A class of Markov processes with non-linear, bounded generators

By

Tadashi UENO

(Received May 23, 1968)

§ 1. Introduction

Let S be a locally compact, Hausdorff space, satisfying the second axiom of countability, and let $\mathbf{B}(S)$ be the topological Borel field.

Consider a non-linear equation

$$\frac{\partial u}{\partial t} = uA^{(u)} \quad (1.1)$$

$u = u(t, \cdot) \rightarrow f$, as $t \rightarrow 0$, *)

where the initial value f and the solution u are probability measures on $(S, \mathbf{B}(S))$. $uA^{(u)}$ is given by a kernel $A^{(u)}(x, \cdot)$:

$$uA^{(u)}(\cdot) = \int_S u(dx)A^{(u)}(x, \cdot),$$

$$(1.2) \quad A^{(u)}(x, \cdot) = \sum_{N=0}^{\infty} \int_{S^N} \prod_{k=1}^N u(dx_k) A_N(x_1, \dots, x_N | x, \cdot)$$

$A_N(x_1, \dots, x_N | x, E)$ is a measurable function of $(x_1, \dots, x_N, x)$ for each $E \in \mathbf{B}(S)$, symmetric in $(x_1, \dots, x_N)$,**) and it is a signed measure in E for each $(x_1, \dots, x_N, x)$ such that

$$C_N = \|A_N\| = \sup_{(x_1, \dots, x_N, x)} \text{total variation of } A_N(x_1, \dots, x_N | x, \cdot)$$

satisfies

$$(1.3) \quad \sum_{N=0}^{\infty} C_N N^P \leq L^P \cdot P!, \quad P=1, 2, \dots.$$

Moreover, A_N is a stochastic generator kernel with parameter $(x_1, \dots, x_N)$, that is,

$$A_N(x_1, \dots, x_N | x, E) = q_N(x_1, \dots, x_N | x) (\pi_N(x_1, \dots, x_N | x, E) - \delta_{\{x\}}(E)),$$

where π_N is a probability measure in E with mass 0 on $\{x\}$, and q_N is non-negative. In a word, $A^{(u)}$ is a stochastic generator kernel bounded by (1.3) and depends multilinearly on u in the sense of (1.2).

We solve equation (1.1) by constructing a fundamental solution $P^f(t, x, E)$, which satisfies

$$\frac{\partial}{\partial t} P^f(t, x, \cdot) = \int_S P^f(t, x, dy) A^f_t(y, \cdot),$$

$$(1.1') \quad f_t(\cdot) = \int_S f(dx) P^f(t, x, \cdot).$$

More precisely, we have

THEOREM 1. For a probability measure f on $(S, \mathbf{B}(S))$ and $t > 0$, there is a

*) $u(t, \cdot) \rightarrow f(\cdot)$ means $u(t, E) \rightarrow f(E)$ and $\frac{\partial u}{\partial t}(E)$ is $\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} (u(t+\Delta, E) - u(t, E))$ for each E in $\mathbf{B}(S)$.

**) Symmetry is not necessary. Any A_N can be replaced by a symmetric kernel $(N!)^{-1} \sum A_N(x_{\sigma(1)}, \dots, x_{\sigma(N)} | x, \cdot)$, where σ ranges over all permutations of $\{1, 2, \dots, N\}$.

probability measure $P^f(t, x, E)$, which is measurable in x for each $E \in \mathcal{B}(S)$, and satisfies

$$(1.4) \quad \begin{aligned} P^f(s+t, x, \cdot) &= \int_S P^f(s, x, dy) P^f_s(t, y, \cdot) \\ f_s(\cdot) &= \int_S f(dx) P^f(s, x, \cdot) \end{aligned}$$

$$(1.5) \quad \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_S P^f(t, x, dy) \varphi(y) - \varphi(x) \right) = A^f \varphi(x) = \int_S A^f(x, dy) \varphi(y)$$

for any bounded measurable function φ on S . Here, the convergence is uniform in $x \in S$. Moreover,

$$(1.6) \quad \frac{\partial}{\partial t} \int_S P^f(t, x, dy) \varphi(y) = \int_S P^f(t, x, dy) A^f_t \varphi(y)$$

$$(1.7) \quad \int_S P^f(t, x, dy) \varphi(y) \text{ is analytic in } t \geq 0.$$

Thus, it is clear that $u(t, \cdot) = f_t(\cdot)$ satisfies (1.1).

We now define, by (1.4), a *temporally inhomogeneous* Markov process $M = (W = S^{[0, \infty)}, P^f, x(t, w))$ with state space S , initial distribution f and transition probability

$$(1.8) \quad \begin{aligned} P(s, x, t, E) &= P^f_s(t-s, x, E) \quad *) \\ f_s(\cdot) &= \int_S f(dx) P^f(s, x, \cdot). \end{aligned}$$

Attached to this Markov process, we define a series of *n-dimensional temporally homogeneous* Markov processes $M_n = (W_n = (S^n)^{[0, \infty)}, P_n^f, x_n(t, w))$ with state space S^n , initial distribution $f^{(n)}$, the n -fold direct product measure of f , and with generator G_n , where

$$(1.9) \quad G_n \varphi(x_1, \dots, x_n) = \sum_{N=0}^{n-1} n^{-N} \sum_{(j_1, \dots, j_N, i)}^{(n)} \int_S A_N(x_{j_1}, \dots, x_{j_N}, \overset{i}{y}, \dots, x_n) |x_i, dy) \varphi(x_1, \dots, x_n)$$

for φ in $B(S^n)$ which is the space of all bounded measurable functions on S^n . Here $\sum_{(j_1, \dots, j_N, i)}^{(n)}$ is the sum over $j_1, \dots, j_N, i$ ranging over $1, \dots, n$, except that no value is taken at the same time. This is possible, because G_n is a bounded operator, and e^{tG_n} defines a non-negative, strongly continuous semigroup on $B(S^n)$, and $e^{tG_n} 1 = 1$. Then, it is clear that the coordinate process $x_n^{(i)}(t, w)$ of $x_n(t) = (x_n^{(1)}(t), \dots, x_n^{(n)}(t))$ are not mutually independent, except at time $t=0$.**) But, on the other hand, we have the following result.

THEOREM 2. For $0 < t_1 < \dots < t_k < 1/2L(3+C_0)$

$$(1.10) \quad \begin{aligned} & \lim_{n \rightarrow \infty} P_n^f(x_n^{(1)}(t_1) \in A_1, \dots, x_n^{(1)}(t_k) \in A_k, x_n^{(2)}(t_1) \in B_1, \dots, x_n^{(2)}(t_k) \in B_k) \\ &= P^f(x(t_1) \in A_1, \dots, x(t_k) \in A_k) \cdot P^f(x(t_1) \in B_1, \dots, x(t_k) \in B_k) \\ &= \lim_{n \rightarrow \infty} P_n^f(x_n^{(1)}(t_1) \in A_1, \dots, x_n^{(1)}(t_k) \in A_k) \lim_{n \rightarrow \infty} P_n^f(x_n^{(2)}(t_1) \in B_1, \dots, x_n^{(2)}(t_k) \in B_k) \end{aligned}$$

*) The proof of Chapman-Kolmogorov equation for $P(s, x, t, \cdot)$ is easy by (1.4). This model shows that for temporally inhomogeneous Markov processes it is not always natural to discuss transition probability and initial distribution separately, while this has been rather customary.

**) We neglect the trivial case, where $A_N(x_1, \dots, x_n | x, \cdot) \equiv 0$ for all $N \geq 1$.

Here, the number of coordinate processes of $x_n(t)$ can be replaced by any fixed number.

Considered as motions of n different particles on the space S , the coordinate processes $x_n^{(1)}(t), \dots, x_n^{(n)}(t)$ of $x_n(t)$ have mutual interactions. But, THEOREM 2 shows that these motions get gradually independent as the number of particles increases, and in the limit, they are mutually independent, and each of these particles moves by the law determined by the equation (1.1). This is the so-called *propagation of chaos* discovered by Mark Kac [3]. Moreover, the N th term $\int_S \prod_{k=1}^N f(dx_k) A_N(x_1, \dots, x_N | x, \cdot)$ of kernel $A^{(f)}$ can be considered as the influence of other N particles on the motion of one particle under observation. So, if the interactions among $N+1$ particles do not actually exist, then $A_N=0$. For instance, in the case of the one dimensional model of Boltzmann equation (5.1) for Maxwellian gas by Kac, the kernel A^f is given by

$$\begin{aligned} A^f(x, dy) &= \int_{R^1} f(dx_1) A_1(x_1 | x, dy) \\ A_1(x_1 | x, dy) &= \frac{1}{\pi} \cdot \frac{1}{\sqrt{x^2 + x_1^2 - y^2}} dy - \delta_x(dy), \text{ if } x^2 + x_1^2 > y^2 \\ &= 0, \quad \text{if otherwise,} \end{aligned}$$

where the effective interactions are limited to binary collisions alone.

Now it is clear that the model we have discussed in this section is a special case of the class of Markov processes with non-linear generators, the field on which Henry McKean [4] made a first step of systematic study. I also note here that this article is a natural extension of Dudley Johnson's works [1, 2], where he made a complete description of 2-state space case. I wish to express my thanks to Henry McKean for his interesting talks and discussions on the field, and to Dudley Johnson who sent me his manuscript before publication, and also to Keniti Sato who read my manuscript very carefully and kindly suggested me necessary corrections.

§ 2. Derivation D and e^{tD}

Let I be an infinite set of indices and let S^I be the direct product $\prod_{j \in I} S_j$ of copies S_j of S with the product topology induced by S . We consider real valued bounded functions on S^I , measurable with respect to the topological Borel field $\mathcal{B}(S^I)$ of S^I and consider the norm

$$\|\varphi\| = \sup_{\xi \in S^I} |\varphi(\xi)|.$$

We say that $\sum_{j=1}^{\infty} \varphi_j$ is convergent in norm, if $\sum_{j=1}^{\infty} \|\varphi_j\| < \infty$. F_0 is the set of all functions which depend only on finite variables and F is the set of all function φ which can be written as

$$\varphi = \sum_{j=1}^{\infty} \varphi_j, \quad \varphi_j \in F_0,$$

where the convergence is in norm.

