On the mean-value theorem in an algebraic number field

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[1]

Vinogradov's mean value theorem is very deep and fundamental in some problems of the analytic number theory ^{[1],[3],[4]}. O. Krner succeeded in dealing with this theorem in an algebraic number field, and he extended the generalized Waring problem for polynom sums to the case of number fields^[6]. Using the Karacuba-Korobov^[5], we shall prove, in the present paper, the following main theorem.

Let K be an algebraic number field of degree n . Let $K^{(l)} (1 \leq l \leq n_1)$ be n_1 real conjugate fields and $K^{(m)}, K^{(m+n_2)} (n_1+1 \leq m \leq n_1+n_2)$ be n_2 pairs of complex conjugate fields, so that $n_1+2n_2=n$. We denote by $\gamma^{(i)} (1 \leq i \leq n)$ the conjugates of $\gamma \in K$. If $\xi =$

$\sum_{\nu=1}^n \gamma_{\nu} x_{\nu}$, then we define

$$\xi^{(i)} = \sum_{\nu=1}^n \gamma_{\nu}^{(i)} x_{\nu}, \quad (1 \leq i \leq n)$$

and

$$||\gamma|| = \text{Max}_{1 \leq i \leq n} |\gamma^{(i)}|, \quad ||\xi - \gamma|| = \text{Max}_{1 \leq i \leq n} |\xi^{(i)} - \gamma^{(i)}|.$$

A number γ of K is called totally positive or totally non-negative according to

$$\gamma^{(l)} > 0 \quad \text{or} \quad \gamma^{(l)} \geq 0, \quad (1 \leq l \leq n_1)$$

respectively. If γ is a totally non-negative integer of K and satisfies $||\gamma|| \leq T$ for some positive T , then we write

$$0 < \gamma < T.$$

Let $\mathfrak{o}$ be the integral domain consisting of all algebraic integers in K . We use a letter c (and similarly $c_0, c_1, \dots$) to denote a positive constant depending only on K , not necessarily the same each time it occurs. The symbol

$$f(x) = o(g(x)) \text{ for } g(x) \neq 0$$

means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$. $[x]$ is the greatest integer in x . As usual, $a|b$ means

that a divides b , and $a \nmid b$ means that a does not divide b . $a^i | b$ means that $a^i | b$ but $a^{i+1} \nmid b$. Let $k \geq 2$.

Let $N_s(P; \nu_1, \nu_2, \dots, \nu_k) = N_s^*(Q; \nu_1, \nu_2, \dots, \nu_k)$, where $Q^n = P$, be the number of solutions of the system of k Diophantine equations

$$\begin{cases} \lambda_1 + \lambda_2 + \dots + \lambda_s = \mu_1 + \mu_2 + \dots + \mu_s + \nu_1, \\ \lambda_1^2 + \lambda_2^2 + \dots + \lambda_s^2 = \mu_1^2 + \mu_2^2 + \dots + \mu_s^2 + \nu_2, \\ \dots\dots\dots \\ \lambda_1^k + \lambda_2^k + \dots + \lambda_s^k = \mu_1^k + \mu_2^k + \dots + \mu_s^k + \nu_k, \end{cases}$$

with the conditions

$$0 < \lambda_j, \mu_j < Q, \quad \lambda_j, \mu_j \in \mathfrak{o} \quad (1 \leq j \leq s),$$

where $k \geq 2$ and $\nu_1, \nu_2, \dots, \nu_k$ are arbitrary fixed integers in $\mathfrak{o}$. Let $N_s(P; 0, 0, \dots, 0) = N_s(P) = N_s^*(Q)$, $Q^n = P$.

MAIN THEOREM. Let $0 < \theta \leq 1$ and $k \geq 2$. k denotes a positive integer. R is a "sufficiently large" positive number with the condition that

$$(1) \quad R > \text{Max}(k, (2^n \theta^{-1})^{1/k}).$$

$Q = Q_0$ is a positive real number with the condition that

$$(2) \quad (2R)^{k(1+\frac{1}{k-1})^{\tau-1}} < Q = Q_0$$

with a positive integer $\tau \geq 1$. Let $Q_0^n = P_0 = P$. Further, let s be a positive integer with

$$(3) \quad s \geq \frac{1}{2}k(k+1) + \tau k - 1.$$

Then we have the following estimate:

$$N_s(P) \leq c^{(k^2+2s+\frac{(\tau-1)s}{k}-\frac{1}{4}(\tau-1)(k+1))\tau n} (1+2\theta)^{2s\tau+\frac{1}{k}\tau(\tau-1)s-\frac{1}{4}\tau(\tau-1)(k+1)} \\ \times s^{2k\tau} \cdot P^{2s-\frac{1}{2}k(k-1)+\delta}$$

where

$$\delta = \delta(k) = \frac{1}{2}k(k+1)(1-1/k)^\tau.$$

This result can be described as the following form:

$$N_s(P) \leq c^{(k^2+2s+\frac{(\tau-1)s}{k}-\frac{1}{2}(\tau-1)(k+1))\tau(n+1)} s^{2k\tau} \\ \times P^{2s-\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)(1-1/k)^\tau}.$$

I should like to express my indebtedness and gratitude to Professor Tikao Tatuzaawa, now at University of Tokyo.

[2] *Lemmata.*

Lemma 1. If we denote by $\pi_K(x)$ the number of prime ideals $\mathfrak{p}$ of K , satisfying the condition $N\mathfrak{p} \leq x$, then we have

$$\pi_K(x) = \sum_{N\mathfrak{p} \leq x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

where $R \leq x$.

PROOF. This is the prime ideal theorem^[7].

Lemma 2. Let R be a sufficiently large positive number. If for any real positive number ε , we have

$$x, \quad \frac{1}{1+\varepsilon}x \geq R,$$

then there exists at least one prime ideal $\mathfrak{p}$ in K such that

$$x > N\mathfrak{p} \geq \frac{1}{1+\varepsilon}x.$$

PROOF. By Lemma 1 we have

$$\pi_K(x) - \pi_K\left(\frac{1}{1+\varepsilon}x\right) = \sum_{\frac{1}{1+\varepsilon} \leq N\mathfrak{p} < x} 1 \\ = \frac{x}{\log x} - \left(\frac{1}{1+\varepsilon}x\right) / \log\left(\frac{1}{1+\varepsilon}x\right) + o\left(\frac{x}{\log x}\right) \\ = \frac{\varepsilon}{1+\varepsilon} \frac{x}{\log x} (1+o(1)) > 0,$$

our lemma is accordingly proved.

