Squaring operations on truncated polynomial algebras

Dedicated to Professor Atuo Komatu
for his 60-th birthday

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Introduction.

In algebraic topology there are many important examples of cohomology rings having the structure of a polynomial algebra over Steenrod algebra. It is a basic problem to classify such algebras. In the present paper we discuss the following problem: under what conditions on the set A of the degrees of the variables does a polynomial algebra $P(A) = \mathbb{Z}_2[x_1, \dots, x_l]$ admit Steenrod squaring operations? For simplicity of discussions, throughout the paper we assume $A = (a_1, \dots, a_l)$ is a monotone increasing sequence of positive integers, i. e., $P(A) = \mathbb{Z}_2[x_a; a \in A]$, $\deg x_a = a$, has the generators of distinct degrees.

If a polynomial algebra $P(A)$ admits squaring operations, then so does the derived truncated polynomial algebra $TP(A) = P(A)/D^3$. We say that a sequence A is *allowable* if $TP(A)$ admits squaring operations. A disjoint union $\sum A_i$ of allowable sequences A_i is again allowable, and we say that A is *irreducible* if it is not a disjoint union of two non-empty allowable sequences. We denote $2^s A = (2^s a_j)$ for $A = (a_j)$, then $2^s A$ is allowable, if and only if A is so. We say that a sequence A is *simple* if it is allowable, irreducible and contains an odd integer. It is a direct consequence of the definitions that every allowable sequence A is of the form $\sum 2^{s_i} A_i$ for some integers $s_i \geq 0$ and some simple sequences A_i .

Our main purpose is to classify the simple sequences, and the result is the following

Main Theorem. *The simple sequences are the followings:*

$$T_1 = (1),$$

$$A_l = (2, 3, \dots, l+1), \quad l \geq 2, \quad l+1 \text{ is not a power of } 2,$$

$$G_m = A_{2^{m+1}-2} - (2^t + 1; 0 \leq t \leq m), \quad m \geq 2,$$

and $E_n = (2^{n+1} - 2^s; 0 \leq s \leq n), \quad n \geq 3.$

Examples of squaring operations on $P(A)$ for $A = T_1$, $A = A_l$ and $A = G_2 = E_2 = (4, 6, 7)$ are the cohomology rings of RP^∞ , $BSO(l+1)$ and BG_2 respectively. For other simple sequences we shall give squaring operations on $P(A)$ or $TP(A)$ as subquotient algebras of $H^*(BSO; \mathbb{Z}_2)$. As an application we have

Corollary. *Let X be an H -space. If the cohomology ring $H^*(X; \mathbb{Z}_2)$ is generated by primitive classes $\{u_j\}$ of distinct degrees, then the set $A = \{\deg u_j + 1\}$ is a disjoint union of $2^{s_i} A_i$ for some simple sequences A_i .*

1. A theorem of Thomas.

In the following we always assume that a sequence $A = (a_1, \dots, a_l)$ of integers satisfies

$$(*) \quad 1 \leq a_1 < a_2 < \cdots < a_l.$$

Denote by

$$P(A) = Z_2[x_{a_1}, \dots, x_{a_l}], \deg x_{a_i} = a_i,$$

the polynomial algebra over Z_2 with the variables x_a , $a \in A$. Let D be the positive part of $P(A)$, then $D^2 = D \cdot D$ consists of the decomposable elements. We denote the derived truncated polynomial algebra by

$$TP(A) = P(A)/D^3, \quad D^3 = D^2 \cdot D.$$

We say that $TP(A)$ (or $P(A)$) *admits squaring operations* if for each integer n there is a linear map $Sq^n: TP(A) \longrightarrow TP(A)$ (or $P(A) \longrightarrow P(A)$) satisfying the following conditions:

$$(1.1), \quad (i). \quad \deg(Sq^n x) = \deg x + n, \quad Sq^0 = id, \quad Sq^n = 0 \text{ for } n < 0.$$

$$(ii). \quad (\text{Cartan formula}) \quad Sq^n(x \cdot y) = \sum_{i+j=n} Sq^i x \cdot Sq^j y.$$

$$(iii). \quad (\text{Adem relation}) \quad Sq^a Sq^b = \sum_i \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i \text{ for } a < 2b.$$

$$(iv). \quad Sq^n x = x^2 \text{ if } \deg x = n \text{ and } Sq^n x = 0 \text{ if } \deg x < n.$$

Here $\binom{a}{b}$ stands for the binomial coefficient with the convention: $\binom{a}{b} = 0$ if $a < b$ or if $b < 0$.

We say that A is *allowable* (resp. *strictly allowable*) if $TP(A)$ (resp. $P(A)$) admits squaring operations. By Cartan formula, if $P(A)$ admits squaring operations then D^3 is closed under the squaring operations, and $TP(A)$ admits squaring operations.

(1.2). *If A is strictly allowable then it is allowable.*

Thomas has proved a very strong restriction on the behavior of Sq^n in $TP(A)$ [7; Theorem 1.4]. In our case this is restated as follows.

Theorem 1.1 (Thomas). *Let A be an allowable sequence. If $n \in A$, $k \geq 0$ and $\binom{n-k-1}{k} \equiv 1 \pmod{2}$ then $n-k \in A$, $Sq^k x_{n-k} \equiv x_n$ and $Sq^k x_n \equiv 0 \pmod{D^2}$.*

This theorem determines the squaring operations on $TP(A)/D^2 = P(A)/D^2$, that is, with the convention

$$(1.3) \quad x_n = 0 \text{ if } n \notin A$$

we have

$$\textbf{Lemma 1.2.} \quad Sq^k x_n \equiv \binom{n-1}{k} x_{n+k} \pmod{D^2}.$$

Proof. We shall prove by induction on k . First consider the case $k = 2^t$. Then $\binom{n-1}{2^t} \equiv 0$ is equivalent to $\binom{n-2^t-1}{2^t} \equiv 1$, and the lemma follows from the theorem. Next let $k = 2^{s+1}d + 2^s$ for some $d > 0$ and $s \geq 0$, and consider Adem relation: $Sq^{2^s} Sq^{2^{s+1}d} = Sq^k + \sum_{r=0}^{s-1} Sq^{k-2^r} Sq^{2^r}$. We have easily $\binom{n-1}{2^r} \binom{n+2^r-1}{k-2^r} \equiv 0$ and $\binom{n-1}{2^{s+1}d} \binom{n+2^{s+1}d-1}{2^s} \equiv \binom{n-1}{k}$. Then by induction hypothesis we have $Sq^{k-2^r} Sq^{2^r} x_n \equiv 0$ and $Sq^k x_n \equiv Sq^{2^s} Sq^{2^{s+1}d} x_n \equiv \binom{n-1}{k} x_{n+k}$.