For a subset J of I and functions φ and ψ in F , we write

$$\varphi \equiv \phi \pmod{J}$$

and say that φ and ϕ are equivalent modulo J , if there are expansions

$$\varphi = \sum_{j=1}^{\infty} \varphi_j, \quad \phi = \sum_{j=1}^{\infty} \phi_j, \quad \varphi_j, \phi_j \in F_0,$$

converging in norm, and there is a class of one to one mappings T_j from J onto J such that ϕ_j coincides with the function obtained from φ_j by the change of variables induced by T_j . Then, the following are clear by definition.

$$(2.1) \quad \varphi_1 \equiv \phi_1, \quad \varphi_2 \equiv \phi_2 \pmod{J} \text{ imply}$$

$$\varphi_1 + \varphi_2 \equiv \phi_1 + \phi_2 \pmod{J}.$$

$$(2.2) \quad \varphi \equiv \phi \pmod{J}, \text{ and } I(\varphi') \subset J^c \text{ imply}^*)$$

$$\varphi \cdot \varphi' \equiv \phi \cdot \phi' \pmod{J},$$

where $I(\varphi)$ is the set of indices on which φ actually depends.

$$(2.3) \quad \varphi \equiv \phi \pmod{J}, \text{ and } J \subset J' \text{ imply}$$

$$\varphi \equiv \phi \pmod{J'}.$$

For a probability measure f on $(S, \mathbf{B}(S))$, subset J of I and $\varphi \in F$, we write

$$\int_J f^\infty \varphi = \int_{S^J} \prod_{i \in J} f(dx_i) \varphi(\xi),$$

where the integration is limited to variables with indices in J and hence the integral is a function of variables with indices in J^c . It is clear by definition that

$$(2.4) \quad \varphi \equiv \phi \pmod{J}, \text{ and } J \subset J' \text{ imply}$$

$$\int_{J'} f^\infty \varphi = \int_{J'} f^\infty \phi.$$

If $\varphi = \sum_{j=1}^{\infty} \varphi_j$, $\varphi_j \in F_0$, is convergent in norm,

$$(2.5) \quad \int_J f^\infty \varphi = \sum_{j=1}^{\infty} \int_{J \cap I(\varphi_j)} f^\infty \varphi_j = \sum_{j=1}^{\infty} \int_{S^{J \cap I(\varphi_j)}} \prod_{i \in J \cap I(\varphi_j)} f(dx_i) \varphi_j$$

holds by the bounded convergence theorem. In this expression we have to consider only the finite product measure of f .

Take a finite subset I_0 of I . We define for $\varphi \in F_0$,

$$Q(N)\varphi = \sum_{i \in I(\varphi)} A_N^{(i)} \varphi(x_{i_1}, x_{i_2}, \dots, x_{i_N} | x_i).$$

Here, $A_N^{(i)}\varphi$ is the function obtained by applying A_N to φ as a function of x_i , keeping other variables fixed. $i_1, i_2, \dots, i_N$ are N distinct indices which are *not contained* in $I(\varphi) \cup I_0$. $Q(N)\varphi$, having variables with indices in $I(\varphi)$ and $i_1, \dots, i_N$, depends on the choice of $i_1, \dots, i_N$. But, it is determined uniquely modulo I_0^c , independent from the choice of $i_1, i_2, \dots, i_N$. We simply write $Q(N)\varphi$ for an arbitrary representative of the equivalence class.

We define $Q(N)\varphi$ for $\varphi \in F$ by

$$Q(N)\varphi = \sum_{j=1}^{\infty} Q(N)\varphi_j,$$

*) J^c denotes the complement of J , that is, $I - J$.

if there is a convergent series $\sum_{j=1}^{\infty} \varphi_j = \varphi$, $\varphi_j \in F_0$, such that the series $\sum_{j=1}^{\infty} \sum_{i \in I(\varphi)} A_N^{(i)} \varphi_j$ ($x_{i_1}, \dots, x_{i_N} | x_i$) is convergent in norm. $Q(N)\varphi$, if exists, is determined uniquely modulo I_0^c , independent of the expression $\varphi = \sum_{j=1}^{\infty} \varphi_j$. In fact, by the assumption on the convergence, $Q(N)\varphi$ coincides with $\sum_{i \in I(\varphi)} A_N^{(i)} \varphi(x_{i_1}, \dots, x_{i_N} | x_i)$ which is convergent and clearly independent on $\{\varphi_j\}$. $D\varphi$ is defined for $\varphi \in F$ by

$$D\varphi = \sum_{N=0}^{\infty} Q_N(\varphi),$$

if there is an expression $\varphi = \sum_{i=0}^{\infty} \varphi_i$, $\varphi_j \in F_0$ such that

$$\sum_{j=1}^{\infty} \sum_{N=0}^{\infty} \sum_{i \in I(\varphi_j)} A_N^{(i)} \varphi_j(x_{i_1}, \dots, x_{i_N} | x_i)$$

is convergent in norm. $D^p\varphi$ and $e^{tD}\varphi$ are defined for $\varphi \in F$ similarly by

$$D^p\varphi = D(D^{p-1}\varphi), \quad D^1\varphi = D\varphi, \quad e^{tD}\varphi = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n\varphi,$$

if there is an expression $\sum_{j=1}^{\infty} \varphi_j$ such that the corresponding series in terms of $A_N^{(i)}$ are convergent in norm for new indices.

These quantities clearly depend on the choice of variables, but are determined modulo I_0^c . It is clear that $D^p\varphi$ and $e^{tD}\varphi$ again belong to F when they exist. Note that variables are added entirely anew for each application of $Q(N)$, and may possibly differ for different φ_j 's in $\sum_{j=1}^{\infty} \varphi_j$ even for a same $Q(N)$. Here, we always assume that I is sufficiently large so that we can pick up new indices freely from I .

When $I(\phi)$ is contained in I_0 , then

$$\phi \cdot Q(N)\varphi, \quad \phi \cdot D^p\varphi, \quad \phi \cdot e^{tD}\varphi$$

are determined modulo I_0^c , by virtue of (2.2). Similarly, if all variables of $\varphi_1, \varphi_2, \dots, \varphi_n$ have indices contained in I_0 , then

$$Q(N_1)\varphi_1 \dots Q(N_n)\varphi_n, \quad D^{p_1}\varphi_1 \dots D^{p_n}\varphi_n, \quad e^{t_1 D}\varphi_1 \dots e^{t_n D}\varphi_n$$

are all determined modulo I_0^c . Here, we write, for instance,

$$D^{p_1}\varphi_1 \dots D^{p_n}\varphi_n \quad \text{for } D^{p_1}(\varphi_1 \cdot D^{p_2}(\varphi_2 \cdot \dots \cdot D^{p_{n-1}}(\varphi_{n-1} \cdot D^{p_n}\varphi_n)) \dots)$$

for abbreviation.

Now, following are the fundamental properties of e^{tD} .

LEMMA 1. *If φ and $\phi \in F$ have no indices in common, there are those representatives of $e^{tD}\varphi$ and $e^{tD}\phi$ which have no indices in common. Then, $e^{tD}(\varphi \cdot \phi)$ is equivalent to the product of such representatives modulo I_0^c . This fact is written symbolically by*

$$(2.6) \quad e^{tD}(\varphi \otimes \phi) \equiv e^{tD}\varphi \otimes e^{tD}\phi \pmod{I_0^c}.$$

For $\varphi \in F$, we have

$$(2.7) \quad e^{(s+t)D}\varphi \equiv e^{sD} e^{tD}\varphi. \pmod{I_0^c}.$$

Here, the existence of the above quantities are assumed.

Note that (2.6) depends on the choice of representatives on the right.

COROLLARY. By the notation of Lemma 1, we have

$$(2.8) \quad e^{tD}(\varphi_1 \otimes \cdots \otimes \varphi_n) \equiv e^{tD}\varphi_1 \otimes \cdots \otimes e^{tD}\varphi_n \pmod{I_0^c}.$$

If $\varphi_1, \dots, \varphi_n$ and $\phi_1, \dots, \phi_n$ have variables with indices in I_0 and

$I(\varphi_1) \cup \cdots \cup I(\varphi_n)$ and $I(\phi_1) \cup \cdots \cup I(\phi_n)$ are disjoint, then,

$$(2.9) \quad e^{t_1D}(\varphi_1 \otimes \phi_1) e^{t_2D}(\varphi_2 \otimes \phi_2) \cdots e^{t_nD}(\varphi_n \otimes \phi_n) \\ \equiv (e^{t_1D}\varphi_1 e^{t_2D}\varphi_2 \cdots e^{t_nD}\varphi_n) \otimes (e^{t_1D}\phi_1 \cdots e^{t_nD}\phi_n) \pmod{I_0^c},$$

where $\varphi_1, \dots, \varphi_n$ may have indices in common, and so may $\phi_1, \dots, \phi_n$. More precisely, there are those representatives of the right hand factors that have no indices in common, and the left hand side is equivalent to the product of those representatives. Here, the quantities above are assumed to exist.

Proof of Lemma 1. By the definition of $A_N^{(i)}$ and $Q(N)$, we have

$$\begin{aligned} Q(N)(\varphi \otimes \psi) &= \sum_{i \in I(\varphi \otimes \psi)} A_N^{(i)}(\varphi \otimes \psi) \\ &= \sum_{i \in I(\varphi) + I(\psi)} A_N^{(i)}(\varphi \otimes \psi) = \sum_{i \in I(\varphi)} A_N^{(i)}(\varphi \otimes \psi) + \sum_{i \in I(\psi)} A_N^{(i)}(\varphi \otimes \psi) \\ &= \sum_{i \in I(\varphi)} \varphi \otimes A_N^{(i)}\psi + \sum_{i \in I(\psi)} \varphi \otimes A_N^{(i)}\psi \\ &= \varphi \otimes Q(N)\psi + \varphi \otimes Q(N)\psi, \quad *) \end{aligned}$$

for any possible choice of $i, \dots, i_N$. This implies

$$D(\varphi \otimes \psi) \equiv \varphi \otimes D\psi + \psi \otimes D\varphi \pmod{I_0^c},$$

if the indices of φ and ψ are avoided for $D\psi$ and $D\varphi$, respectively.

Choosing indices so that new indices are added never in common for $D^p\varphi$ and $D^q\psi$, and repeating the same procedure, we can extend this relation to

$$D^n(\varphi \otimes \psi) \equiv \sum_{p=0}^n \binom{n}{p} D^{n-p}\varphi \otimes D^p\psi \pmod{I_0^c}.$$

This naturally implies (2.6) under the condition that representatives of $e^{tD}\varphi$ and $e^{tD}\psi$ have no indices in common.