Lemma 3. Let $\mathfrak{a}$ be an integral ideal of K . Then we can take a basis $[\alpha_1, \alpha_2, \dots, \alpha_n]$ of $\mathfrak{a}$ such that

$$c_1 \sqrt[n]{N\mathfrak{a}} \leq |\alpha_{(j)}^i| \leq c_2 \sqrt[n]{N\mathfrak{a}} \quad (1 \leq i, j \leq n).$$

PROOF [8]. Let $\mathfrak{C}$ be the ideal class containing $\mathfrak{a}$. Then $\mathfrak{a}$ is a product of a fixed representative ideal $\mathfrak{b}$ in $\mathfrak{C}$ and a number ρ of K : Let $[\gamma_1, \gamma_2, \dots, \gamma_n]$ be a basis of $\mathfrak{b}$, then $[\alpha_1, \alpha_2, \dots, \alpha_n], \alpha_i = \rho \gamma_i (1 \leq i \leq n)$ is a basis of $\mathfrak{a}$.

This lemma is trivial in the rational number field and imaginary quadratic number fields. We accordingly assume that K is the other field.

If $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) (r = n_1 + n_2 - 1)$ is a system of fundamental units, there exists a system of real numbers $(x_1, x_2, \dots, x_r)$ such that

$$\frac{|\rho^{(j)}|}{\sqrt[n]{|N\rho|}} = \exp\left(\sum_{\nu=1}^r x_\nu \log |\varepsilon_\nu^{(j)}|\right) \quad (1 \leq j \leq n).$$

Multiplying ρ by a suitable unit, we may assume the inequality $0 \leq x_\nu < 1 (1 \leq \nu \leq n)$. Then we obtain

$$|\rho^{(j)}| \leq \sqrt[n]{|N\rho|} \exp\left(\sum_{\nu=1}^r |\log |\varepsilon_\nu^{(j)}||\right) \leq c_3 \sqrt[n]{|N\rho|}$$

and

$$|\rho^{(j)}| \geq \sqrt[n]{|N\rho|} \exp\left(-\sum_{\nu=1}^r |\log |\varepsilon_\nu^{(j)}||\right) \geq c_4 \sqrt[n]{|N\rho|}$$

with positive constants c_3 and c_4 depending on K alone. Therefore we have

$$(4) \quad c_3 \sqrt[n]{|N\rho|} \leq |\rho^{(j)}| \leq c_4 \sqrt[n]{|N\rho|} \quad (1 \leq j \leq n).$$

Further, let

$$(5) \quad \begin{cases} \text{Max}_{1 \leq i, j \leq n} |\gamma_{(j)}^i| = M \sqrt[n]{N\mathfrak{b}} \\ \text{Min}_{1 \leq i, j \leq n} |\gamma_{(j)}^i| = m \sqrt[n]{N\mathfrak{b}}. \end{cases}$$

Then from (4) and (5) we obtain

$$|\alpha_{(j)}^i| = |\rho^{(j)} \gamma_{(j)}^i| \leq c_3 \sqrt[n]{|N\rho|} \cdot M \sqrt[n]{N\mathfrak{b}} = (c_3 M) \sqrt[n]{N\mathfrak{b}} \quad (1 \leq i, j \leq n)$$

and

$$|\alpha_{(j)}^i| \geq (c_4 m) \sqrt[n]{N\mathfrak{b}} \quad (1 \leq i, j \leq n).$$

Because of the finiteness of the class number, we have the proof.

Lemma 4. Let $\mathfrak{p}$ be a prime ideal of K . Then there exists an integer $\pi \in \mathfrak{o}$ with the conditions that

$$\mathfrak{p} \mid \pi$$

$$c_1 \sqrt[n]{N\mathfrak{p}} \leq |\pi^{(j)}| \leq c_2 \sqrt[n]{N\mathfrak{p}} \quad (1 \leq j \leq n).$$

PROOF. Let $[\pi_1, \pi_2, \dots, \pi_n]$ be a basis of $\mathfrak{p}$. By lemma 3 we may assume that

$$c_1 \sqrt[n]{N\mathfrak{p}} \leq |\pi_{(j)}^i| \leq c_2 \sqrt[n]{N\mathfrak{p}} \quad (1 \leq i, j \leq n)$$

with positive constants c_1 and c_2 depending only on K . Here we have at least one $\pi_i (1 \leq i \leq n)$ such that $\pi_i \notin \mathfrak{p}^2$. If otherwise, let $\pi_1, \pi_2, \dots, \pi_n \in \mathfrak{p}^2$, then we have $\mathfrak{p}^2 \mid \mathfrak{p} = [\pi_1, \dots, \pi_n]$. Hence we have π_i such that

$$\pi_i \in \mathfrak{p}, \pi_i \notin \mathfrak{p}^2.$$

Hence the lemma follows by taking $\pi = \pi_i$.

Lemma 5. Let $\mathfrak{a}$ be an integral ideal of K and $\mathfrak{C}$ be an arbitrary residue class mod $\mathfrak{a}$. Then we can take an integral number $\lambda \in \mathfrak{o}$ such that

$$\lambda \in \mathfrak{G}, \quad |\lambda^{(j)}| < c_1 \sqrt[n]{N\mathfrak{a}}. \quad (1 \leq j \leq n)$$

with a positive constant c_1 depending on K alone.

PROOF. Let $[\alpha_1, \alpha_2, \dots, \alpha_n]$ be a basis of $\mathfrak{a}$ with the conditions of lemma 3, i.e.,

$$c_2 \sqrt[n]{N\mathfrak{a}} \leq |\alpha^{(i)}_{(j)}| \leq c_3 \sqrt[n]{N\mathfrak{a}} \quad (1 \leq i, j \leq n).$$

Let λ_0 be an element of $\mathfrak{G}$. Further, let $(x_1, x_2, \dots, x_n)$ be a real solution satisfying the system of equations

$$\alpha^{1}_{(p)} x_1 + \dots + \alpha^{n}_{(p)} x_n = \lambda^{0}_{(p)} \quad (1 \leq p \leq n_1)$$

$$R(\alpha^{1}_{(n_1+q)}) x_1 + \dots + R(\alpha^{n}_{(n_1+q)}) x_n = R(\lambda^{0}_{(n_1+q)})$$

$$I(\alpha^{1}_{(n_1+q)}) x_1 + \dots - I(\alpha^{n}_{(n_1+q)}) x_n = I(\lambda^{0}_{(n_1+q)}) \quad (n_1+1 \leq q \leq n_1+n_2).$$

Then, for this real solution, we have

$$\alpha^{1}_{(i)} x_1 + \dots + \alpha^{n}_{(i)} x_n = \lambda^{0}_{(i)} \quad (1 \leq i \leq n).$$

If we let

$$\alpha = \alpha_1[x_1] + \alpha_2[x_2] + \dots + \alpha_n[x_n],$$

and

$$\lambda_0 - \alpha = \lambda,$$

α is an element of $\mathfrak{a}$, and

$$\lambda_0 \equiv \lambda \pmod{\mathfrak{a}}$$

satisfying the inequalities of our lemma. Therefore we obtain the proof.

Let R be a sufficiently large positive number satisfying the conditions such that

- i) $R > \text{Max}(k, (2^n \theta^{-1})^{1/k}), (0 < \theta \leq 1)$,
- ii) We obtain the prime ideal theorem (lemma 1).

