values of k . Then, from Lemma 6.4, we can prove

$$\text{Im } f_k^* \cap H^{N_k+2l^k(p-1)}(P, Z_p) = 0,$$

and this completes the proof of Theorem 4.3_k.

§ 7. Second proof.

We shall first establish a formula on $\Phi_{-1,k} c(\mathfrak{P}^{l^k(p-1)})$.

Let us take as left A -free modules C_0, C_0', C_1 and C_1' in §1 the following modules respectively: $A, A, C_1^k (k \geq 1), C_1^{p-1} \oplus A[c\mathfrak{P}^{l^k} \mathfrak{P}^{p^{k-1}}]$. Furthermore, define A -map d' of $C_1^{k-1} \oplus A[c\mathfrak{P}^{l^k} \mathfrak{P}^{p^{k-1}}]$ into A as follows:

$$\begin{aligned} d'[c\mathfrak{P}^{p^i}] &= c(\mathfrak{P}^{p^i}) \quad i=0, 1, \dots, k-1, \\ d'[c\Delta] &= c(\Delta) \\ d'[c\mathfrak{P}^{l^k} \mathfrak{P}^{p^{k-1}}] &= c(\mathfrak{P}^{l^k} \mathfrak{P}^{p^{k-1}}). \end{aligned}$$

Now, let m_0 be the A -map of $C_0=A$ into $C_0'=A$ which is defined by $m_0(1)=c(\mathfrak{P}^{p^{k(p-1)}})$. Then by a suitable choice of A -map m_1 of C_1^k into $C_1^{k-1} \oplus A[c\mathfrak{P}^{l^k} \mathfrak{P}^{p^{k-1}}]$, we can obtain the following commutative diagram

$$\begin{array}{ccc}
& d & \\
A & \xleftarrow{\quad} & C_1^k \\
m_0 \downarrow & & \downarrow m_1 \\
& d' & \\
A & \xleftarrow{\quad} & C_1^{k-1} \oplus A[c(\mathfrak{P}^{p^k} \mathfrak{P}^{p^{k-1}})].
\end{array}$$

More precisely, we shall define m_1 as follows for later use. Let us begin by computing the effect of $m_0 d$ for $[c\mathfrak{P}^{p^k}]$:

$$\begin{aligned}
m_0 d[c\mathfrak{P}^{p^k}] &= m_0 c(\mathfrak{P}^{p^k}) \\
&= c(\mathfrak{P}^{p^k}) c(\mathfrak{P}^{p^k(p-1)}) \\
&= c(\mathfrak{P}^k(p-1), p^k).
\end{aligned}$$

Now, by the Adem relation we have

$$\begin{aligned}
\mathfrak{P}(p^k(p-1), p^k) &\equiv \sum_{i=1}^{p-1} (-1)^i \mathfrak{P}(p^k(p-1) + (p-i)p^{k-1}, ip^{k-1}) \pmod{M_*'(k-2)} \\
&\equiv \sum_{i=1}^{p-1} \frac{(-1)^i}{p-i} \mathfrak{P}(p^{k-1}, p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}) \\
&\hspace{15em} \pmod{M_*'(k-2)} \\
&= \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} \mathfrak{P}(p^{k-1}, p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}).
\end{aligned}$$

Define m_1 for $[c\mathfrak{P}^{p^k}]$ as follows:

$$\begin{aligned}
(7.1) \quad m_1[c\mathfrak{P}^{p^k}] &= \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c\mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}) [c\mathfrak{P}^{p^{k-1}}] \\
&\quad + \tau_{k,k},
\end{aligned}$$

where $\tau_{k,k}$ is an element of C_1^{k-2} , so that $m_0 d[c\mathfrak{P}^{p^k}] = d' m_1[c\mathfrak{P}^{p^k}]$,

Similarly, define m_1 for $[c\mathfrak{P}^{p^{k-1}}]$, $[c\mathfrak{P}^{p^{k-2}}]$ as follows:

$$(7.2) \quad m_1[c\mathfrak{P}^{p^{k-1}}] = c\mathfrak{P}^{p^k(p-1)} [c(\mathfrak{P}^{p^k} \mathfrak{P}^{p^{k-1}})] + 2c\mathfrak{P}^{p^k(p-1)} [\mathfrak{P} c^{p^{k-1}}] + \tau_{k-1,k},$$

$$(7.3) \quad m_1[c\mathfrak{P}^{p^{k-2}}] = c\mathfrak{P}^{p^k(p-2) + p^{k-1}(p-1) + p^{k-2}} [c\mathfrak{P}^{p^{k-1}}] + \tau_{k-2,k},$$

where $\tau_{k-1,k}$, $\tau_{k-2,k}$ may be regarded as elements of C_1^{k-2} , so that $m_0 d = d' m_1$. Furthermore, for $0 \leq j < k-2$ we can take an $m_1[c\mathfrak{P}^{p^j}]$ as element of C_1^{k-2} by Lemma 2.1 such that $m_0 d[c\mathfrak{P}^{p^j}] = d' m_1[c\mathfrak{P}^{p^j}]$. With respect to $m_1[c\Delta]$, we may choose $m_1[c\Delta]$ in a similar way as above for $k \geq 2$. If $k=1$, then we may define

$$(7.4) \quad m_1[c\Delta] \equiv c(\Delta \mathfrak{P}^{p(p-2) + p-1}) \pmod{C_1^{-1}}$$

such that $m_0 d[c\Delta] = d' m_1[c\Delta]$.