Since both sides of (2.7) are determined modulo I_0^c independent on the choice of indices, it is sufficient to find those representatives which are equivalent modulo I_0^c . But, this is clear by formal expansions:

$$\begin{aligned} e^{(s+t)D}\varphi &= \sum_{n=0}^{\infty} \frac{1}{n!} (s+t)^n D^n \varphi \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p=0}^n \binom{n}{p} s^{n-p} t^p D^{n-p} D^p \varphi \\ e^{sD}(e^{tD}\varphi) &= e^{sD} \left(\sum_{p=0}^{\infty} \frac{t^p}{p!} D^p \varphi \right) \\ &= \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{q! p!} s^q t^p D^q D^p \varphi = \sum_{n=0}^{\infty} \sum_{p+q=n} \frac{1}{p! q!} s^q t^p D^q D^p \varphi \end{aligned}$$

*) We denote by $\varphi \otimes \psi$ the product of φ and ψ which have no indices in common. By $A+B$, we mean $A \cup B$ when $A \cap B = \emptyset$.

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p=0}^n \binom{n}{p} s^{n-p} t^p D^{n-p} D^p \varphi$$

Proof of the Corollary. (2.8) is obtained by a simple induction.

$$\begin{aligned} e^{tD}(\varphi_1 \otimes \dots \otimes \varphi_n) &\equiv e^{tD}((\varphi_1 \otimes \dots \otimes \varphi_{n-1}) \otimes \varphi_n) \\ &= e^{tD}(\varphi_1 \otimes \dots \otimes \varphi_{n-1}) \otimes e^{tD} \varphi_n \pmod{I_0^c}. \end{aligned}$$

(2.9) is true for $n=1$ by (2.5). Assume that this is true for $n-1$, and let φ_0 and ψ_0 be such representatives of $e^{t_2 D} \varphi_2 \dots e^{t_n D} \varphi_n$ and $e^{t_2 D} \psi_2 \dots e^{t_n D} \psi_n$ in the corollary, respectively. Since there are no indices in common for φ_0 and ψ_1 , and also for ψ_0 and φ_1 , the left hand side of (2.9) is equivalent to

$$\begin{aligned} e^{t_1 D}((\varphi_1 \otimes \psi_1) \cdot (\varphi_0 \otimes \psi_0)) &\equiv e^{t_1 D}((\varphi_1 \cdot \varphi_0) \otimes (\psi_1 \cdot \psi_0)) \\ &\equiv e^{t_1 D}(\varphi_1 \cdot \varphi_0) \otimes e^{t_1 D}(\psi_1 \cdot \psi_0) \pmod{I_0^c}, \end{aligned}$$

which completes the proof.

LEMMA 2. If $I(\varphi_1) \cup \dots \cup I(\varphi_n)$ is a set of M indices contained in I_0 , then the following estimates hold, independently on the possible choice of new indices.

$$(2.10) \quad \|D^{p_1} \varphi_1 \dots D^{p_n} \varphi_n\| \leq 2(p_1 + \dots + p_n + 1)! (2(M+1)L)^{p_1 + \dots + p_n} \|\varphi_1\| \dots \|\varphi_n\|$$

$$(2.11) \quad \|e^{t_1 D} \varphi_1 \dots e^{t_n D} \varphi_n\| \leq (1 - 2L(M+1)(t_1 + \dots + t_n))^{-2} \|\varphi_1\| \dots \|\varphi_n\|$$

for $t_1 + \dots + t_n < (2L(M+1))^{-1}$.

Proof. Taking care of the number of variables, we have

$$\|Q(N)\varphi_1\| \leq \sum_{i \in I(\varphi_1)} \|A_N^{(i)} \varphi_1\| \leq \sum_{i \in I(\varphi_1)} \|A_N\| \cdot \|\varphi_1\| \leq M \cdot C_N \cdot \|\varphi_1\|,$$

$$\begin{aligned} \|Q(N_n)\varphi_n \dots Q(N_1)\varphi_1\| &\leq (M+N_1+\dots+N_{n-1})C_{N_n}\|\varphi_n\| \cdot Q(N_{n-1})\varphi_{n-1} \dots Q(N_1)\varphi_1\| \\ &\leq (M+N_1+\dots+N_{n-1})C_{N_n}\|\varphi_n\| \cdot \|Q(N_{n-1})\varphi_{n-1} \dots Q(N_1)\varphi_1\| \\ &\leq M(M+N_1) \dots (M+N_1+\dots+N_{n-1})C_{N_1} \dots C_{N_n} \|\varphi_1\| \dots \|\varphi_n\|. \end{aligned}$$

We use a formula of Dudley Johnson [2, p. 282];

$$(2.12) \quad \sum_{i_n=1}^n \sum_{i_{n-1}=1}^{n-1} \dots \sum_{i_1=1}^1 \tau_n(i_1, \dots, i_n)! \dots \tau_1(i_1, \dots, i_n)! \\ = 1 \cdot 3 \cdot \dots \cdot (2n-1) \leq 2^n \cdot n!,$$

where $\tau_k(i_1, \dots, i_n)$ is the number of $i_1, \dots, i_n$ which are equal to k .

$$\begin{aligned} &\|D\varphi_n \dots D\varphi_1\| \\ &\leq \sum_{N_n=0}^{\infty} \dots \sum_{N_1=0}^{\infty} \|Q(N_n)\varphi_n \dots Q(N_1)\varphi_1\| \\ &\leq \sum_{N_n=0}^{\infty} \dots \sum_{N_1=0}^{\infty} M(M+N_1) \dots (M+N_1+\dots+N_{n-1})C_{N_1} \dots C_{N_n} \|\varphi_1\| \dots \|\varphi_n\| \\ &\leq (M+1)^n \|\varphi_1\| \dots \|\varphi_n\| \sum_{N_n=0}^{\infty} \dots \sum_{N_1=0}^{\infty} \left(\sum_{i_n=1}^n N_{i_n} + 1 \right) \left(\sum_{i_{n-1}=1}^{n-1} N_{i_{n-1}} + 1 \right) \dots \\ &\dots \left(\sum_{i_1=1}^1 N_{i_1} + 1 \right) C_{N_1} \dots C_{N_n} \\ &= (M+1)^n \|\varphi_1\| \dots \|\varphi_n\| \sum_{N_n=0}^{\infty} \dots \sum_{N_1=0}^{\infty} \sum_{i_n=0}^n \dots \sum_{i_0=0}^1 N_{i_n} \dots N_{i_1} C_{N_1} \dots C_{N_n} \end{aligned}$$

(where $N_0 \equiv 1$)

$$\begin{aligned}
&= (M+1)^n \|\varphi_1\| \cdots \|\varphi_n\| \sum_{i_n=0}^n \cdots \sum_{i_1=0}^1 \left(\sum_{N_n=0}^{\infty} N_n^{\tau_n(i_1, \dots, i_n)} C_{N_n} \right) \cdots \\
&\quad \cdots \left(\sum_{N_1=0}^{\infty} N_1^{\tau_1(i_1, \dots, i_n)} C_{N_1} \right) \\
&\leq (M+1)^n \|\varphi_1\| \cdots \|\varphi_n\| \sum_{i_n=0}^n \cdots \sum_{i_1=0}^1 \tau_n(i_1, \dots, i_n)! L^{\tau_n(i_1, \dots, i_n)} \\
&\quad \cdots \tau_1(i_1, \dots, i_n)! L^{\tau_1(i_1, \dots, i_n)} \tau_0(i_1, \dots, i_n)! L^{\tau_0(i_1, \dots, i_n)} \\
&\leq (M+1)^n \|\varphi_1\| \cdots \|\varphi_n\| L^n \sum_{i_n=0}^n \cdots \sum_{i_0=0}^0 \tau_n(i_0, \dots, i_n)! \\
&\quad \cdots \tau_1(i_0, \dots, i_n)! \tau_0(i_0, \dots, i_n)!
\end{aligned}$$

(by applying (2.12) here)

$$\leq (M+1)^n L^n \cdot 2^{n+1} \cdot (n+1)! \|\varphi_1\| \cdots \|\varphi_n\|.$$

Changing the order of indices, we have

$$\|D\varphi_1 \cdots D\varphi_n\| \leq 2(n+1)! (2(M+1)L)^n \|\varphi_1\| \cdots \|\varphi_n\|.$$

Thus, (2.10) is obtained, noting that

$$D^{p_1}\varphi_1 \cdots D^{p_n}\varphi_n = \underbrace{D_1 D_1 \cdots D_1}_{p_1} D\varphi_1 \cdots \underbrace{D_1 D_1 \cdots D_1}_{p_n} D\varphi_n$$

(2.11) is obtained from (2.10) and the following :

$$\begin{aligned}
\|e^{t_1 D} \varphi_1 \cdots e^{t_n D} \varphi_n\| &\leq \sum_{p_1=0}^{\infty} \cdots \sum_{p_n=0}^{\infty} \frac{t_1^{p_1} \cdots t_n^{p_n}}{p_1! \cdots p_n!} \|D^{p_1} \varphi_1 \cdots D^{p_n} \varphi_n\| \\
&\leq \sum_{p_1=0}^{\infty} \cdots \sum_{p_n=0}^{\infty} \frac{2(p_1 + \cdots + p_n + 1)!}{p_1! \cdots p_n!} t_1^{p_1} \cdots t_n^{p_n} (2(M+1)L)^{p_1 + \cdots + p_n} \|\varphi_1\| \cdots \|\varphi_n\| \\
&= 2 \sum_{N=0}^{\infty} (N+1) (2(M+1)L)^N (t_1 + \cdots + t_n)^N \|\varphi_1\| \cdots \|\varphi_n\| \\
&= 2(1 - 2(M+1)L(t_1 + \cdots + t_n))^{-2} \|\varphi_1\| \cdots \|\varphi_n\|, \\
&\quad \text{if } t_1 + \cdots + t_n < (2(M+1)L)^{-1}.
\end{aligned}$$

§ 3. The motion of one particle

We assume, for simplicity, that $\{1, 2, \dots\}$ is contained in I .