Obviously we have

$$-\frac{1}{1+\theta} P_0^{1/k} > \frac{1}{1+\theta} (2R)^{n(1+\frac{1}{k-1})^{\tau-1}} > \frac{2}{1+\theta} R^n > R.$$

Hence, by lemma 2 and (1), we obtain a prime ideal $\mathfrak{p}_1$ such that

$$(6) \quad -\frac{1}{1+\theta} P_0^{1/k} \leq N\mathfrak{p}_1 < P_0^{1/k}.$$

Furthermore, by lemma 4, we obtain an integral number $\pi_1 \in \mathfrak{o}$ such that

$$(7) \quad \mathfrak{p}_1 || \pi_1$$

$$(8) \quad c_{1,1} N\mathfrak{p}_1^{1/n} < |\pi^{(j)}_1| < c_{1,2} N\mathfrak{p}_1^{1/n} \quad (1 \leq j \leq n)$$

with positive constants $c_{1,1}$ and $c_{1,2}$ depending only on K . Here we may assume that

$$(9) \quad 0 < c_{1,1} \leq 1, \quad 1 \leq c_{1,2}.$$

Let

$$(10) \quad Q_1 = \left[\frac{2}{c_{1,1}} \left(\frac{P_0}{N\mathfrak{p}_1} \right)^{1/n} + 1 \right],$$

and $Q_1^n = P_1$.

Then obviously we have

$$(11) \quad \left(\frac{2}{c_{1,1}} \right)^n \frac{P_0}{N\mathfrak{p}_1} < P_1 \leq \left(\left(\frac{2}{c_{1,1}} \right) \left(\frac{P_0}{N\mathfrak{p}_1} \right)^{1/n} + 1 \right)^n.$$

Using P_1, Q_1 and $\mathfrak{p}_1$, let us define inductively P_ν, Q_ν and $\mathfrak{p}_\nu$ ($2 \leq \nu \leq k$).

Let $\mathfrak{p}_\nu$ be a prime ideal, satisfying the condition such that

$$(12) \quad -\frac{1}{1+\theta} P_{\nu-1}^{1/k} \leq N\mathfrak{p}_\nu < P_{\nu-1}^{1/k}.$$

Let π_ν be an integral number of lemma 4, satisfying the conditions such that

$$(13) \quad \mathfrak{p}_\nu || \pi_\nu,$$

$$(14) \quad c_{\nu,1} N\mathfrak{p}_\nu^{1/n} < |\pi^{(j)}_\nu| < c_{\nu,2} N\mathfrak{p}_\nu^{1/n}$$

with positive constants $c_{\nu,1}$ and $c_{\nu,2}$ depending only on K . We may assume that

$$(15) \quad 0 < c_{\nu,1} \leq 1, \quad 1 \leq c_{\nu,2}.$$

Let P_ν, Q_ν be positive integers, where

$$(16) \quad \left[\frac{2}{c_{\nu,1}} \left(\frac{P_{\nu-1}}{N\mathfrak{p}_\nu} \right)^{1/n} + 1 \right] = Q_\nu, Q_\nu^n = P_\nu.$$

Finally let

$$(17) \quad \begin{cases} \tilde{c}_1 = \text{Min}(c_{1,1}, c_{2,1}, \dots, c_{\tau,1}), \\ \tilde{c}_2 = \text{Max}(c_{1,2}, c_{2,2}, \dots, c_{\tau,2}), \end{cases}$$

then we obtain

$$(18) \quad 0 < \tilde{c}_1 \leq 1, \quad 1 \leq \tilde{c}_2.$$

Lemma 6. For all $\nu (1 \leq \nu \leq \tau)$ we obtain as follows:

$$(19) \quad N\mathfrak{p}_\nu > k^n,$$

$$(20) \quad \frac{1}{1+\theta} P_{\nu-1/k} > R,$$

$$(21) \quad N\mathfrak{p}_\nu \cdot P_\nu > P_{\nu-1},$$

$$(22) \quad P_\nu \leq \left(\frac{2}{c_1} \right)^{\nu n} (1+2\theta)^\nu P_0^{(1-\frac{1}{k})^\nu}$$

$$(23) \quad P_\nu \geq P_0^{(1-\frac{1}{k})^\nu}$$

REMARK. By (20)~(23) and lemma 2, we know the existence of the ideals $\mathfrak{p}_\nu (2 \leq \nu \leq \tau)$.

PROOF OF LEMMA 6.

i) Proof of (21). By (10) we know

$$\left(\frac{2}{c_{\nu,1}} \right)^n \left(\frac{P_{\nu-1}}{N\mathfrak{p}_\nu} \right) < P_\nu \leq \left(\frac{2}{c_{\nu,1}} \left(\frac{P_{\nu-1}}{N\mathfrak{p}_\nu} \right)^{1/n} + 1 \right)^n$$

and from the hypothesis $2/c_{\nu,1} > 1$ we obtain

$$\frac{P_{\nu-1}}{N\mathfrak{p}_\nu} < P_\nu \text{ for all } \nu (1 \leq \nu \leq \tau). \quad \text{q.e.d.}$$

ii) Proof of (23). When $\nu=1$, this result is obviously true:

$$P_1 \geq \left(\frac{2}{c_{1,1}} \right)^n \frac{P}{N\mathfrak{p}_1} \geq P_0^{1-\frac{1}{k}}.$$

Suppose that (23) is true for $\nu \leq i-1 (i \geq 2)$. By (12) and (16) and the hypothesis of the above, we obtain

$$\begin{aligned} P_i &\geq \left(\frac{2}{c_{i,1}} \right)^n \left(\frac{P_{i-1}}{N\mathfrak{p}_i} \right) \geq P_{i-1}^{1-\frac{1}{k}} \\ &\geq P_0^{(1-\frac{1}{k})^i}, \end{aligned}$$

and therefore (23) is proved.

iii) Proof of (19) and (20). By (12) and (16), we obtain

$$\begin{aligned} N\mathfrak{p}_\nu &\geq \frac{1}{1+\theta} P_{\nu-1/k} \geq \frac{1}{1+\theta} \left(\frac{2}{c_{\nu-1,1}} \right)^{n/k} \left(\frac{P_{\nu-2}}{N\mathfrak{p}_{\nu-1}} \right)^{1/k} \\ &\geq \frac{1}{1+\theta} P_{\nu-2}^{\frac{1}{k}(1-\frac{1}{k})}. \end{aligned}$$

Furthermore by (23):

$$\geq \frac{1}{1+\theta} P_0^{\frac{1}{k}(1-\frac{1}{k})^{\nu-1}}$$

By (1) and (2)

$$\begin{aligned} &\geq \frac{1}{1+\theta} \left((2R)^{nk(1+\frac{1}{k-1})^{\nu-1}} \right)^{\frac{1}{k}(1-\frac{1}{k})^{\nu-1}} \\ &= \frac{1}{1+\theta} (2R)^n \geq R^n \geq \begin{cases} R, \\ k^n. \end{cases} \end{aligned} \quad \text{q.e.d.}$$

iv) Proof of (22). By (23) and (2) we know that

$$\begin{aligned} P_{\nu-1}^{1/n} &\geq P_0^{\frac{1}{n}(1-\frac{1}{k})^\nu} \geq (2R)^{k(1+\frac{1}{k-1})^\nu(1-\frac{1}{k})^\nu} \\ &= (2R)^k. \end{aligned}$$