Let us compute $m_1 Z_{-1,k} = m_1 \{c(\Delta[c\mathfrak{P}^{p^k}] - c(\Delta \mathfrak{P}^{p^{k-1}})[c\mathfrak{P}^1] - c(\mathfrak{P}^{p^k})[c\Delta])\}$.

We shall now consider three cases separately.

Case I, $k \geq 3$. We have immediately

$$\begin{aligned}
(7.5) \quad m_1 Z_{-1,k} &\equiv c(\Delta) \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c\mathfrak{P}(p^{k-1}(p-i-1) + p^k(p-1), ip^{k-1}) [c\mathfrak{P}^{p^{k-1}}] \\
&\hspace{15em} \pmod{C_1^{k-2}} \\
&= \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c\mathfrak{P}(p^{k-1}(p-i-1) + p^k(p-1), ip^{k-1}, \Delta) [c\mathfrak{P}^{p^{k-1}}].
\end{aligned}$$

By the repeated applications of (1.3), we can obtain the following

$$(7.6) \quad \mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}, \Delta)$$

$$\begin{aligned}
&= \mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), 1, \Delta, ip^{k-1}-1) \\
&\quad + \mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), \Delta, ip^{k-1}) \\
&= \mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), 1, \Delta, ip^{k-1}-1) \\
&\quad + \mathfrak{P}(1, \Delta, p^k(p-1) + p^{k-1}(p-i-1) - 1, ip^{k-1}) \\
&\quad + \mathfrak{P}(\Delta, p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}) \\
&\equiv \mathfrak{P}(\Delta, p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}) \pmod{\mathfrak{P}^{2p^{k-2}}A + M_*'(1)},
\end{aligned}$$

because for $k > 3$ $\mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), 1)$ is contained in $\mathfrak{P}^{2p^{k-2}}A + M_*'(1)$ and, for $k=3$, it is contained in $\mathfrak{P}^{2p}A + M_*'(0)$ by the Adem relation

$$\begin{aligned}
&\mathfrak{P}(2p, p^3(p-1) + p^2(p-i-1) + 1 - 2p) \\
&\equiv -\mathfrak{P}(p^3(p-1) + p^2(p-i-1), 1) \pmod{M_*'(0)}.
\end{aligned}$$

Thus (7.5), (7.6) imply

PROPOSITION 7.1. *We have*

$$\begin{aligned}
\text{for } k > 3 \quad \Phi_{-1, k} \mathcal{C}(\mathfrak{P}^{p^k(p-1)}) &\equiv \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c \mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}) \Phi_{-1, k-1} \\
&\quad + \lambda_k \mathfrak{P}^{p^{k+1}} \\
&\pmod{A\Phi_{1, k-1} + A\Phi_{0, k-1} + \sum_{s < k-1} A\Phi_{r, s} + \sum_{s < k-1} A\Phi_s^{(r)}}
\end{aligned}$$

$$\begin{aligned}
\text{for } k = 3 \quad \Phi_{-1, k} \mathcal{C}(\mathfrak{P}^{p^k(p-1)}) &\equiv \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c \mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), ip^{k-1}) \Phi_{-1, k-1} \\
&\quad + \lambda_k \mathfrak{P}^{p^{k+1}} \\
&\pmod{A\Phi_{k-1}^{(0)} + A\Phi_{0, k-1} + \sum_{s < k-1} A\Phi_{r, s} + \sum_{s < k-1} A\Phi_s^{(r)}}.
\end{aligned}$$

Case II, $k=2$. From (7.1) and (7.3) we have

$$\begin{aligned}
(7.7) \quad m_1 Z_{-1, 2} &\equiv \left\{ \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c \mathfrak{P}(p^2(p-1) + p(p-i-1), ip, \Delta) \right. \\
&\quad \left. - c \mathfrak{P}(p^2(p-2) + p(p-1) + 1, \Delta, p^2-1) \right\} [c \mathfrak{P}^p] \pmod{C_1^0}.
\end{aligned}$$