LEMMA 3. For a probability measure f on $(S, \mathbf{B}(S))$ and a function $\varphi \in F_0$ of one variable $x \equiv x_1$ where $1 \in I_0$,

$$(3.1) \quad \int_{\{1\}} e^{\varphi} f^{\infty} D\varphi = A^f \varphi(x)$$

$$(3.1') \quad \int f^{\infty} D\varphi = \int_S f(dx) A^f \varphi(x).$$

For $t < 1/4L$, there are probability measures $P^f(t, x, \cdot)$ and $f_t(\cdot)$ on $(S, \mathbf{B}(S))$ such that

$$(3.2) \quad \int_{\{1\}} e^{\varphi} e^{tD} \varphi = \int_S P^f(t, x, dy) \varphi(y),$$

$$(3.2') \quad \int f^{\infty} e^{tD} \varphi = \int_S f_t(dx) \varphi(x).$$

In general, for subsets $J = \{1, 2, \dots, k\}$, $J_1 = \{1, 2, \dots, h\}$ of I_0 *) with $h \leq k$ and

*) Note that I_0 can be chosen arbitrarily on which the definitions of D and e^{tD} depend.

$t < 1/2L(k+1)$, there are probability measures $P_J^f(t, x_1, \dots, x_h, \cdot)$ and $f_{t,J}(\cdot)$ on $(S^J, \mathbf{B}(S^J))$ such that

$$(3.3) \quad \int_{S^J} e^{tD} \varphi = \int_{S^J} P_J^f(t, x_1, \dots, x_h, dx_1, \dots, dx_h) \varphi(x_1, \dots, x_h)$$

$$(3.3') \quad \int_{S^J} e^{tD} \varphi = \int_{S^J} f_{t,J}(dx_1, \dots, dx_h) \varphi(x_1, \dots, x_h)$$

for any function φ in F_0 with $I(\varphi) = J$. $P_J^f(t, x_1, \dots, x_h, E)$ is measurable in $(x_1, \dots, x_h)$ for any $E \in \mathbf{B}(S^J)$, and

$$(3.4) \quad f_{t,J}(E) = \int_{S^J} f(dx_1) \dots f(dx_h) P_J^f(t, x_1, \dots, x_h, E).$$

PROPOSITION 1.*^o) Let m and $q(x, \cdot)$ be signed measures on measurable spaces $(R, \mathbf{B}(R))$ and $(S, \mathbf{B}(S))$ with finite total variations $\|m\|$ and $\|q\|(x)$ respectively, and let $q(x, E)$ be a $\mathbf{B}(R)$ measurable function on R for each fixed $E \in \mathbf{B}(S)$. If $\int_R |m|(dx) \|q\|(x) < \infty$, then, the set function $\int m(dx) q(x, E)$, $E \in \mathbf{B}(S)$, is a signed measure on $(S, \mathbf{B}(S))$ with total variation at most $\int |m|(dx) \|q\|(x)$.

Proof of Proposition. In order that $\int m(dx) q(x, E)$ is a measure, it is sufficient to prove

$$\int m(dx) \int q(x, dy) (\sum_n \varphi_n(y)) = \sum_n \int m(dx) \int q(x, dy) \varphi_n(y),$$

if $\sum |\varphi_n(x)| \leq k < \infty$ and φ_n are measurable. But, by bounded convergence theorem the left side is equal to

$$\int m(dx) (\sum_n \int q(x, dy) \varphi_n(y)).$$

Since $\sum_{h=1}^n \int q(x, dy) \varphi_h(y)$ is convergent for each x and bounded by $k\|q\|$, we can apply the bounded convergence theorem again, and have the equality. Total variation of this measure is

$$\begin{aligned} & \int m(dx) q(x, E^+) - \int m(dx) q(x, E^-) \\ &= \int m(dx) (q(x, E^+) - q(x, E^-)) \leq \int |m|(dx) |q(x, E_x^+) - q(x, E_x^-)| \\ &= \int |m|(dx) \|q\|(x), \end{aligned}$$

where E^+ and E^- , and E_x^+ and E_x^- are the positive set and the negative set of $\int m(x, dy) q(x, \cdot)$ and of $q(x, \cdot)$, respectively.

Proof of Lemma 3. Since $\|D\varphi\| \leq \sum_{N=0}^{\infty} \|Q(N)\varphi\| < \infty$ for $\varphi \in F_0$,

$$\int_{\{1\}} e^{f^\infty} D\varphi = \int_{\{1\}} e^{f^\infty} \sum_{N=0}^{\infty} Q(N)\varphi = \sum_{N=0}^{\infty} \int_{\{1\}} e^{f^\infty} Q(N)\varphi$$

*^o) Note that this is also true for complex valued measures with finite total variation.

$$\begin{aligned}
&= \sum_{N=0}^{\infty} \int \prod_{i=2}^{N+1} f(dx_i) A_N \varphi(x_2, \dots, x_{N+1} | x_1) \\
&= A^f \varphi(x_1) \\
\int f^{\infty} D\varphi_1 &= \int_S f(dx) \int_{\{1\}^c} f^{\infty} D\varphi = \int_S f(dx) A^f \varphi(x).
\end{aligned}$$

For φ with indices in J ,

$$Q(N)\varphi = \sum_{i \in I(\varphi)} A_N^{(i)} \varphi = \sum_{i=1}^k \int_S A_N(x_{i_1}, \dots, x_{i_N} | x_i; dy) \varphi(x_1, \dots, \widehat{y}, \dots, x_k)$$

by definition. Hence, $Q(N)\varphi$ is clearly an integral of φ by a signed measure on $(S^J, \mathbf{B}(S^J))$. Similarly, with Proposition in mind, we know that $Q(N_n) \dots Q(N_1)\varphi$ is an integral of φ by a signed measure on $(S^J, \mathbf{B}(S^J))$ with total variation at most $(N_{n-1} + \dots + N_1 + k) \dots (N_1 + k) k C_{N_n} \dots C_{N_1}$ indexed by $(N_n + \dots + N_1 + k)$ variables. And hence,

$$\int_{J_1^c} f^{\infty} Q(N_n) \dots Q(N_1) \varphi$$

is also given by a signed measure on $(S^J, \mathbf{B}(S^J))$ with the total variation bounded by $(N_{n-1} + \dots + N_1 + k) \dots (N_1 + k) k C_{N_n} \dots C_{N_1}$. Since

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{N_n=0}^{\infty} \dots \sum_{N_1=0}^{\infty} (N_{n-1} + \dots + N_1 + k) \dots (N_1 + k) k C_{N_n} \dots C_{N_1}$$

is convergent for $t < 1/2L(k+1)$ as we saw in the proof of Lemma 2, the following is also given by a signed measure on $(S^J, \mathbf{B}(S^J))$ indexed by $x_1, \dots, x_h$ which we denote by $P_J^f(t, x_1, \dots, x_h, \cdot)$.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{N_n=0}^{\infty} \dots \sum_{N_1=0}^{\infty} \int_{J_1^c} f^{\infty} Q(N_n) \dots Q(N_1) \varphi.$$

But, this is clearly equal to

$$\int_{J_1^c} f^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{N_1=0}^{\infty} \dots \sum_{N_n=0}^{\infty} Q(N_n) \dots Q(N_1) \varphi = \int_{J_1^c} f^{\infty} e^{tD} \varphi.$$

(3.3') is obtained by replacing $\int_{J_1^c}$ by $\int$, and denoting the corresponding measure by $f_{t,J}$, (3.4) is clear, since

$$\int f^{\infty} \phi = \int_{S^{J_1}} f(dx_1) \dots f(dx_h) \int_{J_1^c} f^{\infty} \phi.$$

That the measures $P_J^f(t, x, \dots, x_h, \cdot)$ and $f_{t,J}(\cdot)$ are non-negative with total mass 1, will be proved in the Corollary to Theorem 4.

LEMMA 4. Assume that $e^{tD}\varphi$ can be defined for $\varphi \in F$ and x_1 is a variable of φ . Then,

$$(3.5) \quad \int_{\{1\}^c} f^{\infty} e^{tD} \varphi = \int_{\{1\}^c} f^{\infty} e^{tD} \left(\int_{\{1\}} e f_i^{\infty} \varphi \right)$$

REMARK. We can replace $\{1\}^c$ by J^c in Lemma 4, where J is a finite set contained in $I(\varphi)$. Proof is similar, replacing $\{1\}$ by J .

Proof. First, we consider the following case.

$$(3.6) \quad \varphi = \varphi(x_1, \dots, x_k) = \varphi_1(x_1) \cdots \varphi_k(x_k).$$

Then, (3.5) is true by (2.6) of Lemma and (3.2') of Lemma 3:

$$\begin{aligned} \int_{\{1\}}^c f^\infty e^{tD} \varphi &= \int_{\{1\}}^c f^\infty e^{tD} (\varphi_1 \otimes \cdots \otimes \varphi_k) \\ &= \int_{\{1\}}^c f^\infty (e^{tD} \varphi_1) \otimes \cdots \otimes (e^{tD} \varphi_k) = \int_{I(e^{tD} \varphi_1) - \{1\}} f^\infty e^{tD} \varphi_1 \cdot \prod_{i=2}^k \int_{I(e^{tD} \varphi_i)} f^\infty e^{tD} \varphi_i \\ &= \int_{\{1\}}^c f^\infty e^{tD} \varphi_1 \cdot \prod_{i=2}^k \int_{\{1\}}^c f^\infty e^{tD} \varphi_i = \int_{\{1\}}^c f^\infty e^{tD} \varphi_1 \cdot \prod_{i=2}^k \int_S f_t(dx_i) \varphi_i(x_i) \\ &= \int_{\{1\}}^c f^\infty e^{tD} \varphi_1 \cdot \int f_t^\infty \varphi_2 \cdots \varphi_k = \int_{\{1\}}^c f^\infty e^{tD} \left(\int_{\{1\}}^c f_t^\infty \varphi_1 \cdots \varphi_k \right) \\ &= \int_{\{1\}}^c f^\infty e^{tD} \int_{\{1\}}^c f_t^\infty \varphi. \end{aligned}$$

Hence, (3.5) also holds for linear combinations of functions of type (3.6). On the other hand, for φ in F_0 , (3.5) is equivalent to

$$(3.5') \quad \int_{S^J} P_t^f(t, x_1; dy_1, \dots, dy_k) \varphi(y_1, \dots, y_k) \\ = \int_S P_t^f(t, x_1, dy_1) \int_{S^J - \{1\}} f_t(dy_2) \cdots f_t(dy_k) \varphi(y_1, \dots, y_k),$$

by (3.2) and of Lemma 3 taking $\{1\}$ as J_1 and $\{1, 2, \dots, k\}$ as J . Thus, by a standard technique of measure theory, (3.5) is true for all $\varphi \in F_0^{*)}$.