We shall show various consequences of Thomas' theorem. For $k=1$, we have

$$(1.4). \quad \text{If } 2m+1 \in A \text{ then } 2m \in A \text{ and } Sq^1 x_{2m} \equiv x_{2m+1}, \quad Sq^1 x_{2m+1} \equiv 0 \pmod{D^2}.$$

For the simplicity of discussions we choose generators x_{2m+1} , $m \geq 1$, as follows.

$$(1.4)'. \quad Sq^1 x_{2m} = x_{2m+1} \text{ if } 2m+1 \in A, \text{ thus } Sq^1 x_{2m+1} = 0 \text{ for } m \geq 1.$$

For $k=2$ we have

(1.5). If $n \equiv 0, 3 \pmod{4}$ then $Sq^2 x_n \equiv x_{n+2}$ and $n+2 \in A$ implies $n \in A$. If $n \equiv 1, 2 \pmod{4}$ then $Sq^2 x_n \equiv 0$.

Assume that $n = 2^{s+1}d + 2^s$ for $d > 0$, then $\binom{n-2^s-1}{2^s} \equiv 1$, and moreover $\binom{n-2^{s+1}-1}{2^{s+1}} \equiv d-1$. Thus we have the followings.

(1.6). Let $n = 2^{s+1}d + 2^s$. If $d > 0$, then $n \in A$ implies $n - 2^s \in A$. If d is even > 0 , then $n \in A$ implies $n - 2^{s+1} \in A$.

(1.7). For an allowable sequence $A = (a_1, a_2, \dots, a_l)$, $a_1 = 2^t$ for some t , and if $l \geq 2$ then $a_2 \geq 2^t + 2^{t-1}$.

As is easily seen

(1.8). If $1 \leq h \leq 2^t$ then $2^{t+1} + h \in A$ implies $2^t + h \in A$.

Assume that $2^t + 1 < n < 2^{t+1}$ and $2^{t+1} - n + 1$ is not a power of 2. Let $2^{t+1} - n + 1 = 2^{s+1}d + 2^s$, $d > 0$, then $\binom{n-1}{2^s} \equiv 1$, $2^t + 1 < n + 2^s < 2^{t+1}$, and $n + 2^s \in A$ implies $n \in A$. Therefore, if $2^{t+1} - 2^s + 1 \in A$ for $t > s > 0$ then $n \in A$ for $2^t + 1 < n < 2^{t+1}$ and by (1.8) $2^t - 2^{s-1} + 1 \in A$. Repeating this process we have

(1.9). If $2^{t+1} - 2^s + 1 \in A$ for $t > s > 0$ then $i \in A$ for $4 \leq i < 2^{t+1}$ and $i \neq 2^r + 1$, $r \leq t$, i.e., $G_t \subset A$.

Since $\binom{2^{t+1}-2^s}{2^s} \equiv 1$ for $t \geq s \geq 0$, $2^{t+1} + 1 \in A$ implies the assumption of (1.9) and $2^t + 1, 2^{t+1} \in A$, hence $i \in A$ for $2^t + 1 \leq i \leq 2^{t+1} + 1$. Repeating this process we have

(1.10). $2^t + 1 \in A$ implies $i \in A$ for $2 \leq i \leq 2^t + 1$, i.e., $A_2 \subset A$.

Since $\binom{2^t-2}{k} \equiv 1$ for even k of $2^t - 1 > k > 0$, we have

(1.11). If n is odd, $n \in A$ and $2^{t+1} - 1 > n > 2^t$ then $2^t - 1 \in A$.

(1.12). $n \in A$ and $2^{t+1} > n > 2^t$ imply $2^t \in A$.

Finally we see that Thomas' theorem determines the squaring operations in $D^2 \subset TTP(A)$.

$$(1.13). \quad Sq^k(x_m x_n) = \sum_{i+j=k} Sq^i x_m \cdot Sq^j x_n = \sum_{i+j=k} \binom{m-1}{i} \binom{n-1}{j} x_{m+i} x_{n+j} \text{ in } TTP(A).$$

2. Reduction to simple cases.

Let a sequence A be a disjoint union of two allowable sequences B and C . As the tensor product over mod 2 Steenrod algebra (Hopf algebra), $TP(B) \otimes TP(C)$ admits squaring operations, and $TP(A) = (TP(B) \otimes TP(C)) / D^3$ does so. Thus

(2.1). If B and C are allowable and $B \cap C = \emptyset$, then $A = B + C$ is allowable.

The notation $B < C$ (resp. $B \leq C$) means that $b < c$ (resp. $b \leq c$) for any $b \in B$ and $c \in C$.

Lemma 2.1. If $A = B + C$ is allowable, $B < (2^t) \leq C$ and $n \notin C$ for $2^t < n < 2^t + 2^{t-1}$, then B and C are both allowable.

Proof. Consider squaring operations on $TP(A) = Z_2[x_a; a \in A] / D^3$. Let I_B and I_C be the ideals of $TP(A)$ generated by $\{x_b; b \in B\}$ and $\{x_c; c \in C\}$ respectively. By (1.1), (iv), $Sq^{2^r} x_b = 0$ for $r \geq t$, $b \in B$. Let $r < t$, then $2^r + b < 2^t + 2^{t-1}$. Thus $Sq^{2^r} x_b \in I_B$ if $2^r + b \neq 2^t$. By Lemma 1.2, $Sq^{2^r} x_{2^t-2^r} \equiv \binom{2^t-2^r-1}{2^r} x_{2^t} \equiv 0 \pmod{D^2}$, and $Sq^{2^r} x_{2^t-2^r} \in I_B$. Since the operations Sq^{2^r} , $r \geq 0$ generate the squaring operations we have proved

that I_B is closed under the squaring operations. The ideal $I_B \cap I_C$ is spanned by $\{x_b x_c\}$. Then (1.13) shows that $I_B \cap I_C$ is closed under the squaring operations. Consequently $TP(A)/I_B = TP(C)$ and $I_B/(I_B \cap I_C) = TP(B)$ admit squaring operations, hence B and C are allowable.

We have defined that an allowable sequence is *irreducible* if it is not a disjoint union of two non-empty allowable sequences. Obviously an allowable sequence is a disjoint union $A = \sum A_i$ of irreducible sequences A_i . Let B and C be disjoint irreducible sequences whose minimum numbers are $b_1 = 2^s$ and $c_1 = 2^t$ for some s, t , by (1.7). We may assume $s < t$. By (1.12), any $b \in B$ satisfies $b < 2^t$ or $b \geq 2^{t+1}$.

Lemma 2.1 shows that there is no element of $b \in B$ with $b \geq 2^{t+1}$. Thus $B < C$, and we have

Theorem 2.2. *Any allowable sequence A is decomposed (uniquely) into a disjoint union $A = \sum A_i$ of irreducible sequences A_i satisfying $A_i < A_j$ for $i < j$.*

For each sequence $A = (a_1, a_2, \dots, a_l)$ and each integer $s \geq 0$, we define

$$2^s A = (2^s a_1, 2^s a_2, \dots, 2^s a_l)$$

and also

$$A/2^s = (b_1, \dots, b_k)$$

by that $b \in A/2^s$ if and only if $2^s b \in A$. Let A be allowable and $TP(A) = Z_2[x_a]/D^3$ admit squaring operations. Define ring homomorphisms (epimorphisms) $\theta: TP(A) \rightarrow TP(2A) = Z_2[y_{2a}]/D^3$ and $\lambda: TP(A) \rightarrow TP(A/2) = Z_2[z_b]/D^3$ by $\theta(x_a) = y_{2a}$, $\lambda(x_{2b}) = z_b$ and $\lambda(x_a) = 0$ for odd a . Then it is directly checked that $TP(2A)$ and $TP(A/2)$ admit squaring operations by putting $Sq^{2k} y_{2a} = \theta(Sq^k x_a)$, $Sq^{2k-1} y_{2a} = 0$ and $Sq^k z_b = \lambda(Sq^{2k} x_{2b})$. The same is true for $P(2A)$ and $P(A/2)$. Since $(2A)/2 = A$ we have

(2.2). *A is (strictly) allowable if and only if $2^s A$ is (strictly) allowable. If A is (strictly) allowable then so is $A/2^s$.*

We say that an allowable sequence is *simple* if it is irreducible and contains an odd integer. Obviously any irreducible sequence is of a form $2^s A'$ for some simple sequence A' . Thus we have

Corollary. *Any allowable sequence is a disjoint union $A = \sum 2^{s_i} A_i$ for some simple sequences A_i .*

We shall use the following lemmas.