By the hypothesis (1) of R , we obtain

$$P_{\nu-1}^{\frac{1}{n}(1-\frac{1}{k})} \geq (2(2^n\theta^{-1})^{1/k})^k \geq (2^{n+1}\theta^{-1}).$$

Therefore from the definition of P_ν we obtain

$$\begin{aligned} P_\nu &\leq \left(\frac{2}{c_{\nu,1}} \left(\frac{P_{\nu-1}}{N\mathfrak{p}_\nu} \right)^{1/n} + 1 \right)^n \\ &\leq \left(\frac{2}{c_{\nu,1}} \right)^n \frac{P_{\nu-1}}{N\mathfrak{p}_\nu} + 2^n \left(\left(\frac{2}{c_{\nu,1}} \right) \left(\frac{P_{\nu-1}}{N\mathfrak{p}_\nu} \right)^{1/n} \right)^{n-1} \\ &\leq \left(\frac{2}{\tilde{c}_1} \right)^n P_{\nu-1}^{1-\frac{1}{k}} \left\{ (1+\theta) + 2^n (1+\theta)^{1-\frac{1}{n}} P_{\nu-1}^{-\frac{1}{n}(1-\frac{1}{k})} \right\} \\ &\leq \left(\frac{2}{\tilde{c}_1} \right)^n P_{\nu-1}^{1-\frac{1}{k}} \left\{ (1+\theta) + 2^{n+1-1/n} P_{\nu-1}^{-\frac{1}{n}(1-\frac{1}{k})} \right\} \\ &\leq \left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) P_{\nu-1}^{1-\frac{1}{k}} \leq \dots \\ &\leq \left(\frac{2}{\tilde{c}_1} \right)^{n\nu} (1+2\theta)^\nu P_0^{\left(1-\frac{1}{k}\right)^\nu} \end{aligned} \quad \text{q.e.d.}$$

Lemma 7. Let $k \geq 2$ and let $\nu_1, \nu_2, \dots, \nu_k$ be integers in $\mathfrak{o}$. Let $\mathfrak{p}$ be a prime ideal with $N\mathfrak{p} > k^n$. By lemma 4 we may take an integer $\pi \in \mathfrak{o}$ satisfying the conditions such that

$$(23) \quad \begin{aligned} &\mathfrak{p} \mid \pi, \\ &c_1 N\mathfrak{p}^{1/n} \leq |\pi^{(j)}| \leq c_2 N\mathfrak{p}^{1/n} \quad (1 \leq j \leq n) \\ &0 < c_1 \leq 1, \quad 1 \leq c_2, \end{aligned}$$

where c_1 and c_2 depend only on K .

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be integers in $\mathfrak{o}$ satisfying the conditions such that

$$(24) \quad \begin{aligned} &\lambda_i \not\equiv \lambda_j \pmod{\mathfrak{p}} \quad (i \neq j) \\ &0 < \lambda_\nu < m N\mathfrak{p}^{k/n} \quad (1 \leq \nu \leq k) \end{aligned}$$

with a positive number $m \geq 1$. Let T denote the number of solutions of the system of congruences

$$(25) \quad \begin{cases} \lambda_1 + \lambda_2 + \dots + \lambda_k \equiv \nu_1 \pmod{p} \\ \lambda_1^2 + \lambda_2^2 + \dots + \lambda_k^2 \equiv \nu_2 \pmod{p^2} \\ \dots\dots\dots \\ \lambda_1^k + \lambda_2^k + \dots + \lambda_k^k \equiv \nu_k \pmod{p^k}. \end{cases}$$

Then the following estimate holds:

$$(26) \quad T \ll k! (c_3^k m)^{nk} N p^{\frac{1}{2}k(k-1)}$$

with positive constant c_3 depending only on K .

PROOF. If λ_{ji} ($1 \leq i, j \leq k$) take from a fixed complete system of incongruent residues $\pmod{p}$, then by lemma 5, we may assume that

$$(27) \quad 0 < \lambda_{ji} < c_4 \sqrt[n]{Np} \quad (1 \leq i, j \leq k).$$

Furthermore, by (23) we may assume the following inequalities

$$(28) \quad 0 < \lambda_{ji} < c_5 \pi, \quad c_5 > 1, \quad (1 \leq i, j \leq k).$$

Let us write the number λ_j ($1 \leq j \leq k$) in the form

$$(29) \quad \lambda_j = \lambda_{j0} + \lambda_{j1}\pi + \dots + \lambda_{j,k-1}\pi^{k-1} + \lambda_{j,k}\pi^k \quad (1 \leq j \leq k)$$

with the conditions (28).

By (23) and (28), we obtain

$$\begin{aligned} |\lambda_j - \lambda_{j,k}\pi^k| &\leq c_5 |\pi| + \dots + c_5 |\pi|^k \\ &= c_5 |\pi| \frac{|\pi|^k - 1}{|\pi| - 1} < 2c_5 |\pi|^k \\ &< 2c_5 c_2^k N p^{k/n}. \end{aligned}$$

Therefore

$$\begin{aligned} |\lambda_{j,k}| |\pi^k| &\leq (|\lambda_j| + 2c_5 c_2^k N p^{k/n}) \\ &\leq m N p^{k/n} + 2c_5 c_2^k N p^{k/n}. \end{aligned}$$

Hence by (23)

$$\begin{aligned} |\lambda_{j,k}| &\leq \frac{m + 2c_5 c_2^k}{c_1^k} \leq \frac{3c_5 c_2^k}{c_1^k} m \\ &\leq c_3^k m, \end{aligned}$$

where $c_3 > 2$.

Note that in agreement with the supplementary condition (24) stated in the lemma, we have

$$(31) \quad \lambda_{i0} \equiv \lambda_{j0} \pmod{p}$$

for $i \neq j$.