For φ in F for which $e^{tD}\varphi$ exists, there is a sequence $\{\varphi_n\} \subset F_0$, such that $\sum_1^n \varphi_j$ and $e^{tD}(\sum_1^n \varphi_j)$ converge to φ and $e^{tD}\varphi$ in norm for any fixed choice of variables, respectively. Then, the bounded convergence theorem applies to both sides of (3.5), completing the proof.

REMARK. $P_t^f(t, x_1, \dots, x_h, dy_1, \dots, dy_k)$ in Lemma 3 is the product measure

$$\prod_{i \in J_1} P^f(t, x_i, dy_i) \times \prod_{j \in J - J_1} f_t(dy_j),$$

(3.5') is a special case when $J_1 = \{1\}$. This comes from the property (2.6) of exponential e^{tD} .

LEMMA 5. For functions $\varphi_1, \varphi_2, \dots, \varphi_n \in F_0$ of one common variable $x_1 \equiv x$, and for $t_1 + \dots + t_n < 1/4L$,

$$(3.7) \quad \int_{\{1\}}^c f^\infty e^{t_1 D} \varphi_1 e^{t_2 D} \varphi_2 \cdots e^{t_n D} \varphi_n \\ = \int_S P^f(t_1, x, dx_1) \varphi_1(x_1) \int_S P^f(t_1, x_1, dx) \varphi_2(x_2) \cdots \\ \cdots \int_S P^{f_{t_1 + \dots + t_{n-1}}}(t_n, x_{n-1}, dx_n) \varphi_n(x_n),$$

$$(3.7') \quad \int f^\infty e^{t_1 D} \varphi_1 \cdots e^{t_n D} \varphi_n \\ = \int_S f(dx) P^f(t_1, x, dx_1) \varphi_1(x_1) \int_S \cdots \int_S P^{f_{t_1 + \dots + t_n}}(t_n, x_{n-1}, dx_n) \varphi_n(x_n).$$

$$(3.8) \quad P^f(s+t, x, \cdot) = \int_S P^f(s, x, dy) P^f_s(t, y, \cdot),$$

*) The reader can apply, for instance, I. T 20 of Meyer: Probability and Potentials.

$$(3.8') \quad f_{s+t} = (f)_t, \quad \text{for } s+t < 1/4L.$$

Proof. Applying (3.5) of Lemma 4 to $\varphi_i e^{t_{i+1}D} \varphi_{i+1} \cdots e^{t_n D} \varphi_n$ ($1 \leq i < n$) repeatedly, we have

$$\begin{aligned} \int_{\{1\}}^c f^\infty e^{t_1 D} \varphi_1 e^{t_2 D} \varphi_2 \cdots e^{t_n D} \varphi_n &= \int_{\{1\}}^c f^\infty e^{t_1 D} \int_{\{1\}}^c f^\infty e^{t_1 D} \varphi_1 e^{t_2 D} \varphi_2 \cdots e^{t_n D} \varphi_n \\ &= \int_{\{1\}}^c f^\infty e^{t_1 D} \varphi_1 \int_{\{1\}}^c f^\infty e^{t_2 D} \varphi_2 \cdots e^{t_n D} \varphi_n \\ &= \int_{\{1\}}^c f^\infty e^{t_1 D} \varphi_1 \int_{\{1\}}^c f^\infty e^{t_2 D} \varphi_2 \int_{\{1\}}^c (f_{t_1})_{t_2}^\infty e^{t_3 D} \varphi_3 \cdots e^{t_n D} \varphi_n \\ &= \int_{\{1\}}^c f^\infty e^{t_1 D} \varphi_1 \int_{\{1\}}^c \cdots \int_{\{1\}}^c ((f_{t_1})_{t_2} \cdots)_{t_{n-1}}^\infty e^{t_n D} \varphi_n \\ &= \int_S P^f(t_1, x, dx) \varphi_1(x_1) \int_S P^f(t_2, x_1, dx_2) \varphi_2(x_2) \int_S \cdots \\ &\quad \cdots \int_S P^{(\cdots (f_{t_1})_{t_2} \cdots)_{t_{n-1}}} (t_n, x_{n-1}, dx_n) \varphi_n(x_n), \end{aligned}$$

where the last equality sign owes to Lemma 3. The proof of (3.7) is complete, if (3.8') is proved. (3.7') is obtained by integrating the both sides of (3.7) by f .

(3.8) is proved by (2.7) of Lemma 1 and Lemma 4.

$$\begin{aligned} \int_S P^f(s+t, x, dy) \varphi(y) &= \int_{\{1\}}^c f^\infty e^{(s+t)D} \varphi \\ &= \int_{\{1\}}^c f^\infty e^{sD} e^{tD} \varphi = \int_{\{1\}}^c f^\infty e^{sD} \int_{\{1\}}^c f^\infty e^{tD} \varphi \\ &= \int_{\{1\}}^c f^\infty e^{sD} \int_S P^f(s, x_1, dz) \varphi(z) \\ &= \int_S P^f(s, x, dx) \int_S P^f(s, x_1, dz) \varphi(z). \end{aligned}$$

(3.8') is clear by integrating (3.8) by f .

Now, we are in the position to define $P^f(t, x, \cdot)$ and $f_t(\cdot)$ for an arbitrary $t > 0$. In fact, for any $t > 0$, take a sequence $t_1, \dots, t_n > 0$ such that

$$t = t_1 + \cdots + t_n, \quad \text{and } t_k < 1/4L, \quad k=1, \dots, n,$$

and define

$$\begin{aligned} P^f(t, x, \cdot) &= \int_S P^f(t_1, x, dx_1) \int_S P^f(t_2, x_1, dx_2) \int_S \cdots \\ &\quad \cdots \int_S P^{(\cdots (f_{t_1})_{t_2} \cdots)_{t_{n-2}}} (t_{n-1}, x_{n-2}, dx_{n-1}) P^{(\cdots (f_{t_1})_{t_2} \cdots)_{t_{n-1}}} (t_n, x_{n-1}, \cdot) \\ f_t(\cdot) &= \int_S f(dx) P^f(t, x, \cdot) \end{aligned}$$

By virtue of Chapman-Kolmogorov equation (3.8) for small t , it is easy to prove that this is independent of the choice of $t_1, \dots, t_n$. And it is clear that (3.8) and (3.8') hold true for arbitrary s and t .

Proof of Theorem 1. Chapman-Kolmogorov equation has been discussed above. For $t < \frac{1}{4L}$, we have

$$\begin{aligned} \frac{1}{t} \left(\int_S P^f(t, x, dy) \varphi(y) - \varphi(x) \right) &= \frac{1}{t} \left(\int_{\{1\}}^c f^\infty e^{tD} \varphi(x) - \varphi(x) \right) \\ &= \frac{1}{t} \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\{1\}}^c f^\infty (D^n \varphi(x) - \varphi(x)) \right) \end{aligned}$$

$$= \int_{\{1\}} c f^\infty D\varphi(x) + \frac{1}{t} \sum_{n=2}^{\infty} \frac{t^n}{n!} \int_{\{1\}} c f^\infty D^n \varphi(x),$$

by Lemma 3. But, the second term is bounded by

$$\begin{aligned} \frac{1}{t} \sum_{n=2}^{\infty} \frac{t^n}{n!} \int_{\{1\}} c f^\infty \|D^n \varphi\| &\leq \frac{2}{t} \sum_{n=2}^{\infty} \frac{t^n}{n!} \cdot (n+1)! (2(M+1)L)^n \|\varphi\| \\ &= \frac{2}{t} \sum_{n=2}^{\infty} t^n (n+1)(4L)^n \|\varphi\| = \frac{2}{t} \left(\frac{1}{(1-4tL)^2} - 1 - 8tL \right) \|\varphi\| \end{aligned}$$

and hence converges to 0 as $t \rightarrow 0$, where the estimate of $\|D^n \varphi\|$ is given in Lemma 2. Thus, the left-hand side of (1.5) converges to $\int_{\{1\}} c f^\infty D\varphi(x)$ uniformly in x which is equal to $A^f \varphi(x)$ by (3.1) of Lemma 3.

Since the convergence of (1.5) is uniform in x , the right derivative (in t) of $\int_S P^f(t, x, dy) \varphi(y)$ exists and is equal to $\int_S P^f(t, x, dy) A^f \varphi(y)$, by

$$\begin{aligned} \frac{1}{\Delta} \left(\int_S P^f(t+\Delta, x, dy) \varphi(y) - \int_S P^f(t, x, dy) \varphi(y) \right) \\ = \int_S P^f(t, x, dy) \frac{1}{\Delta} \left(\int_S P^f \iota(\Delta, y, dz) \varphi(z) - \varphi(y) \right). \end{aligned}$$

In order that it is also the derivative from the left (in case $t > 0$), it is sufficient to prove (1.7). But, this follows from

$$\begin{aligned} \int_S P^f(s+t, x, dy) \varphi(y) &= \int_S P^f(s, x, dy) \int_S P^f s(t, y, dz) \varphi(z) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_S P^f(s, x, dy) \int_{\{1\}} c f_s^\infty D^n \varphi(y), \text{ for } t < 1/4L, \end{aligned}$$

where the coefficient of t^n is bounded by $(n+1)(4L)^n \|\varphi\|$ for any $s > 0$, completing the proof.

Let W be the set of all S -valued functions defined on $[0, \infty)$, that is, $S^{[0, \infty)}$. Let $\mathbf{B} = \mathbf{B}(W)$ be the σ -algebra of subsets of W generated by all cylinder sets. P_x^f is the probability measure on $(W, \mathbf{B})$ given as the unique extensions of a set function for cylinder sets:

$$\begin{aligned} P_x^f(w_{t_1} \in A_1, \dots, w_{t_n} \in A_n) \\ = \int_{A_1} P^f(t, x, dx_1) \int_{A_2} P^f t_1(t_2 - t_1, x_1, dx_2) \int_{A_3} \dots \int_{A_n} P^f t_{n-2}(t_{n-1} - t_{n-2}, x_{n-2}, dx_{n-1}) \\ P^f t_{n-1}(t_n - t_{n-1}, x_{n-1}, A_n) \text{ for } A_1, \dots, A_n \in B(S) \text{ and } 0 < t_1 < t_2 < \dots < t_n. \\ P^f(A) = \int_S f(dx) P_x^f(A). \end{aligned}$$

E_x^f and E^f are expectations by P_x^f and P^f , respectively. We write the value of w at time t as w_t or $x_t(w)$. Then, $(W, \mathbf{B}, P^f, x_{t_{n-1}})$ is a temporally inhomogeneous Markov process with initial distribution f and transition probability

$$P(s, x, t, E) = P^f s(t-s, x, E), \quad s < t,$$

since

$$\begin{aligned} P^f(x_{t_1} \in A_1, \dots, x_{t_n} \in A_n / \mathbf{B}_s) \\ = P_{x_s}^{f s}(x_{t_1-s} \in A_1, \dots, x_{t_n-s} \in A_n) \\ = P^f(x_{t_1} \in A_1, \dots, x_{t_n} \in A_n / \mathbf{B}(x_s)), \quad P^f - a.e.. \end{aligned}$$

Here, $\mathbf{B}_i$ and $\mathbf{B}(x_s)$ are the σ -subalgebra of $\mathbf{B}$ generated by the cylinder sets of type $\{x_{r_1} \in A_1, \dots, x_{r_n} \in A_n\}$, $r_1, \dots, r_n \leq s$, and $\{x_s \in A\}$, respectively.