Now, by the repeated applications of (1.3) we can obtain the following relations

$$\begin{aligned}
(7.8) \quad &\mathfrak{P}(p^2(p-1) + p(p-i-1), ip, \Delta) \\
&= \mathfrak{P}(p^2(p-1) + p(p-i-1), 1, \Delta, ip-1) \\
&\quad + \mathfrak{P}(p^2(p-1) + p(p-i-1), \Delta, ip) \\
&= \mathfrak{P}(p^2(p-1) + p(p-i-1), 1, \Delta, ip-1) - \mathfrak{P}(2, \Delta, p^2(p-1) \\
&\quad + p(p-i-1) - 2, ip) + \mathfrak{P}(\Delta, p^2(p-1) + p(p-i-1), ip) \\
&\equiv \mathfrak{P}(p^2(p-1) + p(p-i-1), 1, \Delta, ip-1) \\
&\quad + \mathfrak{P}(\Delta, p^2(p-1) + p(p-i-1), ip) \pmod{\mathfrak{P}^3 A}.
\end{aligned}$$

Also, by the Adem relation we have

$$\begin{aligned}
(7.9) \quad &\mathfrak{P}(p^2(p-2) + p(p-1) + 1, \Delta, p^2-1) \\
&= \sum_{i=1}^{p-2} (-1)^{i+1} \mathfrak{P}(p^2(p-1) + p(p-i-2) + 1, \Delta, ip + p-1) \\
&\quad + \sum_{0 < p, i} (-1)^j (j+1) \binom{j}{i} \mathfrak{P}(\Delta, p^2(p-1) + p(p-i-2) + p-j, ip+j) \\
&\pmod{\mathfrak{P}^3 A}.
\end{aligned}$$

Thus we can obtain the following equalities.

$$\begin{aligned}
 (7.10) \quad & \sum_{i=1}^{p-1} \frac{(-1)^i}{i} \mathfrak{P}(p^2(p-1) + p(p-i-1), 1, \Delta, ip-1) \\
 & + \sum_{i=0}^{p-2} (-1)^{i+1} \mathfrak{P}(p^2(p-1) + p(p-i-2) + 1, \Delta, ip+p-1) \\
 & = \sum_{i=1}^{p-1} \frac{(-1)^i}{i} \left\{ \mathfrak{P}(p^2(p-1) + p(p-i-1), 1, \Delta, ip-1) \right. \\
 & \quad \left. + i \mathfrak{P}(p^2(p-1) + p(p-i-1) + 1, \Delta, ip-1) \right\} \\
 & \equiv 0 \quad \text{mod } (2\mathfrak{P}^{p+1} - \mathfrak{P}^p \mathfrak{P}^1) A,
 \end{aligned}$$

because $(2\mathfrak{P}^{p+1} - \mathfrak{P}^p \mathfrak{P}^1) \mathfrak{P}(p^2(p-1) + p(p-i-2))$
 $= -\mathfrak{P}(p^2(p-1) + p(p-i-1), 1) - i \mathfrak{P}(p^2(p-1) + p(p-i-1) + 1).$

That is, from (7.8), (7.9) and (7.10) we have

$$\begin{aligned}
 (7.11) \quad & \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} \mathfrak{P}(p^2(p-1) + p(p-i-1), ip, \Delta) \\
 & - \mathfrak{P}(p^2(p-2) + p(p-1) + 1, \Delta, p^2-1) \\
 & \equiv \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} \mathfrak{P}(\Delta, p^2(p-1) + p(p-i-1), ip) \\
 & + \sum_{i, 0 \leq j < p} (-1)^{j+1} (j+1) \binom{j}{i} \mathfrak{P}(\Delta, p^2(p-1) + p(p-i-2) + p-j, ip+j) \\
 & \quad \text{mod } \mathfrak{P}^3 A + (2\mathfrak{P}^{p+1} - \mathfrak{P}^p \mathfrak{P}^1) A.
 \end{aligned}$$

(7.7) and (7.11) imply

$$\begin{aligned}
 \text{PROPOSITION 7.2. } & \Phi_{-1, 2} c(\mathfrak{P}^{p^2(p-1)}) \\
 & \equiv \left\{ \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c \mathfrak{P}(p^2(p-1) + p(p-i-1), ip) \right. \\
 & \quad \left. + \sum_{i, 0 \leq j < p} (-1)^{j+1} (j+1) \binom{j}{i} c \mathfrak{P}(p^2(p-1) + p(p-i-2) + p-j, ip+j) \right\} \Phi_{-1, 1} \\
 & + \lambda_2 \mathfrak{P}^3 \quad \text{mod } A\Phi_1^{(0)} + A\Phi_1^{(1)} + A\Phi_0^{(1)} + A\Phi_0^{(2)} + A\Phi_{-1, -1}.
 \end{aligned}$$