It is easy to see that the numbers $\lambda_{11}, \lambda_{21}, \dots, \lambda_{n1}$ satisfy the system of congruences

$$(32) \quad \left. \begin{aligned} \lambda_{10} + \lambda_{20} + \dots + \lambda_{k0} &\equiv \nu_1 \\ \lambda_{10}^2 + \lambda_{20}^2 + \dots + \lambda_{k0}^2 &\equiv \nu_2 \\ \dots\dots\dots \\ \lambda_{10}^k + \lambda_{20}^k + \dots + \lambda_{k0}^k &\equiv \nu_k \end{aligned} \right\} \pmod{p}.$$

Denote by T_1 the number of this system. Then we have

$$(33) \quad T_1 \leq k!.$$

This result is obviously true when $k=1$. We now proceed by induction on k . Suppose that the conclusion is true for $k-1$. Let $\lambda_{k,1}$ be known and $T_1^{(k-1)}$ the number of solutions of the system of congruences

we obtain a system of $k-l+1$ congruences:

$$\nu_j \equiv \sum_{q=0}^{l-1} f_{j,q} \pi^q + j \sum \lambda_{p,0}^{j-1} \lambda_{p,l-1} \pi^{l-1} \pmod{\mathfrak{p}^l} \quad (l-1 \leq j \leq k-1),$$

where $f_{j,q}$ ($0 \leq q \leq l-1$) is a polynomial of $\lambda_{p,q}$ ($1 \leq p \leq k$, $0 \leq q \leq l-2$) with rational integral coefficients. From these congruences, we obviously obtain

$$\sum_{p=1}^k \lambda_{p,0}^{j-1} \lambda_{p,l-1} \equiv \nu_{l,j-1} \pmod{\mathfrak{p}} \quad (l-1 \leq j \leq k-1).$$

$$\nu_{0,j-1} \in \mathfrak{O}$$

Here we have used $N\mathfrak{p} > k^n$. Then, denoting by T_l the number of solutions of this system, we have similarly

$$(35) \quad T_l = N\mathfrak{p}^{l-1} \quad (1 \leq l \leq k).$$

Finally, let

$$\begin{bmatrix} \lambda_{1,0}, \dots, \lambda_{1,k-1} \\ \dots \\ \lambda_{k,0}, \dots, \lambda_{k,k-1} \end{bmatrix}$$

be known and let T_{k-1} be the number of solutions

$$\begin{bmatrix} \lambda_{1,k} \\ \vdots \\ \lambda_{k,k} \end{bmatrix}$$

of the system of equations (29). Obviously, by (30) we obtain

$$(36) \quad T_{k-1} \leq (c_3^k m)^{nk}.$$

By (33), (35) and (36), we have

$$\begin{aligned} T &\leq T_1 T_2 \dots T_k T_{k+1} \\ &\leq k! N\mathfrak{p}^{1+2+\dots+k} (c_3^k m)^{nk} \\ &\leq k! (c_3^k m)^{nk} N\mathfrak{p}^{\frac{1}{2}k(k+1)}. \end{aligned}$$

The lemma is proved.

[3] *Proof of the main theorem.* By lemma 4 and lemma 5, we may assume that

$$\lambda_i = \varphi_i + \xi_i \pi_1, \quad \mu_i = \phi_i + \eta_i \pi_1 \quad (1 \leq i \leq s)$$

and

$$(37) \quad 0 < \varphi_i, \phi_i < c_1 \pi_1$$

$$(38) \quad c_2 \sqrt[N]{N\mathfrak{p}_1} \leq |\pi^{(j)}| \leq c_3 \sqrt[N]{N\mathfrak{p}_1}, \quad \mathfrak{p}_1 \nmid \pi_1,$$

where φ_i and ϕ_i run separately through a complete set of residues mod $\mathfrak{p}_1$ and are decided uniquely by λ_i and μ_i respectively.

By (1) and (2), we obtain

$$(39) \quad (\varphi_1 + \xi_1 \pi_1)^\nu + \dots + (\varphi_s + \xi_s \pi_1)^\nu = (\phi_1 + \eta_1 \pi_1)^\nu + \dots + (\phi_s + \eta_s \pi_1)^\nu, \quad (1 \leq \nu \leq k),$$

where

$$(40) \quad 0 < \varphi_i + \xi_i \pi_1, \quad \phi_i + \eta_i \pi_1 < Q, \quad (1 \leq i \leq s)$$

We shall put the system

$$\{\varphi_1, \dots, \varphi_s\}, \quad \{\phi_1, \dots, \phi_s\}$$

in the first class if we can find k distinct φ_i, ϕ_i . All the remaining systems $\{\varphi_1, \dots, \varphi_s\}, \{\phi_1, \dots, \phi_s\}$ will be placed in the second class.

Since, by definition of $\mathfrak{d}$:

$$\mathfrak{d}^{-1} = \{\rho; E(\rho\xi) = 1 \text{ for every } \xi \in \mathfrak{o}\}$$

we obtain

$$N_s^{(1)} = N_s^{(1)}(N\mathfrak{p}_1 P_1) = \int_{U_1} \cdots \int_{U_k} \left| \sum_{\varphi} S(\varphi) \right|^{2s} dy_1 \cdots dy_k,$$

where $\sum_{\varphi}$ runs through a complete system of residues mod $\mathfrak{p}_1$.

Hence we have

$$\begin{aligned} N_s^{(1)} \leq & \left(\frac{s}{k} \right)^2 \int_{U_1} \cdots \int_{U_k} \sum_{\varphi_1} \cdots \sum_{\varphi_k} \sum_{\phi_1} \cdots \sum_{\phi_k} S(\varphi_1) \cdots S(\varphi_k) \overline{S(\phi_1)} \cdots \overline{S(\phi_k)} \\ & \times \sum_{\varphi} |S(\varphi)|^{2(s-k)} dy_1 \cdots dy_k. \end{aligned}$$

By Hölder's inequality we have

$$(42) \quad N_s^{(1)} \leq \left(\frac{s}{k} \right)^2 N\mathfrak{p}_1^{2(s-k)-1} \int_{U_1} \cdots \int_{U_k} A(\alpha_1, \dots, \alpha_k) dy_1 \cdots dy_k,$$

where

$$A(\alpha_1, \dots, \alpha_k)$$

$$\begin{aligned} (43) \quad & = \left| \sum_{(\varphi_1, \dots, \varphi_k)} S(\varphi_1) \cdots S(\varphi_k) \right|^{2(s-k)} \sum_{\varphi} |S(\varphi)|^{2(s-k)} \\ & = \sum_{(\varphi_1, \dots, \varphi_k)} \sum_{\varphi} \sum_{\xi_1} \cdots \sum_{\xi_k} \prod_{\nu=1}^k E(f(\varphi_{\nu} + \xi_{\nu} \pi_1)) \prod_{\nu=1}^k E(f(\phi_{\nu} + \eta_{\nu} \pi_1)) \\ & \quad \times \sum_{\xi_{k+1}} \cdots \sum_{\xi_s} \prod_{\nu=k+1}^s E(f(\varphi + \xi_{\nu} \pi_1)) \prod_{\nu=k+1}^s E(f(\varphi + \eta_{\nu} \pi_1)). \end{aligned}$$

Let

$$(44) \quad A(\alpha_1, \dots, \alpha_k) = \sum_{\varphi_{\nu}, \psi_{\nu}} \cdots \sum_{\varphi} \sum_{\xi_{\nu}} \cdots \sum_{\eta_{\nu}} \sum_{\xi_{\nu}} \cdots \sum_{\eta_{\nu}} E(B).$$

Then we have

$$(45) \quad B = \sum_{\nu=1}^k f(\varphi_{\nu} + \xi_{\nu} \pi_1) - \sum_{\nu=1}^k f(\psi_{\nu} + \eta_{\nu} \pi_1) + \sum_{\nu'=k+1}^s f(\varphi + \xi_{\nu'} \pi_1) - \sum_{\nu'=k+1}^s f(\varphi + \eta_{\nu'} \pi_1).$$

Furthermore,

$$\int_{U_1} \cdots \int_{U_k} A(\alpha_1, \dots, \alpha_k) dy_1 \cdots dy_k$$

is equal to the number of the system of equations

$$(46) \quad \sum_{\nu=1}^k (\varphi_{\nu} + \xi_{\nu} \pi_1)^p - \sum_{\nu=1}^k (\psi_{\nu} + \eta_{\nu} \pi_1)^p = \sum_{\nu'=k+1}^s (\varphi + \xi_{\nu'} \pi_1)^p - \sum_{\nu'=k+1}^s (\varphi + \eta_{\nu'} \pi_1)^p \quad (1 \leq p \leq k)$$

with the conditions (40).