By these notations, a part of Lemma 5 is restated in the following way.

THEOREM 3. *For functions $\varphi_1, \dots, \varphi_n$ in F_0 of one variable and for $t_1, \dots, t_n$ satisfying $0 < t_1 < \dots < t_n < 1/4L$,*

$$\begin{aligned} E^f(\varphi_1(x_{t_1}) \dots \varphi_n(x_{t_n})) &= \int f^\infty e^{t_1 D} \varphi_1, \dots, e^{(t_n - t_{n-1})D} \varphi_n, \\ E_x^f(\varphi_1(x_{t_1}) \dots \varphi_n(x_{t_n})) &= \int_{\{1\}^c} f^\infty e^{t_1 D} \varphi_1, \dots, e^{(t_n - t_{n-1})D} \varphi_n. \end{aligned}$$

§ 4. Propagation of chaos

First, we note that G_n is bounded, that is,

$$\begin{aligned} \|G_n\| &= \sup_{(x_1, \dots, x_n)} \text{total variation of } G_n(x_1, \dots, x_n; \cdot) \\ &\leq \sum_{N=0}^{n-1} n^{-N} \sum_{(j_1, \dots, j_N, i)}^{(n)} \|A_N^{(i)}(x_{j_1}, \dots, x_{j_N})\| = \sum_{N=0}^{n-1} n^{-N} \sum_{(j_1, \dots, j_N, i)}^{(n)} C_N \\ &= \sum_{N=0}^{n-1} n^{-N} \binom{n}{N+1} C_N = \sum_{N=0}^{n-1} n^{-N} \frac{n!}{(n-N-1)! (N+1)!} C_N \leq n \sum_{N=0}^{n-1} \frac{C_N}{(N+1)!} \leq n \sum_{N=0}^{\infty} C_N \leq nL \end{aligned}$$

Moreover, by looking at the definition (1.9) carefully, we know that G_n is a stochastic generator kernel, that is,

$$\begin{aligned} G_n \varphi(x_1, \dots, x_n) &= q_n(x_1, \dots, x_n) \int_{S^n} \pi_n(x_1, \dots, x_n; dy_1, \dots, dy_n) \\ &\quad (\varphi(y_1, \dots, y_n) - \varphi(x_1, \dots, x_n)), \end{aligned}$$

where π_n is a probability measure on $(S^n, \mathbf{B}(S^n))$ and q_n is non-negative. Hence, e^{tG_n} defines a non-negative, strongly continuous semi-group on $B(S^n)$ and $e^{tG_n} 1 = 1$. Then, there is a probability measure on $P_n(t, x_1, \dots, x_n; \cdot)$ on $(S^n, \mathbf{B}(S^n))$ such that

$$e^{tG_n} \varphi(x_1, \dots, x_n) = \int_{S^n} P_n(t, x_1, \dots, x_n; dy_1, \dots, dy_n) \varphi(y_1, \dots, y_n) \in B(S^n)$$

Thus, the construction of M_n in § 1 is clearly possible. We introduce $Q_n(N)$ on $B(S^n)$ by

$$\begin{aligned} Q_n(N) \varphi(x_1, \dots, x_n) &= n^{-N} \sum_{(j_1, \dots, j_N, i)}^{(n)} \int_S A_N(x_{j_1}, \dots, x_{j_N} | x_i, dy) \\ &\quad \varphi(x_1, \dots, \overset{i}{\wedge}, \dots, x_n). \end{aligned}$$

Then,

$$\begin{aligned} G_n \varphi(x_1, \dots, x_n) &= \sum_{N=0}^{n-1} Q_n(N) \varphi(x_1, \dots, x_n), \\ G_n \varphi_1 G_n \varphi_2 \dots G_n \varphi_k &= \sum_{N_k=0}^{n-1} \dots \sum_{N_1=0}^{n-1} Q_n(N_k) \varphi_k \dots Q_n(N_1) \varphi_1. \end{aligned}$$

Here, we mean, for instance, $G_n \varphi_2 G_n \varphi_1 = G_n(\varphi_2 \cdot G_n \varphi_1)$

LEMMA 6. *If $\varphi_1, \dots, \varphi_k$ have indices in $I_0 = \{1, 2, \dots, M\}$ and $n > M$, then,*

$$(4.2) \quad Q_n(N_k) \varphi_k \dots Q_n(N_1) \varphi_1 \equiv \frac{(n-M)!}{(n-M-N_1-\dots-N_k)! n^{N_1+\dots+N_k}} Q(N_k) \varphi_k \dots$$

$$\dots\dots Q(N_1)\varphi_1 + e_n(N_k, \dots\dots, N_1; \varphi_k, \dots\dots, \varphi_1) \pmod{I_0^c}, *)$$

where

$$\sum_{N_k=0}^{n-1} \dots\dots \sum_{N_1=0}^{n-1} \|e_n(N_k, \dots\dots, N_1; \varphi_k, \dots\dots, \varphi_1)\| \leq \frac{2}{n} (h+2)! (2L(2+C_0))^{k+2} e^{M/L} \|\varphi_1\| \dots\dots \|\varphi_k\|$$

for any possible choice of indices for $Q(N_k), \dots\dots, Q(N_1)$.

Proof. We have, by definition,

$$\begin{aligned} & Q_n(N_k)\varphi_k \dots\dots Q_n(N_1)\varphi_1 \\ &= n^{-(N_1+\dots+N_k)} \sum_{(j_1^k, \dots, j_{N_k}^k, i_k)}^{(n)} \dots\dots \sum_{(j_1^2, \dots, j_{N_2}^2, i_2)}^{(n)} \sum_{(j_1^1, \dots, j_{N_1}^1, i_1)}^{(n)} \\ & A_{N_k}^{(i_k)}(x_{j_1^k}, \dots, x_{j_{N_k}^k}) \varphi_k \cdot A_{N_{k-1}}^{(i_{k-1})}(x_{j_1^{k-1}}, \dots, x_{j_{N_{k-1}}^{k-1}}) \varphi_{k-1} \dots\dots \varphi_2 \cdot A_{N_1}^{(i_1)} \\ & (x_{j_1^1}, \dots, x_{j_{N_1}^1}) \varphi_1, \end{aligned}$$

where $\sum_{(j_1^h, \dots, j_{N_h}^h, i_h)}^{(n)}$ is the same as explained in § 1, and $A_{N_h}^{(i_h)}(x_{j_1^h}, \dots, x_{j_{N_h}^h}) \varphi_h \cdot \phi$ means the following function:

$$\int_S A_{N_h}(x_{j_1^h}, \dots, x_{j_{N_h}^h} | x_{i_h}, dy)(\varphi_h \cdot \phi)(x_1, \dots, \widehat{y}, \dots, x_n).$$

Now, the number of such $J=(j_1^1, \dots, j_{N_1}^1, \dots, j_1^k, \dots, j_{N_k}^k)$, where no number appears more than once and each j_i^h belongs to $\{M+1, \dots, n\}$, is equal to $(n-M)(n-M-1)\dots(n-M-N_1-\dots-N_k+1)=(n-M)!/(n-M-N_1-\dots-N_k)!$ and the sum over $(i_1, \dots, i_k)$ of those terms with one fixed J of this sort is equivalent modulo I_0^c to

$$n^{-(N_1+\dots+N_k)} Q(N_k)\varphi_k \dots\dots Q(N_1)\varphi_1.$$

Hence, the sum of all terms with such J is the first summand of (4.2).

On the other hand, the number of J 's where at least one of j_i^h belongs to $\{1, \dots, M\}$ is bounded by

$$(N_1+\dots+N_k)Mn^{N_1+\dots+N_k-1}$$

The number of J 's, where some j_i^h and $j_{i'}^{h'}$ coincide for some $h \neq h'$ is bounded by

$$\begin{aligned} & [N_1(N_2+\dots+N_k)+N_2(N_3+\dots+N_k)+\dots+N_{k-1}N_k]n^{N_1+\dots+N_k-1} \\ & \leq (N_1+\dots+N_k)^2 \cdot n^{N_1+\dots+N_k-1}, \end{aligned}$$

since the number of J 's, where some j_i^h coincides with some $j_{i'}^{h'}$ for $h' > h$, but not for $h' < h$, is bounded by $N_h(N_{h+1}+\dots+N_k)n^{N_1+\dots+N_k-1}$. And each partial sum over $(i_1, \dots, i_k)$ with one fixed J is bounded by

$$n^{-(N_1+\dots+N_k)} M(M+N_1)\dots(M+N_1+\dots+N_{k-1})C_{N_1}\dots C_{N_k}\|\varphi_1\|\dots\|\varphi_k\|.$$

*) By the first term of the right hand side of (4.2) we mean 0 when $N_1+\dots+N_k > n-M$.