Case III, $k=1$. Let us begin with specifying basic d' -cycles of $C_1^0 \oplus A[c(\mathfrak{P}^p \mathfrak{P}^1)]$. Define λ, μ by

$$\begin{aligned}
 \lambda &= c(R) [c(\mathfrak{P}^p \mathfrak{P}^1)] + 2c(\mathfrak{P}^{p+2} - \mathfrak{P}^{p+1} \mathfrak{P}^1) [c\Delta] \\
 \mu &= c(\mathfrak{P}^1) [c(\mathfrak{P}^p \mathfrak{P}^1)] + c(-2\mathfrak{P}^p \mathfrak{P}^1 + \mathfrak{P}^{p+1}) [c\mathfrak{P}^1],
 \end{aligned}$$

where $R = 2\Delta \mathfrak{P}^1 - \mathfrak{P}^1 \Delta$. Furthermore, denote the stable secondary cohomology operations associated with (d', λ) and (d', μ) by Φ_λ and Φ_μ respectively. Then Φ_λ and Φ_μ which are defined for $\text{Ker } \mathfrak{P}^{p+1} \cap \text{Ker } \mathfrak{P}^1 \cap \text{Ker } \Delta$ are uniquely determined operations, because $A/dC_1' = A/d'(C_1^0 \oplus A[c(\mathfrak{P}^p \mathfrak{P}^1)])$ is zero in degrees $2(p+2)(p-1)$, $2(p+2)(p-1)-1$ (Cf. Theorem 1.1(2)).

Now, by (7.1), (7.2) and (7.4) we have

$$\begin{aligned}
 (7.12) \quad & m_1 Z_{-1, 1} \equiv -c \mathfrak{P}(p(p-2), \Delta, p-1) [c(\mathfrak{P}^p \mathfrak{P}^1)] \\
 & + \left\{ \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c \mathfrak{P}(p(p-1) + p-i-1, i, \Delta) \right. \\
 & \quad \left. - 2c \mathfrak{P}(p(p-1), \Delta, p-1) \right\}
 \end{aligned}$$

$$-c\mathfrak{P}(\Delta, p(p-2)+p-1, p)\} [c\mathfrak{P}^1] \bmod C_1^{-1}.$$

Next, we shall express $m_1 Z_{-1, 1}$ as an element of $A\lambda + A\mu + AW_0 + AU_0 \bmod C_1^{-1}$. By the Adem relation we obtain

$$\mathfrak{P}(p(p-2), \Delta, p-1) = \sum_{i=0}^{p-2} (-1)^{i+1} \mathfrak{P}(\Delta, p(p-2)+p-i-1, i).$$

Further, utilizing $1/2(R+\mathfrak{P}^1\Delta)\mathfrak{P}^s = (s+1)\mathfrak{P}(\Delta, s+1)$, we have

$$\mathfrak{P}(p(p-2), \Delta, p-1) = \frac{1}{2} \sum_{i=0}^{p-2} \frac{(-1)^i}{i+1} (R+\mathfrak{P}^1\Delta) \mathfrak{P}(p(p-2)+p-i-2, i).$$

Thus, our object will be attained by expressing the element

$$(7.13) \quad \begin{aligned} z = & \frac{1}{2} \sum_{i=0}^{p-2} \frac{(-1)^i}{i+1} (\mathfrak{P}^{p+1} - 2\mathfrak{P}^p \mathfrak{P}^1) \mathfrak{P}(\Delta, p(p-2)+p-i-2, i) \\ & + \sum_{i=1}^{p-2} \frac{(-1)^{i+1}}{i} \mathfrak{P}(p(p-1)+p-i-1, i, \Delta) - 2\mathfrak{P}(p-1), \Delta, p-1) \\ & - \mathfrak{P}(\Delta, p(p-2)+p-1, p) \end{aligned}$$

as an element of $RA + \mathfrak{P}^{p-1}A$. By the Adem relation we have

$$\begin{aligned} z = & \frac{1}{2} \sum_{i=0}^{p-2} \frac{(-1)^i}{i+1} \left\{ (i-1)(i+2) \mathfrak{P}(\Delta, p(p-1)+p-i-1, i) \right. \\ & + (i+1)^2 \mathfrak{P}(\Delta, p(p-1)+p-i-2, i+1) \\ & - (i-1) \mathfrak{P}(p(p-1)+p-i-2, \Delta, i+1) \\ & \left. - (i-1) \mathfrak{P}(p(p-1)+p-i-1, \Delta, i) \right\} \\ & - 2\mathfrak{P}(p(p-1), \Delta, p-1) + \mathfrak{P}(\Delta, p(p-1)+p-1) \\ = & \sum_{i=1}^{p-2} \frac{(-1)^i}{i(i+1)} R \mathfrak{P}(p(p-1)+p-i-2, i) - \frac{1}{2} R \mathfrak{P}(p(p-1)+p-2). \end{aligned}$$

Thus we obtain finally

$$\begin{aligned} m_1 Z_{-1, 1} \equiv & \frac{1}{2} \sum_{i=0}^{p-1} \frac{(-1)^{i+1}}{i+1} c\mathfrak{P}(p(p-2)+p-i-2, i) (\lambda + c(\Delta)\mu) \\ & + \left[\sum_{i=1}^{p-2} \frac{(-1)^i}{i(i+1)} c\mathfrak{P}(p(p-1)+p-i-2, i) - \frac{1}{2} c\mathfrak{P}(p(p-1)+p-2) \right] V_0 \\ & \bmod C_1^{-1}. \end{aligned}$$

This implies the following proposition.