For all p ($1 \leq p \leq k$), we have

$$(47) \quad \sum_{\nu=1}^k (\varphi_{\nu} + \xi_{\nu} \pi_1)^p = \sum_{l=0}^p \left\{ \sum_{\nu=1}^k (\varphi_{\nu} + \xi_{\nu} \pi_1 - \varphi_{\nu})^{p-l} \right\} \binom{p}{l} \varphi_{\nu}^l$$

and

$$(48) \quad \sum_{\nu'=k+1}^s (\varphi + \xi_{\nu'} \pi_1)^p = \sum_{l=0}^p \left\{ \left(\sum_{\nu'=k+1}^s \xi_{\nu'}^{p-l} \right) \pi_1^{p-l} \right\} \binom{p}{l} \varphi^l.$$

By (47) and (48), we find by induction that

$$(49) \quad \sum_{\nu=1}^k (\varphi_{\nu} + \xi_{\nu} \pi_1 - \varphi)^p - \sum_{\nu=1}^k (\varphi_{\nu} + \eta_{\nu} \pi_1 - \varphi)^p = \pi_1^p \sum_{\nu'=k+1}^s (\xi_{\nu'}^p - \eta_{\nu'}^p), \quad (1 \leq p \leq k).$$

Let $N'(P)$ be the number of solutions of the system of equations (49) with the conditions (40).

Letting

$$(50) \quad \sum_{\nu'=k+1}^s \xi_{\nu'}^p - \sum_{\nu'=k+1}^s \eta_{\nu'}^p = \kappa_p \quad (1 \leq p \leq k),$$

we obtain

$$(51) \quad \sum_{\nu=1}^k (\varphi_{\nu} + \xi_{\nu} \pi_1 - \varphi)^p - \sum_{\nu=1}^k (\varphi_{\nu} + \eta_{\nu} \pi_1 - \varphi)^p = \pi_1^p \kappa_p \quad (1 \leq p \leq k).$$

Let us consider the system of equations (51), where $\kappa_1, \dots, \kappa_k$ are arbitrary fixed integers in $\mathfrak{o}$.

Let $N''(\pi_1 \kappa_1, \dots, \pi^k \kappa_k)$ be the number of solutions of system (51) and let T' be the number of solutions of the system of equations

$$(52) \quad \sum_{\nu=1}^k (\varphi_{\nu} + \xi_{\nu} \pi_1 - \varphi)^p - \sum_{\nu=1}^k (\varphi_{\nu} + \eta_{\nu} \pi_1 - \varphi)^p \equiv 0 \pmod{\mathfrak{p}_1^p} \quad (1 \leq p \leq k).$$

By (37) and (38) we may assume that

$$(53) \quad \varphi_{\nu}, \psi_{\nu} < c_4 N \mathfrak{p}_1^{1/n} (c_4 > 1) \quad (1 \leq \nu \leq s).$$

Hence, by (12) and (53), we obtain

$$(54) \quad \varphi, \psi < c_4 N \mathfrak{p}_1^{1/n} < c_4 P_0^{(1/k)(1/n)} = c_4 Q_0^{1/k} < Q_0.$$

By (40) and (54) we have

$$(55) \quad |\xi \pi_1| \leq |Q_0| + |\varphi| \leq 2Q_0.$$

By (8) and (55) we have

$$(56) \quad \eta, \xi < \frac{2Q_0}{c_{1,1} N \mathfrak{p}_1^{1/n}} < Q_1 = \left[\frac{2}{c_{1,1}} - \frac{Q_0}{N \mathfrak{p}_1^{1/n}} + 1 \right].$$

Therefore by (50), (51) and (52), we obtain

$$(57) \quad \begin{aligned} N_s' &\leq \sum_{\kappa_1} \dots \sum_{\kappa_k} N_s''(\kappa_1 \pi_1, \dots, \kappa_k \pi^k) N_{s-k} \left(\frac{2Q_0}{c_{1,1} N \mathfrak{p}_1^{1/n}}; \kappa_1, \dots, \kappa_k \right) \\ &\leq \sum_{\kappa_1} \dots \sum_{\kappa_k} N_s''(\kappa_1 \pi_1, \dots, \kappa_k \pi^k) N_{s-k}(Q_1, \kappa_1, \dots, \kappa_k) \\ &\leq N_{s-k}(P_1) \sum_{\kappa_1} \dots \sum_{\kappa_k} N_s''(\kappa_1 \pi_1, \dots, \kappa_k \pi^k) \\ &= N_{s-k}(P_1) T'. \end{aligned}$$

In the congruences (52) the number of solutions of $\phi_1, \dots, \phi_k, \psi$ is at most $N \mathfrak{p}_1^{k+1}$ and the number of solutions of $\eta_1, \dots, \eta_k$ is at most P_1^k . For any fixed values of

$\phi_1, \dots, \phi_k, \psi$ and $\eta_1, \dots, \eta_k$ let

$$(58) \quad \sum_{\nu=1}^k (\phi_\nu + \eta_\nu \pi_1 - \varphi)^p = \omega_p \quad (1 \leq p \leq k)$$

and

$$(59) \quad \varphi_\nu + \xi_\nu \pi_1 - \varphi = \alpha_\nu \quad (1 \leq p \leq k).$$

Then, by (52), (58) and (59), we have

$$(60) \quad \begin{cases} \alpha_1 + \alpha_2 + \dots + \alpha_k \equiv \omega_1 \pmod{\mathfrak{p}_1} \\ \alpha_1^2 + \alpha_2^2 + \dots + \alpha_k^2 \equiv \omega_2 \pmod{\mathfrak{p}_2^1} \\ \dots\dots\dots \\ \alpha_1^k + \alpha_2^k + \dots + \alpha_k^k \equiv \omega_k \pmod{\mathfrak{p}_k^1}. \end{cases}$$

Since

$$\varphi_\nu + \xi_\nu \pi_1 = \varphi + \alpha_\nu \quad (1 \leq \nu \leq k),$$

for any fixed value of $\alpha_1, \dots, \alpha_k, \varphi_\nu$ and $\xi_\nu (1 \leq \nu \leq k)$ are determined uniquely.