Lemma 2.3. *An allowable sequence $A = (a_1, \dots, a_l)$ is irreducible if and only if the following condition is satisfied: if $2^t \in A$, $2^t > a_1$, then there exists an $n \in A$ such that $2^t < n < 2^t + 2^{t-1}$.*

Proof. If A is not irreducible, then by Theorem 2.2 $A = B + C$ for some allowable B and C with $B < C$. By (1.7), $C = (2^t, \dots)$ and $n \notin C$ for $2^t < n < 2^t + 2^{t-1}$, and the condition is not satisfied. Conversely, if the condition is not satisfied then A is not irreducible by Lemma 2.1.

Lemma 2.4. *If A is simple and $2 \notin A$, then $A/2$ is also simple or empty.*

Proof. By (2.2), $A/2$ is allowable. Consider $2^t \in A/2$ which is not minimum. Then $2^{t+1} \in A$ and it is not minimum. By Lemma 2.3 there exists an $m \in A$ such that $2^{t+1} < m < 2^{t+1} + 2^t$. If $m = 2n$ then $n \in A/2$ and $2^t < n < 2^t + 2^{t-1}$. Since $2 \notin A$,

$2^{t+1}+1 \notin A$ by (1.10). Thus, if $m=2n+1$ then $2^t < n < 2^t+2^{t-1}$, $2n \in A$ by (1.4), and $n \in A/2$. It follows from Lemma 2.3 that $A/2$ is irreducible. If $1 \in A$ then the irreducibility implies $A=(1)$, whence $A/2=\phi$. By (1.10), $3, 5 \notin A$. So, we may assume that A contains an odd integer $2n+1 \geq 7$, whence $n \in A/2$. If n is odd this shows that $A/2$ is simple. If n is even then $2n+1 \in A$ implies $2n-1 \in A$ by (1.5) and hence $n-1 \in A/2$. Thus $A/2$ is simple or empty.

3. A lemma.

The following lemma is a key to the discussions in the subsequent sections. If $n+1 \in A$ and $2n+1 \notin A$ for an allowable sequence A , then $Sq^n x_{n+1}$ is decomposable, i. e., we can write down

$$(3.1) \quad Sq^n x_{n+1} = \sum a_j x_{n-j} x_{n+1+j} \text{ for some } a_j \in \mathbb{Z}_2, j \geq 0.$$

Lemma 3.1. Assume that $i \in A$ for $n-k \leq i \leq n+k+1$, $k \geq 0$, and $2n+1 \notin A$. Then the coefficients a_j in the expansion (3.1) of $Sq^n x_{n+1}$ satisfy

$$\begin{aligned} a_j &= 1 \text{ for } 0 \leq j \leq k && \text{if } n+k \text{ is even or if } n \text{ is even and } n+k \equiv 1 \pmod{4}, \\ a_j &= 1 \text{ for } 0 \leq j \leq k-1 && \text{if } n+k \equiv 3 \pmod{4} \text{ or if } n \text{ and } n+k \text{ are odd.} \end{aligned}$$

Proof. Case 1, n : even. First consider the case that n is even. We have

$$Sq^1 Sq^n x_{n+1} = Sq^{n+1} x_{n+1} = x_{n+1}^2$$

and

$$Sq^2 Sq^n x_{n+1} = (n/2+1) Sq^{n+2} x_{n+1} + Sq^{n+1} Sq^1 x_{n+1} = 0$$

by (1.1) and (1.4'). Applying Sq^1 and Sq^2 to (3.1) we have, by (1.13), (1.4) and (1.5),

$$x_{n+1}^2 = Sq^1 Sq^n x_{n+1} = a_0 x_{n+1}^2 + \sum_{s \geq 1} (a_{2s-1} + a_{2s}) x_{n-2s+1} x_{n+2s+1},$$

$$\begin{aligned} \text{when } n \equiv 0 \pmod{4} \quad 0 &= Sq^2 Sq^n x_{n+1} = (a_0 + a_1) x_{n+1} x_{n+2} + \sum_{t \geq 0} \{ (a_{4t+2} + a_{4t+4}) x_{n-4t-2} x_{n+4t+5} \\ &\quad + (a_{4t+3} + a_{4t+5}) x_{n-4t-3} x_{n+4t+6} \} \end{aligned}$$

and when $n \equiv 2 \pmod{4}$

$$0 = Sq^2 Sq^n x_{n+1} = \sum_{t \geq 0} \{ (a_{4t} + a_{4t+2}) x_{n-4t} x_{n+4t+3} + (a_{4t+1} + a_{4t+3}) x_{n-4t-1} x_{n+4t+4} \}.$$

Thus we have the following sequences of equalities up to $j \leq k$.

$$1 = a_0 = a_1 = a_2 = \dots = a_{4t+2} = a_{4t+4} = a_{4t+3} = a_{4t+5} = a_{4t+6} = \dots \quad (n \equiv 0),$$

$$1 = a_0 = a_2 = a_4 = \dots = a_{4t} = a_{4t+2} = a_{4t+1} = a_{4t+3} = a_{4t+4} = \dots \quad (n \equiv 2).$$

It follows that $a_j = 1$ for $0 \leq j \leq k$ if $n+k \equiv 3 \pmod{4}$ and $a_j = 1$ for $0 \leq j \leq k-1$ if $n+k \equiv 1 \pmod{4}$.