PROPOSITION 7.3. $\Phi_{-1, 1} c\mathfrak{P}^{p(p-1)}$

$$\begin{aligned} = & \frac{1}{2} \sum_{i=0}^{p-2} \frac{(-1)^{i+1}}{i+1} c\mathfrak{P}(p(p-2)+p-i-2, i) (\Phi_\lambda + c(\Delta)\Phi_\mu) \\ & + \left[\sum_{i=1}^{p-2} \frac{(-1)^i}{i(i+1)} c\mathfrak{P}(p(p-1)+p-i-2, i) - \frac{1}{2} c\mathfrak{P}(p(p-1)+p-2) \right] \Phi_0^{(1)} \\ & + \lambda_1 \mathfrak{P}^{p^2} \bmod A\Phi_{-1, -1} \end{aligned}$$

REMARK 7.4. In the same manner as in §3 we shall define, in case $p=2$, cycles in $C_1^k = \sum_{i=0}^k A[\text{Sq}^{2^i}]$, which are contained in the kernel of A -map d

$$d[c\text{Sq}^{2^i}] = c(\text{Sq}^{2^i}) \quad (i=0, \dots, k)$$

as follows:

$$z_{i, j} = c(\text{Sq}^{2^i}) [c\text{Sq}^{2^j}] + \dots \quad i \neq j-1, i \leq j$$

Then we have a similar result to Proposition 3.2 which is stated as follows :

The kernel of $d: C_1^k \rightarrow A$ is just $\sum_{\substack{i \leq j \leq k \\ i \neq j-1}} Az_{i,j}$ (Cf. [10]). Also if we define

$$z_{0,k} = c(\text{Sq}^1)[c\text{Sq}^{2k}] + c[\text{Sq}^{2k-1}][c\text{Sq}^2] + c(\text{Sq}^{2k})[c\text{Sq}^1] \quad (k \geq 2),$$

then we can obtain

$$\begin{aligned} \text{for } k > 3 \quad \Phi_{0,k} c(\text{Sq}^{2k}) &= c(\text{Sq}^{2k} \text{Sq}^{2k-1}) \Phi_{0,k-1} + \sum_{i \neq 0} \alpha_{i,k-1} \Phi_{k-1} \\ &\quad + \sum_{j \leq k-2} \alpha'_{i,j} \Phi_{i,j} + \lambda'_k \text{Sq}^{2k+1} \\ \text{for } k=3 \quad \Phi_{0,k} c(\text{Sq}^{2k}) &= c(\text{Sq}^{2k} \text{Sq}^{2k-1} + \text{Sq}^{2k+2k-1} + \text{Sq}^{2k+2k-2} \text{Sq}^{2k-2}) \Phi_{0,k-1} \\ &\quad + \sum_{i \neq 0} \alpha'_{i,k-1} \Phi_{i,k-1} + \sum_{j \leq k-1} \alpha'_{i,j} \Phi_{i,j} + \lambda'_3 \text{Sq}^{2k+1}. \end{aligned}$$

Now, Let us compute $\Phi_0^{(1)} y^{pn}$. For our purpose we refer to Toda's result [17] which may be stated in basic operations as follows.

$$(7.14) \quad \mathfrak{P}^p = -\Delta \Phi_0^{(2)} - \mathfrak{P}^{p-2} \Phi_0^{(1)}.$$

Apply this theorem for y^{pn} , then we obtain

$$\mathfrak{P}^p y^{pn} = n y^{p(n+p-1)} = -\mathfrak{P}^{p-2} \Phi_0^{(1)} y^{pn}.$$

This implies Theorem 4.30 :

$$\Phi_0^{(1)} y^{pn} = -n y^{p(n+2(p-1))} \pmod{\text{zero}}.$$

Next, we shall consider effects of Φ_λ and Φ_μ which are defined for a cohomology class y^{pn} . It is easily verified that y^{pn} is contained in $\text{Ker } \mathfrak{P}^{p+1} \cap \text{Ker } \mathfrak{P}^1 \cap \text{Ker } \Delta$. Obviously Φ_μ operates on y^{pn} trivially as $H^{2j+1}(P, Z_p) = 0$. Thus we are only interested in $\Phi_\lambda y^{pn}$. $\Phi_\lambda y^{pn}$ is determined with indeterminacy of the following subgroup

$$(7.15) \quad \text{Im } c(R) + \text{Im } c(\mathfrak{P}^{p+2} - \mathfrak{P}^{p+1} \mathfrak{P}^1).$$

The first term is zero, as $c(R)$ 'lifts' the dimension by $2(p-1)+1$. As for the second term, we notice

$$c(\mathfrak{P}^{p+2} - \mathfrak{P}^{p+1} \mathfrak{P}^1) y^{pn} = (-\mathfrak{P}^p \mathfrak{P}^2 + \mathfrak{P}^1 \mathfrak{P}^p \mathfrak{P}^1) y^{pn} = 0.$$

Thus $\Phi_\lambda y^{pn}$ is settled without any indeterminacy.