By (50) we have

$$|\alpha_\nu| \leq |\varphi_\nu + \xi_\nu \pi_1| + |\varphi| \leq 2Q_0 = \frac{2Q_0}{N\mathfrak{p}_1^{k/n}} N\mathfrak{p}_1^{1/n}$$

Therefore m of lemma 7 is equal to

$$(61) \quad \begin{aligned} m &= \frac{2Q_0}{N\mathfrak{p}_1^{k/n}} = 2 \frac{Q_0}{N\mathfrak{p}_1^{1/n}} N\mathfrak{p}_1^{-\frac{k}{n} + \frac{1}{n}} \\ &= c_{1,1} \left(\frac{2}{c_{1,1}} \frac{Q_0}{N\mathfrak{p}_1^{1/n}} \right) N\mathfrak{p}_1^{-\frac{1-k}{n}}. \end{aligned}$$

Hence, by (24) and (61) we have

$$(62) \quad m < c_{1,1} Q_1 N\mathfrak{p}_1^{-\frac{k-1}{n}}.$$

By lemma 7 and (52) we obtain

$$(63) \quad \begin{aligned} T' &\leq N\mathfrak{p}_1^{k+1} P_1^k \left(k! \left(c_5^k c_{1,1} Q_1 N\mathfrak{p}_1^{-\frac{k-1}{n}} \right)^{kn} N\mathfrak{p}_1^{\frac{1}{2}k(k-1)} \right) \\ &\leq k! (c_5^k c_{1,1})^{kn} P_1^k Q_1^{kn} N\mathfrak{p}_1^{-k^2 + k + \frac{1}{2}k(k-1) + k+1} \\ &\leq k! c_6^{k^2 n} P_1^{2k} N\mathfrak{p}_1^{-\frac{1}{2}k(k+1) + 2k+1}. \end{aligned}$$

Therefore, by (42), (43), (57) and (63) we finally obtain

$$\begin{aligned} N_{s^{(1)}} &\leq \left(\frac{s}{k} \right)^2 N\mathfrak{p}_1^{2(s-k)-1} \int_{U_1} \dots \int_{U_k} A(\alpha_1, \dots, \alpha_k) dy_1 \dots dy_k \\ &= \left(\frac{s}{k} \right)^2 N\mathfrak{p}_1^{2(s-k)-1} N_{s'} \\ &\leq \left(\frac{s}{k} \right)^2 N\mathfrak{p}_1^{2(s-k)-1} T' N_{s-k}(P_1) \\ &\leq \left(\frac{s}{k} \right)^2 N\mathfrak{p}_1^{2s-2k-1} k! c_6^{k^2 n} P_1^{2k} N\mathfrak{p}_1^{-\frac{1}{2}k(k+1) + 2k+1} N_{s-k}(P_1). \end{aligned}$$

Hence

$$(64) \quad N_s^{(1)} \leq \binom{s}{k}^2 k! c_6^{k^2 n} P_1^{2k} N_{p_1}^{2s - \frac{1}{2}k(k+1)} N_{s-k}(P_1).$$

Let us now estimate $N_s^{(2)}$. By the definition of $N_s^{(2)}$ we obtain

$$(65) \quad N_s^{(2)} = \int_{U_1} \cdots \int_{U_k} C(\alpha_1, \dots, \alpha_k) dy_1 \cdots dy_k$$

with

$$(66) \quad C(\alpha_1, \dots, \alpha_k) = \sum_{\varphi_1} \cdots \sum_{\varphi_s} \sum_{\phi_1} \cdots \sum_{\phi_s} S(\varphi_1) \cdots S(\varphi_s) \overline{S(\phi_1)} \cdots \overline{S(\phi_s)},$$

where the integers $\{\varphi_1, \dots, \varphi_s\}$, $\{\phi_1, \dots, \phi_s\}$ vary in such a way that at least one of the systems $\{\varphi_1, \dots, \varphi_s\}$, $\{\phi_1, \dots, \phi_s\}$ belong to the second class and φ_ν, ϕ_ν ($1 \leq \nu \leq s$) satisfy the conditions (37), (38) and (40).

Let M be the number of the systems $\{\varphi_1, \dots, \varphi_s\}$ such that they belong to the second class. Further M_ν be the number of the systems $\{\varphi_1, \dots, \varphi_s\}$ in which we can find ν different φ_i ($1 \leq i \leq s$). Obviously we have $M_0 = N_{p_1}$, $M_1 = \binom{N_{p_1}}{2} 2^s, \dots, M_{s-1} = \binom{N_{p_1}}{k-1} (k-1)^s$. From $N_{p_1} > k^n$ ($k \geq 2$), we have

$$(67) \quad \begin{aligned} M &\leq M_0 + M_1 + \cdots + M_{k-1} \\ &\leq N_{p_1} + \binom{N_{p_1}}{2} 2^s + \cdots + \binom{N_{p_1}}{k-1} (k-1)^s \\ &\leq \binom{N_{p_1}}{k-1} k(k-1)^s \\ &\leq N_{p_1}^{k-1} k^s. \end{aligned}$$

Taking $\{\phi_1, \dots, \phi_s\}$ arbitrarily, the number of the systems $\{\phi_1, \dots, \phi_s\}$ is $N_{p_1}^s$. Hence, by (66) and (67) we have

$$(68) \quad \begin{aligned} |C(\alpha_1, \dots, \alpha_k)| &\leq N_{p_1}^{k-1} \cdot k^s \cdot N_{p_1}^s \max_{1 \leq i \leq s} |S(\varphi_i)|^{2s} \\ &\leq k^s N_{p_1}^{s+k-1} \sum_{\varphi} |S(\varphi)|^{2s} \\ &\leq k^s N_{p_1}^{s+k-1} P_1^{2k} \sum_{\varphi} |S(\varphi)|^{2(s-k)}. \end{aligned}$$

Further, by (65) and (68) we have

$$(69) \quad N_s^{(2)} \leq 2k^s P_1^{2k} N_{p_1}^{s+k-1} N_{s-k}(P_1).$$

Since by (3)

$$s \geq \frac{1}{2}k(k+1) + \tau(k-1), \quad (\tau=1)$$

it follows that

$$(70) \quad N_{p_1}^{2s - \frac{1}{2}k(k+1)} \geq N_{p_1}^{s+k-1}$$

and

$$\begin{aligned} \binom{s}{k}^2 k! &\geq (s(s-1) \cdots (s-k+1))^2 / k! \\ &\geq (s-k+1)^{2k} / k! \\ &\geq \left(\left(\frac{1}{2}(k+1) \right)^{2k} / 2k! \right) 2k^{2k} \end{aligned}$$

$$(71) \quad \geq 2k^{2k}.$$

Hence by (64),(69),(70) and (71) we have

$$N_s^{(2)} < N_s^{(1)}.$$

Therefore, by (41) and (64), we have

$$\begin{aligned} N_s(P) &\leq N_s^{(1)} + N_s^{(2)} \leq 2N_s^{(1)} \\ &\leq 2c_6^{k^2n} \left(\frac{s}{k}\right)^2 \cdot k! P_1^{2k} N_{\mathfrak{p}_1}^{2s-\frac{1}{2}k(k+1)} N_{s-k}(P_1). \end{aligned}$$