Case 2, n : odd. Next consider the case that n is odd. By (1.5) $2n+1 \notin A$ implies $2n+3 \notin A$. By what we have just proved, in

$$Sq^{n+1} x_{n+2} = \sum_{i \geq 0} b_i x_{n+1-i} x_{n+2+i}$$

$$b_i = 1 \text{ for } 0 \leq i < k \text{ when } n+k \equiv 2 \pmod{4}$$

and

$$b_i = 1 \text{ for } 0 \leq i < k-1 \text{ when } n+k \equiv 0 \pmod{4}.$$

From the relation $0 = Sq^1 Sq^n x_{n+1} = \sum_{s \geq 0} (a_{2s} + a_{2s+1}) x_{n-2s} x_{n+2s+2}$, it follows

$$(*) \quad a_{2s} = a_{2s+1} \text{ for } 0 \leq s \leq (k-1)/2.$$

Also we have when $n \equiv 1 \pmod{4}$

$$Sq^2 Sq^n x_{n+1} = \sum_{t \geq 0} \{ a_{4t+1} (x_{n-4t+1} x_{n+4t+2} + x_{n-4t-1} x_{n+4t+4})$$

$$+ a_{4t+2}(x_{n-4t}x_{n+4t+3} + x_{n-4t-2}x_{n+4t+5})\}$$

and when $n \equiv 3 \pmod{4}$

$$Sq^2 Sq^n x_{n+1} = \sum_{t \geq 0} \{a_{4t}(x_{n-4t+2}x_{n+4t+1} + x_{n-4t}x_{n+4t+3})$$

$$+ a_{4t+3}(x_{n-4t-1}x_{n+4t+4} + x_{n-4t-3}x_{n+4t+6})\}.$$

In the equality $Sq^2 Sq^n x_{n+1} = ((n-1)/2) Sq^{n+2} x_{n+1} + Sq^{n+1} Sq^1 x_{n+1} = Sq^{n+1} x_{n+2}$, comparing the coefficients we have when $n \equiv 1 \pmod{4}$

$$a_{4t+1} = b_{4t} = 1 \text{ for } 4t \leq k-1 \text{ and } a_{4t+2} = b_{4t+1} = 1 \text{ for } 4t+1 \leq k-2,$$

and when $n \equiv 3 \pmod{4}$

$$a_{4t} = b_{4t-1} = 1 \text{ for } 4t-1 \leq k-2 \text{ and } a_{4t+3} = b_{4t+2} = 1 \text{ for } 4t+2 \leq k-1.$$

Combining with (*) we have that $a_j = 1$ for $0 \leq j \leq k$ if $n+k$ is even and $a_j = 1$ for $0 \leq j \leq k-1$ if $n+k$ is odd.

The proof of the lemma has been established.

We shall use the following cases of Lemma 3.1 ($t \geq 2$).

$$(3.2). \quad n = 2^t, k = 1; Sq^{2^t} x_{2^t+1} = x_{2^{t-1}} x_{2^t+2} + \text{other terms.}$$

$$(3.3). \quad 2^{t-1} + 1 \leq n < 2^t - 1, n+k+1 = 2^t + 1; Sq^n x_{n+1} = x_{2^{n+1}-2^t} x_{2^t} + \text{other terms.}$$

$$(3.4). \quad n \text{ is even, } k = 0; Sq^n x_{n+1} = x_n x_{n+1} + \text{other terms.}$$

$$(3.5). \quad n = 2^t, k = 2^t - 2; Sq^{2^t} x_{2^t+1} = x_2 x_{2^t+1-1} + \text{other terms.}$$

The following special modification of the lemma will be used.

$$(3.6). \quad \text{If } 2^t + 2^{t-1} - 1 \in A \text{ and } 2^{t+1} - 1 \notin A, t \geq 2, \text{ then } Sq^{2^t-1} x_{2^t} = x_{2^{t-1}} x_{2^t} + \text{other terms.}$$

In fact by use of (1.1), $Sq^{2^t-1} Sq^{2^t-1} x_{2^t} = \sum_{s=0}^{t-1} Sq^{2^{t+1}-2^s-1} Sq^{2^s-1} x_{2^t} = Sq^{2^t+2^{t-1}} Sq^{2^{t-1}-1} x_{2^t} = Sq^{2^t+2^{t-1}-1} (x_{2^{t+2}-1-1} + \text{decomposable terms}) = x_{2^{t+2}-1-1}$. Assume $x_{2^{t+2}-1-1} = Sq^{2^{t-1}+j} x_{2^{t-j-1}}$. $Sq^{2^{t-1}-j-1} x_{2^{t+j}} = \binom{2^t-j-2}{2^{t-1}+j} \binom{2^t+j-1}{2^{t-1}-j-1} x_{2^{t+2}-1-1}$ by (1.13). If $j > 0$, then $0 \equiv \binom{2^t+j-1}{2^{t-1}-j-1} \equiv \binom{j-1}{2^{t-1}-j-1}$ implies $j-1 \geq 2^{t-1}-j-1$, $2^{t-1}+j \geq 2^t-j-2$ and hence $\binom{2^t-j-2}{2^{t-1}+j} \equiv 0$. Thus $x_{2^{t+2}-1-1}$ is only covered by $Sq^{2^t-1} (x_{2^{t-1}} x_{2^t})$, and the coefficient of $x_{2^{t-1}} x_{2^t}$ must be 1 modulo 2.

4. Classical cases.

The purpose of this section is to prove the following

Theorem 4.1. *If a simple sequence A contains $2 \in A$, then*

$$A = A_l = (2, 3, \dots, l+1)$$

for some $l+1$ which is not a power of 2.

First consider $Sq^{2a} Sq^n x_{n+1}$. If $a > n$ it is trivial by (1.1), (iv). Let $a < n$. By (iii) and (iv) of (1.1) we have $Sq^{2a} Sq^n x_{n+1} = Sq^{a+n} Sq^a x_{n+1}$. Now assume $\binom{n}{a} \equiv 0$, then $Sq^a x_{n+1}$ is decomposable. It follows easily from (ii) and (iv) of (1.1) that $Sq^{a+n} Sq^a x_{n+1} = 0$ (in $TP(A)$). Similar arguments hold if $a+n+1 \notin A$, and we have

$$(4.1). \quad \text{If } \binom{n}{a} \equiv 0 \pmod{2} \text{ or if } a+n+1 \notin A \text{ then } Sq^{2a} Sq^n x_{n+1} = 0.$$

Lemma 4.2. *Let $t \geq 2$ and A be an allowable sequence. If $2^t + 1 \in A$ and $2^{t+1} + 2 \in A$ then $2^{t+1} + 1 \in A$.*

Proof. By (1.10) and (1.8) the assumption implies $i \in A$ for $2 \leq i \leq 2^t + 2$. Assume

that $2^{t+1}+1 \notin A$, then we can apply Lemma 3.1 for the case (3.2), and $Sq^{2^t}x_{2^t+1} = x_{2^t-1}x_{2^t+2} + \text{other terms}$. To this equality apply Sq^{2^t} . Since $\binom{2^t}{2^{t-1}} \equiv 0$, $Sq^{2^t}Sq^{2^t}x_{2^t+1} = 0$ by (4.1). Thus $0 = Sq^{2^t}(\sum_{j \geq 0} a_j x_{2^t-j} x_{2^t+1+j})$ where $a_1 = 1$. Comparing the coefficients in $x_{2^t-1}x_{2^t+1+2}$ we have

$$0 \equiv a_1 + \sum_{j \geq 2} a_j \binom{2^t-j-1}{j-1} \binom{2^t+j}{2^t-j+1} \pmod{2}.$$

Let $j \geq 2$, then $0 \equiv \binom{2^t+j}{2^t-j+1} \equiv \binom{j}{2^t-j+1}$ implies $j \geq 2^t-j+1$, $j-1 > 2^t-j-1$ and hence $\binom{2^t-j-1}{j-1} \equiv 0$. It follows a contradiction $0 \equiv a_1 = 1 \pmod{2}$. Therefore we obtain $2^{t+1}+1 \in A$.