Thus, the partial sum $e_n(N_k, \dots, N_1; \varphi_k, \dots, \varphi_1)$ is bounded by

$$\left\{ \frac{1}{n} M(N_1 + \dots + N_k) + \frac{1}{n} (N_1 + \dots + N_k)^2 M(M + N_1) \dots (M + N_1 + \dots + N_{k-1}) \right\} \\ C_{N_1} \dots C_{N_k} \|\varphi_1\| \dots \|\varphi_k\| \leq \frac{2}{n} (N_1 + \dots + N_k + M)^{k+2} C_{N_1} \dots C_{N_k} \|\varphi_1\| \dots \|\varphi_k\|$$

Then, by a formula of Dudley Johnson [2, p.283]:

$$\sum_{N_1=1}^{\infty} \dots \sum_{N_k=1}^{\infty} (M + N_1 + \dots + N_k)^q C_{N_1} \dots C_{N_k} \leq q! (2L)^q e^{M/L} \cdot 2^k,$$

we have the following estimate:

$$\sum_{N_k=0}^{n-1} \dots \sum_{N_1=0}^{n-1} \|e_n(N_k, \dots, N_1; \varphi_k, \dots, \varphi_1)\| \\ \leq \frac{2}{n} \|\varphi_1\| \dots \|\varphi_k\| \sum_{N_1=0}^{\infty} \dots \sum_{N_k=0}^{\infty} (N_1 + \dots + N_k + M)^{k+2} C_{N_1} \dots C_{N_k} \\ \leq \frac{2}{n} \|\varphi_1\| \dots \|\varphi_k\| \sum_{h=0}^k \binom{k}{h} C_0^h \sum_{N_{h+1}=1}^{\infty} \dots \sum_{N_k=1}^{\infty} (N_{h+1} + \dots + N_k + M)^{k+2} C_{N_{h+1}} \dots C_{N_k} \\ \leq \frac{2}{n} \|\varphi_1\| \dots \|\varphi_k\| (k+2)! (2L)^{k+2} e^{M/L} \sum_{h=0}^k \binom{k}{h} C_0^h 2^{k-h} \\ \leq \frac{2}{n} (k+2)! (2L(2+C_0))^{k+2} e^{M/L} \|\varphi_1\| \dots \|\varphi_k\|.$$

THEOREM 4. *If $\varphi_1, \dots, \varphi_k$ have indices in $I_0 = \{1, \dots, M\}$, and $t_1 + \dots + t_k < 1/2L(M+C_0+1)$, then,*

$$e^{t_k G_n} \varphi_k \dots e^{t_1 G_n} \varphi_1 - e^{t_k D} \varphi_k \dots e^{t_1 D} \varphi_1 \\ \equiv e_n(t_1, \dots, t_k; \varphi_1, \dots, \varphi_k) \pmod{I_0^c},$$

where

$$\|e_n(t_1, \dots, t_k; \varphi_1, \dots, \varphi_k)\| \rightarrow 0, \text{ as } n \rightarrow \infty,$$

for any possible choice of indices.

Proof. We have, by Lemma 6,

$$e^{t_k G_n} \varphi_k \dots e^{t_1 G_n} \varphi_1 - e^{t_k D} \varphi_k \dots e^{t_1 D} \varphi_1 \\ = \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} (G_n^{p_k} \varphi_k \dots G_n^{p_1} \varphi_1 - D^{p_k} \varphi_k \dots D^{p_1} \varphi_1) \\ = \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} \left(\sum_{N_{p_1}+\dots+p_k=0}^{n-1} \dots \sum_{N_1=0}^{n-1} Q_n(N_{p_1} + \dots + p_k) \dots \right. \\ \left. Q_n(N_{p_1+\dots+p_{k-1}+1}) \varphi_k \dots \varphi_2 Q_n(N_{p_1}) \dots Q_n(N_1) \varphi_1 \right. \\ \left. - \sum_{N_{p_1}+\dots+p_k=0}^{\infty} \dots \sum_{N_1=0}^{\infty} Q(N_{p_1} + \dots + p_k) \dots Q(N_{p_1+\dots+p_{k-1}+1}) \varphi_k \dots \varphi_2 Q(N_{p_1}) \dots Q(N_1) \varphi_1 \right) \\ \equiv \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} \sum_{N_{p_1}+\dots+p_k=0}^{n-1} \dots \sum_{N_1=0}^{n-1} \\ \left(\frac{(n-M)!}{(n-M-N_1-\dots-N_{p_1+\dots+p_k})! n^{N_1+\dots+N_{p_1+\dots+p_k}}} - 1 \right) \\ Q(N_{p_1+\dots+p_k}) \dots Q(N_{p_1+\dots+p_{k-1}+1}) \varphi_k \dots \varphi_2 Q(N_{p_1}) \dots Q(N_1) \varphi_1$$

$$\begin{aligned}
& + \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} \sum_{N_{p_1+\dots+p_k}=0}^{n-1} \dots \sum_{N_1=0}^{n-1} e_n(N_{p_1+\dots+p_k}, \dots, N_2, N_1; \\
& \quad \underbrace{1, \dots, 1}_{p_k-1}, \varphi_k, 1, \dots, \varphi_2, \underbrace{1, \dots, 1}_{p_1-1}, \varphi_1) \\
& - \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} \sum_{(N_1 < n, \dots, N_{p_1+\dots+p_k} < n)^c} Q(N_{p_1+\dots+p_k}) \dots \\
& \quad \dots Q(N_{p_1+\dots+p_{k-1}+1}) \varphi_k \dots Q(N_1) \varphi_1 \\
& = I_1 + I_2 + I_3 \pmod{I_0^c},
\end{aligned}$$

Here, $\sum_{(N_1 < n, \dots, N_{p_1+\dots+p_k} < n)^c}$ means the sum over $(N_1, \dots, N_{p_1+\dots+p_k})$, where at least one of N_i is larger than n ($1 \leq i \leq p_1 + \dots + p_k$).

By Lemma 6, $\|I_2\|$ is bounded by

$$\begin{aligned}
(4.3) \quad & \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} (p_1 + \dots + p_k + 2)! (2L(2+C_0))^{p_1+\dots+p_k+2} \frac{2}{n} e^{M/L} \|\varphi_1\| \dots \|\varphi_1\| \\
& = \frac{2}{n} e^{M/L} 2(1-2L(2+C_0)(t_1 + \dots + t_k))^{-3} (2L(2+C_0))^3 \|\varphi_1\| \dots \|\varphi_k\|,
\end{aligned}$$

and hence, $\|I_2\|$ is arbitrarily small for sufficiently large n . As we have seen in the proof of Lemma 2, we have

$$\begin{aligned}
(4.4) \quad & \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \sum_{N_{p_1+\dots+p_k}=0}^{\infty} \|Q(N_{p_1+\dots+p_k}) \dots \\
& \quad \dots Q(N_{p_1+\dots+p_{k-1}+1}) \varphi_k \dots Q(N_2) Q(N_1) \varphi_1\| < \infty,
\end{aligned}$$

and hence, $\|I_3\|$ is also arbitrarily small for sufficiently large n . Finally, I_1 is written as

$$\begin{aligned}
& \sum_{p_1=0}^{\infty} \dots \sum_{p_k=0}^{\infty} \frac{t_1^{p_1} \dots t_k^{p_k}}{p_1! \dots p_k!} \sum_{N_{p_1+\dots+p_k}=0}^{\infty} \dots \sum_{N_1=0}^{\infty} a(n, M, N_1, \dots, N_{p_1+\dots+p_k}) \\
& \quad Q(N_{p_1+\dots+p_k}) \dots Q(N_{p_1+\dots+p_{k-1}+1}) \varphi_k \dots \varphi_2 Q(N_{p_1}) \dots Q(N_1) \varphi_1,
\end{aligned}$$

where

$$\begin{aligned}
a(n, M, N_1, \dots, N_{p_1+\dots+p_k}) &= \frac{(n-M)!}{(n-M-N_1-\dots-N_{p_1+\dots+p_k})! n^{N_1+\dots+N_{p_1+\dots+p_k}}} - 1, \\
&\quad \text{if } 0 \leq N_1, \dots, N_{p_1+\dots+p_k} \leq n-1, \\
&= 0, \text{ if otherwise.}
\end{aligned}$$

Then, by the bounded convergence theorem, (4.4) and

$$\begin{aligned}
-1 &\leq a(n, M, N_1, \dots, N_{p_1+\dots+p_k}) \leq 0 \\
a(n, M, N_1, \dots, N_{p_1+\dots+p_k}) &\rightarrow 0, \text{ as } n \rightarrow \infty
\end{aligned}$$

imply that $I_1 \rightarrow 0$, as $n \rightarrow \infty$.

COROLLARY. $P_J^f(t, x_1, \dots, x_h; \cdot)$ and $f_{t,J}(\cdot)$ of Lemma 3 are probability measures on $(S^J, \mathbf{B}(S^J))$, where $J = \{1, 2, \dots, k\}$, $h \leq k$, and $t < 1/2[2L(k+C_0+1)]$.

Proof. Let $J_1 = \{1, 2, \dots, h\}$ and take I_0 so that $J \subset I_0$. Since $\phi_1 \equiv \phi_2 \pmod{I_0^c}$ implies $\int_{J_1^c} f^\infty \phi_1 = \int_{J_1^c} f^\infty \phi_2$, we have, by Theorem 4,

$$\int_{J_1^c} f^\infty e^{tD} \varphi = \lim_{n \rightarrow \infty} \int_{J_1^c} f^\infty e^{tG_n} \varphi,$$

for $t < 1/[2L(k+C_0+1)]$ and $\varphi \in F_0$ with $I(\varphi)=J$. But, the left hand side is $\int_{S^J} P_J^f(t, x, \dots, x_k; dy_1 \dots dy_k) \varphi(y_1, \dots, y_k)$, while the limitand on the right is non-negative, if φ is non-negative. The right hand side is clearly 1, if $\varphi \equiv 1$. Thus, $P_J^f(t, x_1, \dots, x_k; \cdot)$ is a probability measure on S^J , implying that $f_{t,J}$ is also a probability measure by (3.4).

Proof of Theorem 2. Let $\varphi_h(x_1) = \chi_{A_h}(x_1)$, $\phi_h(x_2) = \chi_{B_h}(x_2)$, $h=1, 2, \dots, k$, and let $I_0 = \{1, 2\}$, where $\chi_A(x)$ is defined by $\chi_A(x) = 1$ or 0, according as $x \in A$ or not. Now, the left hand side of (1.10) is equal to

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{S^n} f^{(n)} e^{t_1 G_n} \varphi_1 \phi_1 e^{(t_2 - t_1) G_n} \varphi_2 \phi_2 \dots e^{(t_k - t_{k-1}) G_n} \varphi_k \phi_k \\ &= \lim_{n \rightarrow \infty} \int f^\infty e^{t_1 G_n} \varphi_1 \phi_1 e^{(t_2 - t_1) G_n} \varphi_2 \phi_2 \dots e^{(t_k - t_{k-1}) G_n} \varphi_k \phi_k \end{aligned}$$

which is, by Theorem 4, equal to

$$\int f^\infty e^{t_1 D} \varphi_1 \phi_1 \dots e^{(t_k - t_{k-1}) D} \varphi_k \phi_k, \quad \text{for } t_k < \frac{1}{2L(3+C_0)}.$$

But, by (2.9) in the Corollary to Lemma 1, this is equal to

$$\begin{aligned} & \int f^\infty (e^{t_1 D} \varphi_1 e^{(t_2 - t_1) D} \varphi_2 \dots e^{(t_k - t_{k-1}) D} \varphi_k) \otimes (e^{t_1 D} \phi_1 \dots e^{(t_k - t_{k-1}) D} \phi_k) \\ &= \int f^\infty e^{t_1 D} \varphi_1 \dots e^{(t_k - t_{k-1}) D} \varphi_k \cdot \int f^\infty e^{t_1 D} \phi_1 \dots e^{t_k D} \phi_k \\ &= P^f(x(t_1) \in A_1 \dots x(t_k) \in A_k) \cdot P^f(x(t_1) \in B_1 \dots x(t_k) \in B_k), \end{aligned}$$

by Theorem 3. The second equality of (1.10) is now clear.