LEMMA 7.5. *We obtain*

$$\Phi_0^{(1)} c(\mathfrak{P}^p) = \Phi_\lambda.$$

Proof. In Theorem 1.1(4), put $C_0 = C_0' = A$, $C_1 = C_1^0$ and $C_1' = C_1^0 \oplus A[c(\mathfrak{P}^p \mathfrak{P}^1)]$. Let m_0 be the A -map defined by $m_0(1) = c(\mathfrak{P}^p)$ and m_1 the A -map defined as follows:

$$\begin{aligned} m_1[c\mathfrak{P}^1] &= [c(\mathfrak{P}^p \mathfrak{P}^1)] \\ m_1[c\Delta] &= c(\Delta \mathfrak{P}^{p-1})[c\mathfrak{P}^1] + c\mathfrak{P}^p[c\Delta]. \end{aligned}$$

Then we have the following commutative diagram

$$\begin{array}{ccc} A & \xleftarrow{d} & A[c\mathfrak{P}^1] \oplus A[c\Delta] \\ m_0 \downarrow & & \downarrow m_1 \\ A & \xleftarrow{d'} & A[c(\mathfrak{P}^p \mathfrak{P}^1) \oplus A[c\mathfrak{P}^1] \oplus A[c\Delta]]. \end{array}$$

Consider $m_1 V_0 = m_1 \{c(R)[c\mathfrak{P}^1] - 2c(\mathfrak{P}^2)[c\Delta]\}$

$$= c(R)[c(\mathfrak{P}^p \mathfrak{P}^1)] - 2c(\mathfrak{P}^2)c(\Delta \mathfrak{P}^{p-1})[c\mathfrak{P}^1] - 2c(\mathfrak{P}^2)c(\mathfrak{P}^p)[c\Delta]$$

$$\begin{aligned}
&= c(R) [c(\mathfrak{P}^p \mathfrak{P}^1)] + 2c(\mathfrak{P}^{p+2} - \mathfrak{P}^{p+1} \mathfrak{P}^1) [c\Delta] \\
&= \lambda.
\end{aligned}$$

On the other hand, Φ_λ is a uniquely determined operation. Therefore our lemma is proved by utilizing Theorem 1.1 (4).

Now, we shall prove the following proposition by making an essential use of Lemma 7.5 and Theorem 4.3.

PROPOSITION 7.2. $\Phi_\lambda y^{pn} = n(n-1)y^{pn+(p+2)(p-1)} \pmod{\text{zero}}$.

Proof. The computation of $\Phi_\lambda y^{pn}$ is performed as follows:

$$\begin{aligned}
\Phi_\lambda y^{pn} &= \Phi_0^{(1)} c(\mathfrak{P}^p) y^{pn} = -\Phi_0^{(1)} \mathfrak{P}^p y^{pn} \\
&= -n\Phi_0^{(1)} y^{p(n+p-1)} \\
&= n(n-1) y^{pn+(p+2)(p-1)}.
\end{aligned}$$

Next we shall give the effect of $\Phi_{-1,1}$ for y^{p^2n} . For our purpose, we shall first compute $\Phi_{-1,1} y^{p^2(p-1)}$. By $\mathfrak{P}^{p(p-1)} y^{p(p-1)} = y^{p^2(p-1)}$, by Lemma 2.4, and $M'(0) y^{p(p-1)} = 0$ we have

$$\begin{aligned}
(7.16) \quad \Phi_{-1,1} y^{p^2(p-1)} &= \Phi_{-1,1} \mathfrak{P}^{p(p-1)} y^{p(p-1)} \\
&= \Phi_{-1,1} c(\mathfrak{P}^{p(p-1)}) y^{p(p-1)}
\end{aligned}$$

Making use of Proposition 7.3, we have

$$\begin{aligned}
(7.17) \quad \Phi_{-1,1} c(\mathfrak{P}^{p(p-1)}) y^{p(p-1)} &= \frac{1}{2} \sum_{i=0}^{p-2} \frac{(-1)^{i+1}}{i+1} c\mathfrak{P}(p(p-2) + p - i - 2, i) \Phi_\lambda y^{p(p-1)} \\
&+ \sum_{i=1}^{p-2} \frac{(-1)^i}{i(i+1)} c\mathfrak{P}(p(p-1) + p - i - 2, i) - \frac{1}{2} c\mathfrak{P}(p(p-1) + p - 2) \\
&\times \Phi_0^{(1)} y^{p(p-1)} + \lambda_1 \mathfrak{P}^{p^2} y^{p(p-1)} \pmod{\text{zero}}.
\end{aligned}$$