Lemma 4.3. *Let $2 \leq m < 2^{t+1}$ and A be allowable. If $2^t+1 \in A$ and $m \in A$ then $i \in A$ for $2 \leq i \leq m$.*

Proof. We prove by induction on m . For the case $m \leq 2^t+2$ the lemma is obvious by (1.10). If m is odd then $m-1 \in A$ by (1.4) and the lemma is true by the induction hypothesis. So, it is sufficient to prove that $m = 2n+2 \in A$, $2^{t-1} < n < 2^t-1$, and $2^t+1 \in A$ imply $2n+1 \in A$. Assume that $2n+1 \notin A$, then by Lemma 3.1 for the case (3.3) we have $a_{2^t-n-1} = 1$ in $Sq^n x_{n+1} = \sum a_j x_{n-j} x_{n+1+j}$. Since $2^{t-1} < n < 2^t-1$, $\binom{n}{n+1-2^{t-1}} = \binom{n}{2^{t-1}-1} \equiv 0$. By (4.1) it follows $0 = Sq^{2n+2-2^t} Sq^n x_{n+1}$

$$= \sum a_j Sq^{2n+2-2^t} (x_{n-j} x_{n+1+j}), \text{ and comparing the coefficients in } x_{2n-2^t+1} x_{2n+2}$$

$$0 \equiv \sum_{j \geq 0} a_j \binom{n-j-1}{n+j-2^t+1} \binom{n+j}{n-j+1} \pmod{2}.$$

$\binom{n-j-1}{n+j-2^t+1} \equiv 0$ implies $n+j \geq 2^t-1$ and $n-j-1 \geq n+j-2^t+1$. If $n+j = 2^t-1$ then $\binom{n-j-1}{n+j-2^t+1} \binom{n+j}{n-j+1} \equiv 1$. If $n+j \geq 2^t$ then $\binom{n+j}{n-j+1} \equiv \binom{n+j-2^t}{n-j+1} \equiv 0$ since $n+j-2^t < n+j-2^t+1 \leq n-j-1 < n-j+1$. Thus we have a contradiction $0 \equiv a_{2^t-n-1} \equiv 1 \pmod{2}$. Therefore we obtain $2n+1 \in A$.

Proof of Theorem 4.1. If $1 \in A$ then A is not irreducible by Lemma 2.3. Thus $1 \notin A$. If $3 \notin A$ then $A = (2) + (A - (2))$ is not simple by Lemma 2.1. Thus $3 \in A$. Let t be the maximal number such that $2^t+1 \in A$, and $2^{t+1}+1 \notin A$, and let a_t be the maximal number of A . If $a_t < 2^{t+1}$ then it follows from Lemma 4.3 $A = A_t$. Now assume $a_t \geq 2^{t+1}$ and lead to a contradiction. If $2^{t+1} \notin A$ then $i \notin A$ for $2^{t+1} < i < 2^{t+2}$ by (1.12), and A is not irreducible by Lemma 2.1. Thus $2^{t+1} \in A$. By Lemma 2.3 there exists an $n \in A$ such that $2^{t+1} < n < 2^{t+1}+2^t$. Let $n_0 = 2^{t+1}+2^s(2a+1)$ be the minimal of such n . By (1.6), $n_0 \in A$ implies $n_0 - 2^s \in A$. By the choice of n_0 a must be zero and $n_0 = 2^{t+1}+2^s$. Since $2^{t+1}+1 \notin A$, $t > s \geq 1$. By (1.6), $2^{t+1}+2^s \in A$ implies $2^{t+1}-2^s \in A$. Then applying Lemma 4.3 we have $i \in A$ for $2 \leq i \leq 2^{t+1}-2^s$, in particular $2^t+2^{s-1} \in A$. Now consider the allowable sequence $A/2^{s-1}$. Then $2^{t-s+1} \in A/2^{s-1}$ and $2^{t-s+2}+2 \in A/2^{s-1}$. It follows from Lemma 4.2 that $2^{t-s+2}+1 \in A/2^{s-1}$. Since $2^{s-1}(A/2^{s-1}) \subset A$ we have $2^{t+1}+2^{s-1} \in A$ which contradicts to the minimum condition of $n_0 = 2^{t+1}+2^s$. We have proved the theorem.

5. Exceptional cases.

The purpose of this section is to prove the following theorems.

Theorem 5.1. *If a simple sequence A satisfies $2 \notin A$ and $4 \in A$ then, for some $m \geq 2$, $A = G_m = A_{2^{m+1}-2} - (2^t + 1; 0 \leq t \leq m) = (i; 4 \leq i \leq 2^{m+1}-1, i-1 \text{ is not a power of } 2)$.*

Theorem 5.2. *If a simple sequence A does not contain 1, 2 and 4 then, for some $n \geq 3$, $A = E_n = (2^{n+1} - 2^s; 0 \leq s \leq n) = (2^n, 2^n + 2^{n-1}, \dots, 2^{n+1} - 2, 2^{n+1} - 1)$.*

Obviously, if a simple sequence contains 1 then $A = T_1 = (1)$. This and Theorems 4.1, 5.1, 5.2, cover all the cases of simple sequences.

We prepare the following lemmas.

Lemma 5.3. *Let $t > s + 1$ and $s > 0$. If an allowable sequence A satisfies $2 \notin A$, $2^t - 2^s + 1 \in A$ and $2^t + 2^s \in A$ then $2^{t+1} - 2^{s+1} + 1 \in A$.*

Proof. Assume $2^{t+1} - 2^{s+1} + 1 \notin A$ then by Lemma 3.1 of the case (3.4) $Sq^{n x_{n+1}} = x_n x_{n+1} + \dots$ for $n = 2^t - 2^s$. Since $2 \notin A$, $2^s + n + 1 = 2^t + 1 \notin A$ by (1.10). It follows from (4.1) $0 = Sq^{2^{s+1}} Sq^{n x_{n+1}} = Sq^{2^{s+1}} \sum a_j x_{n-j} x_{n+1+j}$. Consider the coefficients in $x_{2^t+2^s} x_{2^t-2^s+1}$ then we have

$$0 \equiv a_0 \binom{2^t - 2^s - 1}{2^{s+1}} + \sum_{j \geq 1} a_j \binom{2^t - 2^s - j - 1}{j+1} \binom{2^t - 2^s + j}{2^{s+1} - j - 1} \pmod{2}$$

where $a_0 = 1$. Since $t > s + 1$, $\binom{2^t - 2^s - 1}{2^{s+1}} \equiv 1$. Let $j \geq 1$ and assume $\binom{2^t - 2^s + j}{2^{s+1} - j - 1} \equiv 1$ then $j + 1 \leq 2^{s+1}$. Consider the case $j + 1 \equiv 2^s$, 2^{s+1} then $j + 1 = 2^r(2a + 1)$, $r < s$, and $\binom{2^t - 2^s + j}{2^{s+1} - j - 1} = \binom{2^t - 2^s + 2^{r+1}a + 2^r - 1}{2^{s+1} - 2^{r+1}(a+1) + 2^r} \equiv 0$. If $j + 1 = 2^s$ then $\binom{2^t - 2^s - j - 1}{j+1} = \binom{2^t}{2^s} \equiv 0$. If $j + 1 = 2^{s+1}$ then $\binom{2^t - 2^s - j - 1}{j+1} = \binom{2^t - 2^{s+2} + 2^s}{2^{s+1}} \equiv 0$. Therefore we have a contradiction $0 \equiv a_0 \equiv 1 \pmod{2}$, and $2^{t+1} - 2^{s+1} + 1$ must be contained in A .