Hence

$$(72) \quad N_s(P) \leq D_1 P^{2k} N_{\mathfrak{p}_1}^{2s-\frac{1}{2}k(k+1)} N_{s-k}(P_1),$$

where

$$(73) \quad D_1 = c_6^{k^2n} \left(\frac{s}{k}\right)^2 k!, \quad c_6 > 2.$$

Applying this inequality τ times, we have

$$\begin{aligned} N_s(P) &\leq D_1 D_2 (P_1 P_2)^{2k} N_{\mathfrak{p}_1}^{2s-\frac{1}{2}k(k+1)} N_{\mathfrak{p}_2}^{2(s-k)-\frac{1}{2}k(k+1)} N_{s-2k}(P_2) \\ &\leq \dots \\ &\leq D_1 D_2 \dots D_\tau (P_1 P_2 \dots P_\tau)^{2k} (N_{\mathfrak{p}_1} N_{\mathfrak{p}_2} \dots N_{\mathfrak{p}_\tau})^{2s-\frac{1}{2}k(k+1)} \\ &\quad \times (N_{\mathfrak{p}_2} N_{\mathfrak{p}_3}^2 \dots N_{\mathfrak{p}_\tau^{\tau-1}})^{-2k} P_\tau^{2(s-\tau k)}. \end{aligned} \quad (74)$$

By lemma 6, we have

$$\begin{aligned} (P_1 P_2 \dots P_\tau)^{2k} &\leq (N_{\mathfrak{p}_2} P_2 P_2 \dots P_\tau)^{2k} \\ &\leq \dots \\ &\leq (N_{\mathfrak{p}_2} N_{\mathfrak{p}_2^3} \dots N_{\mathfrak{p}_{\tau-1}^\tau})^{2k} P_\tau^{2k\tau}. \end{aligned} \quad (75)$$

By (12) and (22), we have

$$\begin{aligned} N_{\mathfrak{p}_1} N_{\mathfrak{p}_2} \dots N_{\mathfrak{p}_\tau} &\leq (P_0 P_1 \dots P_{\tau-1})^{1/k} \\ &\leq \left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{\frac{1}{k} (1+2+\dots+\tau-1)} P_0^{\frac{1}{k} \left(1 + \left(1 - \frac{1}{k} \right) + \dots + 1 - \frac{1}{k}^{\tau-1} \right)} \\ &\leq \left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{\frac{1}{2k} \tau(\tau-1)} P_0^{\frac{1}{k} \frac{1-(1-1/k)^\tau}{1-(1-1/k)}} \\ &\leq \left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{\frac{1}{2k} \tau(\tau-1)} P_0^{1-(1-1/k)^\tau} \end{aligned} \quad (76)$$

Furthermore, by (22) we obtain

$$\begin{aligned} P_\tau^{2(s-\tau k)} &= P_\tau^{2s} P_\tau^{-2\tau k} \\ &\leq \left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{2s\tau} P^{2s(1-1/k)^\tau} P^{-2\tau k} \end{aligned} \quad (77)$$

Summing up the results of (74),(75),(76) and (77), we obtain

$$N_s(P) \leq D_1 D_2 \dots D_\tau ((N_{\mathfrak{p}_2} N_{\mathfrak{p}_2^3} \dots N_{\mathfrak{p}_\tau^{\tau-1}})^{2k} P_\tau^{2k\tau})$$

$$\begin{aligned}
& \times \left(\left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{\frac{1}{2k} \tau (\tau-1)} P^{1 - \left(1 - \frac{1}{k}\right)^\tau} \right)^{2s - \frac{1}{2} k(k+1)} \\
& \times (Np_2 Np_3^2 \cdots Np_\tau^{\tau-1})^{-2k} \left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{2s\tau} P^{2s(1-1/k)^\tau} P_\tau^{-2s\tau} \\
& \leq D_1 D_2 \cdots D_\tau \left(\left(\frac{2}{\tilde{c}_1} \right)^n (1+2\theta) \right)^{2s\tau + \frac{1}{2k} \tau (\tau-1) \left(2s - \frac{1}{2} k(k+1)\right)} \\
& \quad \times P^{2s - \frac{1}{2} k(k+1) + \frac{1}{2} k(k+1)(1-1/k)^\tau} \\
(78) \quad & = D(1+2\theta)^\sigma P^\eta
\end{aligned}$$

where

$$(79) \quad \sigma = 2s\tau + \frac{1}{k} \tau (\tau-1) s - \frac{1}{4} \tau (\tau-1) (k+1),$$

$$(80) \quad \eta = 2s - \frac{1}{2} k(k+1) + \frac{1}{2} k(k+1) (1-1/k)^\tau$$

and by (73)

$$\begin{aligned}
D &= D_1 D_2 \cdots D_\tau \left(\frac{2}{\tilde{c}_1} \right)^{n \left(2s\tau + \frac{1}{2k} \tau (\tau-1) \left(2s - \frac{1}{2} k(k+1) \right) \right)} \\
&\leq c_7^{\left(k^2 \tau + 2s\tau + \frac{s}{k} \tau (\tau-1) - \frac{1}{4} \tau (\tau-1) (k+1) \right)} \\
&\quad \times \left(\binom{s}{k} \binom{s-2k}{k} \cdots \binom{s-(\tau-1)k}{k} \right)^2 (k!)^\tau \\
&\leq c_7^{\left(k^2 + 2s + \frac{(\tau-1)s}{k} - \frac{1}{4} (\tau-1) (k+1) \right) \tau n} \left(\frac{s!}{(s-\tau k)!} \right)^{2\tau} \frac{1}{(k!)^\tau} \\
(81) \quad &\leq c_7^{\left(k^2 + 2s + \frac{(\tau-1)s}{k} - \frac{1}{4} (\tau-1) (k+1) \right) \tau n} s^{2\tau k}
\end{aligned}$$

with positive constant c_7 depending only on K .

By (78), (79), (80) and (81) we have

$$\begin{aligned}
(87) \quad N_s(P) &\leq c_7^{\left(k^2 + 2s - \frac{(\tau-1)s}{k} - \frac{1}{4} (\tau-1) (k+1) \right) \tau n} \\
&\quad \times (1+2\theta)^{2s\tau + \frac{1}{k} \tau (\tau-1) s - \frac{1}{4} \tau (\tau-1) (k+1)} \cdot s^{2\tau k} \\
&\quad \times P^{2s - \frac{1}{2} k(k+1) + \frac{1}{2} k(k+1)(1-1/k)^\tau}
\end{aligned}$$

Hence the proof of the main theorem is complete.

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