By Proposition 7.6 we have $\Phi_\lambda y^{p(p-1)} = 2y^{2(p+1)(p-1)}$. Moreover, we have

$$\begin{aligned}
c\mathfrak{P}(p(p-2) + p - i - 2, i) &= c\mathfrak{P}(p - i - 2, p(p-2), i) \\
&= c(\mathfrak{P}^i) c(\mathfrak{P}^{p(p-2)}) c(\mathfrak{P}^{p-i-2}) \\
&= -\mathfrak{P}^i c(\mathfrak{P}^{p(p-2)}) \mathfrak{P}^{p-i-2}.
\end{aligned}$$

$$\begin{aligned}
\text{Thus } c\mathfrak{P}(p(p-2) + p - i - 2, i) y^{2(p+1)(p-1)} &= (-1)^{i+1} (i+1) \mathfrak{P}^i c(\mathfrak{P}^{p(p-2)}) y^{(3p-i)(p-1)}.
\end{aligned}$$

Here we have

$$\begin{aligned}
(7.18) \quad \mathfrak{P}^i c(\mathfrak{P}^{p(p-2)}) y^{(3p-i)(p-1)} &= -\mathfrak{P}^i \mathfrak{P}^{p(p-2)} y^{(3p-i)(p-1)} \\
&= -\binom{2p-i-3}{p-2} \mathfrak{P}^i y^{(p^2+p-i)(p-1)}
\end{aligned}$$

by successive applications of

$$\begin{aligned}
-j\mathfrak{P}^j c(\mathfrak{P}^{p(p-j)}) y^{lp-j} &= \mathfrak{P}^j c(\mathfrak{P}^p \mathfrak{P}^{p(p-j-1)}) y^{lp-j} \\
&= \mathfrak{P}^j c(\mathfrak{P}^{p(p-j-1)}) c(\mathfrak{P}^p) y^{lp-j} \quad (0 < j < p).
\end{aligned}$$

Therefore, if $i \neq p-2$, the right-hand side of (7.18) is equal to zero. Continuing the computation, we have

$$\frac{1}{2} \sum_{i=0}^{p-1} \frac{(-1)^{i+1}}{i+1} c\mathfrak{P}(p(p-2) + p - i - 2, i) \Phi_\lambda y^{p(p-1)} = y^{p(p-1)(p+1)}.$$

In the same manner as above, we can obtain the result that the second term of the right-hand side in (7.17) is equal to zero, and that $\mathfrak{P}^{p^2}y^{p(p-1)}=0$. Finally we have

$$(7.19) \quad \Phi_{-1,1}y^{p^2(p-1)}=y^{p(p-1)(p+1)}.$$

Then we have the first part of Theorem 5.2 (see Proof of Theorem 5.2). From this theorem we can easily deduce Theorem 4.3₁:

$$(7.20) \quad \Phi_{-1,1}y^{p^2n}=-ny^{p(pn+p-1)} \pmod{\text{zero}},$$

by operating the both sides of Theorem 5.2 ($k=1$) on y^{p^2n} .

Now, we shall prove Theorem 5.4_k by induction on k .

Assume that our theorem is proved for $j < k$, and notice that we have then for $j < k-1$

$\Phi_{-1,j}y^{p^kn}=0$ without any modulus. Our first object will be

$$(7.21) \quad \Phi_{-1,k}p^{p^{k-1}(p-1)}=y^{p^k(p-1)(p-1)}$$

Now we have $\mathfrak{P}^{p^k(p-1)}y^{p^k(p-1)}=y^{p^{k+1}(p-1)}$, and $M'(k-1)y^{p^k(p-1)}=0$. So we have

$$(7.22) \quad \begin{aligned} \Phi_{-1,k}y^{p^{k+1}(p-1)} &= \Phi_{-1,k}\mathfrak{P}^{p^k(p-1)}y^{p^k(p-1)} \\ &= \Phi_{-1,k}c(\mathfrak{P}^{p^k(p-1)})y^{p^k(p-1)}. \end{aligned}$$