Lemma 5.4. *Let $t \geq 3$. If an allowable sequence A satisfies $2 \notin A$, $2^t + 2 \in A$ and $2^{t+1} - 3 \in A$ then $2^{t+1} - 1 \in A$.*

Proof. $2 \notin A$ implies $2^t + 1 \notin A$. Since $\binom{2^t + 2^{t-1} - 2}{2^{t-1} - 2} \equiv 1$, $2^{t+1} - 3 \in A$ implies $2^t + 2^{t-1} - 1 \in A$. Now assume that $2^{t+1} - 1 \notin A$, then by (3.6) we have $Sq^{2^t-1} x_{2^t} = x_{2^t-1} x_{2^t} + a_2 x_{2^t-3} x_{2^t+2} + \dots$. By use of (1.5) $Sq^2 Sq^{2^t-1} x_{2^t} = x_{2^t-1} x_{2^t+2} + \text{other terms}$. On the other hand we have by (4.1) $Sq^2 Sq^{2^t-1} x_{2^t} = 0$ since $2^{t+1} + 1 \notin A$. This is a contradiction and we conclude $2^{t+1} - 1 \in A$.

Lemma 5.5. *Let $t \geq 2$. If an allowable sequence A satisfies $2 \notin A$, $2^t + 2 \in A$ and $2^{t+1} - 1 \in A$ then $2^t + 3 \in A$ or $2^{t+1} + 2 \in A$.*

Proof. Assume that $2^t + 3 \notin A$ and $2^{t+1} + 2 \in A$. $2^{t+1} - 1 \in A$ implies $2^{t+1} - 2 \in A$. Then $A/2$ is allowable and satisfies $2^{t-1} + 1 \in A/2$, $2^t - 1 \in A/2$ and $2^t + 1 \in A/2$. By Lemma 4.3, $i \in A/2$ for $2 \leq i \leq 2^t - 1$. Then we can apply Lemma 3.1 for the case

(3.5) and we have $Sq^{2^t-1}x_{2^t-1+1}=x_2x_{2^t-1}+\text{other terms in }TP(A/2)$. Passing to A , we have $Sq^{2^t}x_{2^t+2}=x_4x_{2^t+1-2}+\text{other terms}$. Thus $Sq^{2^t+1}x_{2^t+2}=x_4x_{2^t+1-1}+\text{other terms}$. By Adem relation $Sq^{2^t+1}x_{2^t+2}=Sq^2Sq^{2^t-1}x_{2^t+2}+Sq^{2^t}Sq^1x_{2^t+2}$. Since $2 \notin A$ we have $2^{t+1}+1 \notin A$ and $Sq^{2^t-1}x_{2^t+2}$ is decomposable. By the assumption $2^t+3 \in A$, $Sq^1x_{2^t+2}$ is decomposable. Since $x_4 \equiv Sq^kx_{4-k}$ holds only for $k=0$, the possibility to cover $x_4x_{2^t+1-1}$ by $Sq^2D^2+Sq^{2^t}D^2$ is $x_4Sq^2x_{2^t+1-3}$ and $x_4Sq^{2^t}x_{2^t-1}$ both of which are trivial by (1.5) and (1.1), (iv). So, we arrive to a contradiction. Thus $2^t+3 \in A$ or $2^{t+1}+2 \in A$.

Proof of Theorem 5.1.

Case 1: $7 \in A$. Obviously $1 \notin A$. Since $2 \notin A$ $2^t+1 \notin A$ for all $t \geq 0$. $7 \in A$ implies $4, 6 \in A$ and $G_2=(4, 6, 7) \subset A$. Let m be the maximal number of m such that $G_m \subset A$, $m \geq 2$. Now assume that $G_m \not\subset A$. Then, by (1.4), there exists an integer n such that $2n \in A$ and $2n \geq 2^{m+1}$. By Lemma 2.4, $A/2$ is simple and $2 \in A/2$. Then by Theorem 4.1, $A/2 = A_l = (2, 3, \dots, l+1)$ for some $l \equiv 2^m-1$. Since $n \in A/2$ and $n \geq 2^m$, $A/2$ contains 2^m+1 , hence $2^{m+1}+2 \in A$. Since $2^{m+1}-1 \in G_m \subset A$, we have $2^{m+2}-3 \in A$ by Lemma 5.3 of $t=m+1$, $s=1$, and hence $2^{m+2}-1 \in A$ by Lemma 5.4. Then $2^{m+2}-2 \in A$, $2^{m+1}-1 \in A/2$ and $l+1 \geq 2^{m+1}-1$. Thus $2^m+2^{s-1} \in A/2$ and $2^{m+1}+2^s \in A$ for $m > s \geq 1$. Again by use of Lemma 5.3 of $t=m+1$, we have $2^{m+2}-2^{s+1}+1 \in A$ for $m > s \geq 1$ and also for $s=0$ as above. Then it follows from (1.9) $G_{m+1} \subset A$ contradicting the maximality of m . Thus we have proved $A=G_m$ for some $m \geq 2$.

Case 2: $7 \notin A$. By use of (1.8), we have $2^r+3 \notin A$ for $r \geq 2$. By (1.11), there exists some t with $2^{t+1}-1 \in A$, where $t \geq 3$ since $7 \notin A$. We choose the maximal t of $2^{t+1}-1 \in A$. By a similar argument to the case we see $2^t+2 \in A$. Then by Lemma 5.5 $2^{t+1}+2 \in A$, and by Lemmas 5.3, 5.4 $2^{t+2}-3 \in A$ and $2^{t+2}-1 \in A$ contradicting to the choice of t . Thus the case $7 \notin A$ does not occur, and the Theorem 5.1 has been established.

By (1.4) and (1.5), $4 \notin A$ implies $5, 6, 7 \notin A$. Then by (1.8) $9, 10, 11 \notin A$, and $11 \notin A$ implies $13 \notin A$ by (1.5). Again using (1.8) repeatedly we have

(5.1). $4 \notin A$ implies $2^t+i \notin A$ for $i=1, 2, 3, 5$ and $t \geq 2$.