§ 5. Examples and comments

a) Consider an equation

$$(5.1) \quad \frac{\partial u(t, x)}{\partial t} = \frac{1}{2\pi} \int_{R^1 \times [0, 2\pi)} [u(t, x^*) u(t, y^*) - u(t, x) u(t, y)] dy d\theta$$

where

$$x^* = x \cos \theta - y \sin \theta, \quad y^* = x \sin \theta + y \cos \theta$$

and $u(t, x)$ is supposed to be a density function of a probability measure on $S = R^1$ with respect to the Lebesgue measure. Clearly, (5.1) corresponds to the one dimensional model of Maxwellian gas given by Mark Kac [3].*) Since this is equivalent to

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{\pi} \iint_{x^{*2} + y^{*2} > x^2} \frac{u(t, x^*) u(t, y^*)}{\sqrt{x^{*2} + y^{*2} - x^2}} dx^* dy^* - u(t, x),$$

we can rewrite (5.1) in the following way.

$$(5.2) \quad \frac{\partial u(t, \cdot)}{\partial t} = \int_{R^2} u(t, dx) u(t, dx_1) A_1(x_1 | x, \cdot),$$

where

$$A_1(x_1 | x, E) = \frac{1}{\pi} \int_{E \cap \{y | x^2 + x_1^2 > y^2\}} \frac{dy}{\sqrt{x^2 + x_1^2 - y^2}} - \delta_x(E).$$

This is just (1.2), where

$$A_0(x, \cdot) \equiv A_2(x_1, x_2 | x, \cdot) \equiv \dots \equiv 0,$$

*) See also McKean [5, 6].

$$q_1(x_1|x) \equiv 1, \quad \pi_1(x_1|x, E) = \frac{1}{\pi} \int_{E \cap (x^2 + x_1^2 > y^2)} \frac{dy}{\sqrt{x^2 + x_1^2 - y^2}}.$$

This corresponds to the situation that the motion is determined by interactions between two particles.....binary collisions alone. Here, the hitting measure depends on the distribution $u(t, \cdot)$ at time t , but the waiting time for jumps is independent on $u(t, \cdot)$.

b) In case $S = \{1, -1\}$, the mass of π_N necessarily concentrates at one point, and hence,

$$\pi_N(x, \dots, x_N | x, \cdot) = \delta_{\{-x\}}(\cdot), \quad x = \pm 1,$$

independently on N and $x_1, \dots, x_N$. q_N being symmetric in $x_1, \dots, x_N$, there is a non-negative function $C_N(x, k)$ in $(x, k) \in S \times \{1, \dots, N\}$ such that

$q_N(x_1, \dots, x_N | x) = C_N(x, \text{the number of } 1' \text{ s in } \{x_1, \dots, x_N\})$, where $C_N(x, \cdot)$ satisfies the growth condition.

$$(5.3) \quad \sum_{N=0}^{\infty} N^P (\max_{x, k} C_N(x, k)) = \frac{1}{2} \sum_{N=0}^{\infty} N^P \|A_N\| \leq \frac{1}{2} L^P \cdot P!$$

Then, $A^{(u)}(x, \cdot)$ is given, writing $u(\{1\}) = p$ and $u(\{-1\}) = q$, $1 - p$ by

$$\begin{aligned} A^{(u)}(x, \cdot) &= \sum_{N=0}^{\infty} \int_S \prod_{i=1}^N u(dx_i) q_N(x_1, \dots, x_N | x) (\delta_{\{x\}}(\cdot) - \delta_{\{-x\}}(\cdot)) \\ (5.4) \quad &= \sum_{N=0}^{\infty} \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} C_N(x, k) (\delta_{\{-x\}}(\cdot) - \delta_{\{x\}}(\cdot)) \\ &= a(x, p) (\delta_{\{-x\}}(\cdot) - \delta_{\{x\}}(\cdot)). \end{aligned}$$

Hence, equation (1.1) can be rewritten as

$$\begin{aligned} \frac{d}{dt} (p_t, q_t) &= (p_t, q_t) \begin{pmatrix} -a(1, p_t) & a(1, p_t) \\ a(-1, p_t) & -a(-1, p_t) \end{pmatrix} \\ (p_t, q_t) &\rightarrow (p_0, q_0), \end{aligned}$$

or, equivalently,

$$\begin{aligned} \frac{d}{dt} p_t &= -a(1, p_t) p_t + a(-1, p_t) (1 - p_t), \\ p_t &\rightarrow p_0. \end{aligned}$$

This is the case which Dudley Johnson described in [1, 2]. He characterized the class of functions $a(x, p)$ of p , expanded as in (5.4) with non-negative $C_N(x, k)$ satisfying the growth condition (5.3), as the set of all functions which are non-negative, analytic on $[0, 1]$ and positive on $(0, 1)$. We easily obtain a similar result, when $S = \{1, 2, \dots, m\}$ and

$$\begin{aligned} A^{(u)}(x, \cdot) &= \sum_{N=0}^{\infty} \sum_{k_1 + \dots + k_m = N} \binom{N}{k_1, \dots, k_m} p_1^{k_1} \dots p_m^{k_m} C_N(k_1, \dots, k_m | x) \cdot (\pi(x, \cdot) - \delta_{\{x\}}(\cdot)) \\ &= a(x, p_1, \dots, p_m) (\pi(x, \cdot) - \delta_x(\cdot)), \end{aligned}$$

where C_N satisfies $\sum_{N=0}^{\infty} N^p (\max C_N) \leq \frac{1}{2} L^p \cdot p!$, and $\pi(x, \cdot)$ is a probability measure on S with $\pi(x, \{x\}) = 0$. It is easy to prove that $a(x, p_1, \dots, p_m) = a(x, p_1, \dots, p_{m-1}, 1 - \sum_{k=1}^{m-1} p_k)$ is non-negative, analytic in $p_1, \dots, p_{m-1}$ on $\{(p_1, \dots, p_{m-1}) | 0 \leq p_1, \dots, p_{m-1}, p_1 + \dots + p_{m-1} \leq 1\}$, and positive at inner points, while the converse is still open for $m > 2$.

c) Let $S = \{\dots, -1, 0, 1, 2, \dots\}$ and consider

$$(5.5) \quad \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} (u(t, x+1)^{N+1} + u(t, x-1)^{N+1} - 2u(t, x)^{N+1}),$$

$$u(t, \cdot) \rightarrow f(\cdot), \quad t \rightarrow 0.$$

This is a discrete model of the non-linear equation

$$\frac{\partial}{\partial t} u(t, x) = \mathcal{A}(u(t, x)^{N+1}), \quad u(t, x) \rightarrow f(x), \quad t \rightarrow 0$$

and can be discussed in our scheme, by taking

$$A_0 \equiv A_1 \equiv \dots \equiv A_{N-1} \equiv A_{N+1} \equiv \dots \equiv 0$$

$$q_N(x_1, \dots, x_N | x) = 1, \text{ if } x_1 = \dots = x_N = x$$

$$= 0, \text{ if otherwise,}$$

$$\pi_N(x_1, \dots, x_N | x, \cdot) = \frac{1}{2} (\delta_{\{x+1\}}(\cdot) + \delta_{\{x-1\}}(\cdot)).$$

Similarly, a discrete model of the following equation can be considered,

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} (F(u(t, x))) + \frac{\partial}{\partial x} (G(u(t, x)))$$

where F and G (or $-G$) are polynomials or analytic functions satisfying a suitable growth condition with positive coefficients. For instance, a discrete model of Burger's equation*) is given by

$$\frac{\partial}{\partial t} u(x) = \frac{1}{2} (u(x+1) + u(x-1) - 2u(x)) - \frac{1}{2} (u(x+1)^2 - u(x-1)^2).$$

d) As far as the equation (1.1) is concerned, an arbitrary signed measure or a complex valued measure f with bounded variation can be an initial value, in which case the solution $u(t, x)$ is a signed measure or a complex valued measure. In fact, (2.5) shows that we do not really need an infinite product measure f^∞ when we integrate a function in F . Only the time interval, in which expansions of exponentials have meaning, needs to be changed by simply multiplying $1/\|f\|$.

Moreover, $A^{(u)}(x, \cdot)$ need not be a stochastic generator kernel for the equation. In fact, $A_N(x_1, \dots, x_N | x, \cdot)$ in §1 can be replaced by an arbitrary complex valued kernel satisfying the measurability in $(x_1, \dots, x_N, x)$, symmetry in $(x_1, \dots, x_N)$ and the condition (1.3) on the total variation. The same is true also for a linear combination of such kernels. Theorem 4 also remains true for such kernels.

References

- [1] D.P. Johnson, thesis, Massachusetts Institute of Technology (unpublished).
- [2] —, On a class of stochastic processes and its relationship to infinite particle gases, Trans. Amer. Math. Soc., **132** (1968), 275–295.
- [3] M. Kac, Probability and related topics in the physical sciences, Wiley, New York, 1959.
- [4] H.P. McKean, Jr., A class of Markov processes associated with non-linear parabolic equations, Proc. Nat. Acad. Sci., **56** (1966), 1907–1911.
- [5] —, Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas, Archive Rat. Mech. Anal., **21** (1966), 343–367.
- [6] —, An exponential formula for solving Boltzmann's equation for a Maxwellian gas, Journal of Combinatorial Theory, **2** (1967), 358–382.

THE UNIVERSITY OF TOKYO and THE ROCKEFELLER UNIVERSITY

*) See McKean [4, 6].