Hence, making use of Theorem 4.3, the hypothesis of induction and Proposition 7.1 ($k \geq 3$) or Proposition 7.2 ($k=2$), we have

$$(7.23) \quad \begin{aligned} \text{for } k=2 \quad \Phi_{-1,2}y^{p^3(p-1)} &= \left\{ \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c\mathfrak{P}(p^2(p-1)+p(p-i-1), i) \right. \\ &\quad \left. + \sum_{i,0 \leq j < p} (-1)^{j+1}(j+1) \binom{j}{i} c\mathfrak{P}(p^2(p-1)+p(p-i-2)+p-j, ip+j) \right\} \\ &\quad \times \Phi_{-1,k-1}y^{p^2(p-1)} + \lambda_k \mathfrak{P}^3 y^{p^2(p-1)} \pmod{\text{zero}}, \\ (7.24) \quad \text{for } k \geq 3 \quad \Phi_{-1,k}y^{p^{k+1}(p-1)} &= \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c\mathfrak{P}(p^k(p-1)+p^{k-1}(p-i-1), ip^{k-1}) \Phi_{-1,k-1}y^{p^k(p-1)} \\ &\quad + \lambda_k \mathfrak{P}^{p^{k+1}} y^{p^k(p-1)} \pmod{\text{zero}}. \end{aligned}$$

Now, let us compute as follows.

$$(7.25) \quad \begin{aligned} c\mathfrak{P}(p^2(p-1)+p(p-i-2)+p-j, ip+j) \Phi_{-1,k-1}y^{p^k(p-1)} \\ &= c\mathfrak{P}(p-j, p^2(p-1)+p(p-i-2), ip+j) \Phi_{-1,k-1}y^{p^k(p-1)} \\ &= (-1)^j c\mathfrak{P}(p^2(p-1)+p(p-i-2), ip+j) \mathfrak{P}^{p-j} y^{p^{k-1}(p+1)(p-1)} \\ &= 0 \quad (p > j > 0). \end{aligned}$$

Thus we have for $k \geq 2$

$$(7.26) \quad \begin{aligned} \Phi_{1,k}y^{p^{k+1}(p-1)} &= \sum_{i=1}^{p-1} \frac{(-1)^{i+1}}{i} c\mathfrak{P}(p^k(p-1)+p^{k-1}(p-i-1), ip^{k-1}) \\ &\quad \times \Phi_{-1,k-1}y^{p^k(p-1)} + \lambda_k \mathfrak{P}^{p^{k+1}} y^{p^k(p-1)} \pmod{\text{zero}}. \end{aligned}$$

The second term of the right-hand side in (7.26) is obviously zero. Furthermore, we have by the hypothesis of induction, Lemma 2.3, Lemma 2.4 and $M'(k-2)y^{p^{k-1}n}=0$

$$\begin{aligned}
& c\mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1))\Phi_{-1, k-1} y^{p^k(p-1)} \\
&= c(\mathfrak{P}^{p^k-1(p-i-1)}\mathfrak{P}^{p^k(p-1)}) y^{p^{k-1}(p+1)(p-1)} \\
&= c\mathfrak{P}^{p^k(p-1)} c\mathfrak{P}^{p^{k-1}(p-i-1)} y^{p^{k-1}(p+1)(p-1)} \\
&= (-1)^i c\mathfrak{P}^{p^k(p-1)}\mathfrak{P}^{p^{k-1}(p-i-1)} y^{p^{k-1}(p^2-1)} \\
&= -c\mathfrak{P}^{p^k(p-1)} y^{p^{k-1}(2p-i)(p-1)}.
\end{aligned}$$

Next, by the same device used in the proof of (7.18) we can obtain

$$\begin{aligned}
& c\mathfrak{P}(p^k(p-1) + p^{k-1}(p-i-1), i p^{k-1})\Phi_{-1, k-1} y^{p^k(p-1)} \\
&= c\mathfrak{P}^{i p^{k-1}}\mathfrak{P}^{p^k(p-1)} y^{p^{k-1}(2p-i)(p-1)} \begin{cases} = 0 & 0 < i \neq p-1 \\ = y^{p^k(p+1)(p-1)} & i = p-1 \end{cases}
\end{aligned}$$

Therefore, we have finally

$$\Phi_{-1, k} y^{p^{k+1}(p-1)} = y^{p^k(p+1)(p-1)} \pmod{\text{zero}}.$$

To compute $\Phi_{-1, k} y^{p^{k+1}} n$ for general n , we can utilize the following expression of $\mathfrak{P}^{p^{k+1}}$ in terms of secondary operations (see the proof of Theorem 5.6).

$$\begin{aligned}
(7.27) \quad \mathfrak{P}^{p^{k+1}} &\equiv c(\Delta)\Phi_k^{(2)} - c(\Delta\mathfrak{P}^{p^k(p-1)-1})\Phi_{0, k} - c(\mathfrak{P}^{p^k(p-1)})\Phi_{-1, k} \\
&\pmod{\sum_{j < k} A\Phi_{i, j} + \sum_{\substack{0 \leq i \leq 2 \\ j < k}} A\Phi_j^{(i)}}.
\end{aligned}$$

Thus, in a similar way as in the proof of (7.20), our theorem is proved.

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