Lemma 5.6. *Let $t \geq 4$. If an allowable sequence A satisfies $4 \notin A$, $2^t-1 \in A$ and $2^t+6 \in A$ then $2^{t+1}-3 \in A$ and $2^{t+1}-1 \in A$.*

Proof. First assume $2^{t+1}-3 \notin A$. Then by (3.4) $Sq^{2^t}x_{2^t-1}=x_{2^t-2}x_{2^t-1}+a_1x_{2^t-3}x_{2^t}+a_5x_{2^t-7}x_{2^t+4}+a_7x_{2^t-9}x_{2^t+6}+\dots$. Since $2^t+3 \notin A$ by (5.1) we have $Sq^8Sq^{2^t-2}x_{2^t-1}=0$ by (4.1). Then we have a contradiction: $0=\left\{\binom{2^t-3}{8}+a_1\binom{2^t-4}{2}+a_5\binom{2^t-8}{6}+a_7\binom{2^t-10}{8}\right\}x_{2^t-1}x_{2^t+6}+\dots=x_{2^t-1}x_{2^t+6}+\dots$. Thus $2^{t+1}-3 \in A$. Next assume $2^{t+1}-1 \notin A$. As in the proof of Lemma 5.4, we have $Sq^{2^t-1}x_{2^t}=x_{2^t-1}x_{2^t}+a_4x_{2^t-5}x_{2^t+4}+\dots$. Since $2^t+2 \notin A$ by (5.1), we have $Sq^4Sq^{2^t-1}x_{2^t}=0$. Then we have a contradiction: $0=\left\{\binom{2^t-1}{4}+a_4\binom{2^t-6}{4}\right\}x_{2^t-1}x_{2^t+4}+\dots=x_{2^t-1}x_{2^t+4}+\dots$, where $2^t+4 \in A$ since $2^t+6 \in A$. Thus $2^{t+1}-1 \in A$.

Lemma 5.7. *Let $t \geq 3$. If an allowable sequence A satisfies $4 \notin A$, $2^t+4 \in A$ and $2^{t+1}-1 \in A$ then $2^t+7 \in A$ or $2^{t+1}+4 \in A$.*

Proof. Assume $2^t+7 \notin A$ and $2^{t+1}+4 \notin A$. By a similar argument to the proof

of Lemma 5.5, we have $Sq^{2^t}x_{2^t+4} = x_8x_{2^t+1-4} + \text{other terms}$. Then $Sq^3Sq^{2^t}x_{2^t+4} = x_8x_{2^t+1-1} + \text{other terms}$. By Adem relation $Sq^3Sq^{2^t} = Sq^4Sq^{2^t-1} + Sq^2Sq^{2^t-1}Sq^2 + Sq^{2^t}Sq^3$. Then by use of (5.1) and the assumption $2^t+7 \notin A$, $Sq^3Sq^{2^t}x_{2^t+4} \in Sq^4D^2 + Sq^2D^2 + Sq^{2^t}D^2$. Thus at least one of $x_8Sq^4x_{2^t+1-5} = \binom{2^{t+1}-6}{4}x_8x_{2^t+1-1}$, $x_8Sq^2x_{2^t+1-3} = \binom{2^{t+1}-4}{2}x_8x_{2^t+1-1}$ and $x_8Sq^{2^t}x_{2^t+1-1} = 0$ coincides with $x_8x_{2^t+1-1}$. But these coefficients are zero modulo 2, and we get a contradiction. Therefore $2^t+7 \in A$ or $2^{t+1}+4 \in A$.

Lemma 5.8. *Let $t \geq 4$. If an allowable sequence A satisfies $4 \notin A$, $2^t+4 \in A$ and $2^{t+1}-3 \in A$ then $2^{t+1}+4 \in A$.*

Proof. Assume $2^{t+1}+4 \notin A$, then as above we have $Sq^{2^t}x_{2^t+4} = x_8x_{2^t+1-4} + \text{other terms}$ and $Sq^1Sq^{2^t}x_{2^t+4} = x_8x_{2^t+1-3} + \text{other terms}$. By Adem relation $Sq^1Sq^{2^t} = Sq^{2^t+1} = Sq^4Sq^{2^t-3} + Sq^{2^t-1}Sq^2 + Sq^{2^t}Sq^1 = Sq^4Sq^{2^t-3} + Sq^{2^t-1}Sq^{2^t-1}Sq^2 + Sq^{2^t}Sq^1$. By (5.1) and Lemma 1.2 $Sq^{2^t-3}x_{2^t+4}$ and $Sq^1x_{2^t+4}$ are decomposable. Also, $Sq^{2^t-1}Sq^2x_{2^t+4} \equiv Sq^{2^t-1}x_{2^t+6} \equiv \binom{2^t+5}{2^{t-1}}x_{2^t+2^t-1+6} \equiv 0 \pmod{D^2}$. Thus $Sq^1Sq^{2^t}x_{2^t+4} \in Sq^4D^2 + Sq^{2^t-1}D^2 + Sq^{2^t}D^2$, and $x_8x_{2^t+1-3}$ coincides at least one of $x_8Sq^4x_{2^t+1-7} = \binom{2^{t+1}-8}{4}x_8x_{2^t+1-3} = 0$, $x_8Sq^{2^t-1}x_{2^t+2^t-1-2} = \binom{2^t+2^{t-1}-3}{2^{t-1}-1}x_8x_{2^t+1-1} = 0$ and $x_8Sq^{2^t}x_{2^t-3} = 0$ which is a contradiction. Therefore we have $2^{t+1}+4 \in A$.

Proof of Theorem 5.2.

Case 1: $8 \in A$. By Lemma 2.4, $A/2$ is simple and $2 \notin A/2$, $4 \in A/2$. It follows from Theorem 5.1 $A/2 = G_m$ for some $m \geq 2$. Thus $2G_m \subset A$ and $A - 2G_m$ consists of odd integers. By (1.11), we have that there exists an integer s such that $2^s - 1 \in A$ and A does not contain odd integers smaller than $2^s - 1$. By (1.4), $2^s - 2 \in A$ and $2^{s-1} - 1 \in A/2 = G_m$. Then we have $3 \leq s - 1 \leq m + 1$ and $4 \leq s \leq m + 2$. Since $2^{t-1} + 2$, $2^{t-1} + 3 \in G_m = A/2$ for $3 \leq t \leq m + 1$ we have $2^t + 4$, $2^t + 6 \in A$ for $3 \leq t \leq m + 1$. Now assume that $4 \leq s < m + 2$. Using Lemma 5.6 for $t = s, \dots, m + 1$ we have from $2^s - 1 \in A$ that $2^{m+2} - 3 \in A$. Then, by Lemma 5.8, $2^{m+2} + 4 \in A$ which contradicts to $2^{m+1} + 2 \notin G_m = A/2$. Therefore we must have $s = m + 2$. By (1.4) and by the structure of $G_m = A/2$, A does not contain any integer greater than $2^{m+2} - 1$. Thus $A = 2G_m \cup (2^{m+2} - 1)$. Next, by Lemma 5.7, $2^{m+2} - 1 \in A$ implies $2^{m+1} + 7 \in A$ or $2^{m+2} + 4 \in A$, but $2^{m+2} + 4 \notin A$ as above. Thus $2^{m+1} + 7 \in A$, hence $2^{m+1} + 7 = 2^{m+2} - 1$. It follows that $m = 2$ and $A = 2G_2 \cup (15) = (8, 12, 14, 15) = E_3$.

Case 2: $8 \notin A$. By (1.7), $a_1 = 2^n$ for some n . Since $1, 2, 4, 8 \notin A$, $n \geq 4$. We shall prove by induction on $n \geq 4$ and assume the theorem is true for $n - 1$, i.e., $A/2 = E_{n-1}$. Then $2E_{n-1} \subset A$ and $A - 2E_{n-1}$ consist of some odd integers i of $i - 1 \in 2E_{n-1}$, i.e., $i = 2^{n+1} - 2^s + 1$ for some s of $n \geq s \geq 1$. Since $\binom{2^{n+1} - 2^s - 2^{s-1}}{2^{s-1}} \equiv 1$, $2^{n+1} - 2^s + 1 \in A$ implies $2^{n+1} - 2^s - 2^{s-1} + 1 \in A$. This shows that s must be 1, and we have $A = 2E_{n-1} \cup (2^{n+1} - 1) = (2^{n+1} - 2^s; n \geq s \geq 0) = E_n$. This completes the proof of Theorem 5.2.

6. Allowability of the sequences G_m and E_n .

In order to complete the proof of our main theorem we shall prove that the

sequences T_1 , A_l , G_m and E_n in the theorem are allowable and simple. By Lemma 2.3, these sequences are simple if it is allowable. We have well-known examples $P(T_1)=H^*(RP^\infty; Z_2)$ and $P(A_l)=H^*(BSO(l+1); Z_2)=Z_2[w_2, w_3, \dots, w_l]$ where w_i denotes the i -th Stiefel-Whitney class. Thus we have

(6.1). $T_1=(1)$ and $A=(2, 3, \dots, l+1)$, $l+1$ is not a power of 2, are strictly allowable hence allowable and simple.

The Stiefel-Whitney classes w_i satisfy the following Wu formula

$$(6.2). \quad Sq^j w_i = \sum_{0 \leq k \leq j} \binom{i-j+k-1}{k} w_{j-k} w_{i+k}, \quad w_0=1, \quad w_1=0.$$

The following statements are probably well-known.

$$(6.3). \quad (i). \quad \binom{2^t-k-1}{k} \equiv 1 \pmod{2} \text{ if and only if } k=0.$$

$$(ii). \quad \binom{2^t-2^s-k-1}{k} \equiv 1 \pmod{2} \text{ if and only if } k=2^q-2^s \text{ for } s \leq q < t.$$

In fact, if $k=2^r(2a+1)$, $a \geq 0$, then $\binom{2^t-k-1}{k} \equiv \binom{2^t-2^{r+1}(a+1)}{2^{r+1}a} \binom{2^r-1}{2^r} \equiv 0$, and we obtain (i). If $r \neq s$ we have similarly $\binom{2^t-2^s-k-1}{k} \equiv 0$. If $r=s$, we can put $k=2^q(2b+1)-2^s$, $s \leq q < t$. Then $\binom{2^t-2^s-k-1}{k} \equiv \binom{2^t-2^{q+1}(b+1)}{2^{q+1}b} \binom{2^q-1}{2^q-2^s} \equiv \binom{2^{t-q-1}-b-1}{b} \equiv 1$ if and only if $b=0$, by (i), i.e., $k=2^q-2^s$.

Now consider the squaring operations on $H^*(BSO(2^{n+1}-1); Z_2)=Z_2[w_2, w_3, \dots, w_{2^{n+1}-1}]$, $n \geq 1$. Let I be the ideal generated by $\{w_i, i \notin E_n\}$ ($E_1=A_2$, $E_2=G_2$).

Theorem 6.1. $H^*(BSO(2^{n+1}-1); Z_2)/I = P(E_n)$, $n \geq 1$, admits squaring operations, hence $E_n(E_1=A_2, E_2=G_2)$ is strictly allowable and it is allowable and simple.

Proof. It is sufficient to prove that I is closed under the squaring operations, or by Cartan formula to prove $Sq^j w_i \in I$ for $i \notin E_n$. Assume that

$$j-k=2^{n+1}-2^r, \quad i+k=2^{n+1}-2^s \quad \text{and} \quad \binom{i-j+k-1}{k} \equiv 1 \pmod{2}$$

for some i, j, k and $0 \leq r \leq n+1$, $0 \leq s \leq n+1$. Since $i-j+k-1=2^r-2^s-k-1$ it follows by (6.3), (ii) that $k=2^q-2^s$ for some $s \leq q < r$. Thus $i=2^{n+1}-2^q$, $0 \leq q \leq n$. This shows, by (6.2), that $Sq^j w_i \in I$ for $i \notin E_n$, completing the proof.

Next consider $H^*(BSO(2^{m+1}+1); Z_2)/D^3 = Z_2[w_2, w_3, \dots, w_{2^{m+1}+1}]/D^3 = TP(A_{2^{m+1}})$. Define Q_k , $k \geq 0$, inductively by

$$Q_0 = Sq^0 \quad \text{and} \quad Q_k = Sq^{2^k-1} Q_{k-1} \quad \text{for } k > 0.$$

By (6.2), $Sq^{2^t} w_{2^t+1} \equiv w_{2^t+1} \pmod{D^2}$. Then by induction on k we have

$$(6.4). \quad Q_k w_2 \equiv w_{2^k+1} \pmod{D^2} \quad \text{for } k \geq 0.$$

Let I' be the ideal generated by $Q_k w_2$, $k \geq 0$. Then it follows from (6.4) that $TP(A_{2^{m+1}})/I'$ is isomorphic to $TP(G_m \cup (2^{m+1}))$. Here remark that $Q_k w_2 = 0$, for $k > m+2$ by dimensional reason and $Q_{m+2} w_2 = w_{2^{m+1}+1} = w_{2^{m+1}} Q_{m+1} w_2$.

Lemma 6.2. $(H^*(BSO(2^{m+1}+1); Z_2)/D^3)/I' = TP(G_m \cup (2^{m+1}))$ admits squaring operations, hence $G_m \cup (2^{m+1})$ is allowable.

Proof. It is sufficient to prove

$$(*) \quad Sq^i Q_k w_2 \in I' \quad \text{for } i > 0.$$

By (1.1), (iv), $Sq^i Q_k w_2 = 0$ for $i > 2^k + 1$, $Sq^{2^k+1} Q_k w_2 = (Q_k w_2)^2$ and by definition $Sq^{2^k} Q_k w_2$

$=Q_k w_2$. Then it is sufficient to prove (*) for $0 < i < 2^k$ by induction on k . The case $k=0$ is trivial. By Adem relation, $Sq^i Q_{k+1} w_2 = Sq^i Sq^{2^k} Q_k w_2$ is a linear combination of $Sq^{2^k+i-j} Sq^j Q_k w_2 \in Sq^{2^k+i-j} I' \subset I'$. Thus (*) is proved and the lemma follows.

Applying Lemma 2.1 to the result of Lemma 6.2 we have

Theorem 6.3. G_m is allowable and simple for $m \geq 1 (G_1 = A_2)$.

Proof of Main Theorem. Let A be a simple sequence. By (1.7), $a_1 = 2^t$ for some $t \geq 0$. Then $A = T_1, A_t, G_m, E_n$ according to $t=0, 1, 2, t \geq 3$ by Theorems 4.1, 5.1, 5.2 ($t+1$ is not a power of 2). Conversely these sequences are simple by (6.1) and Theorems 6.1, 6.2.

Proof of Corollary to Main Theorem. By Theorem (1.1) of [4], $TP(A) = Z_2[x_j]/D^3$, $\alpha x_j = u_j$, admits squaring operations. Then the corollary follows from the main theorem.

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